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Risk-Sensitive Adaptive Trackers for Strict-Feedback Systems With Output Measurements

Gürdal Arslan and Tamer Başar

***Abstract*—This note investigates the control of stochastic nonlinear systems with parametric uncertainty. The class of systems considered are single-input–single-output and in strict-feedback form, with the performance measured with respect to a risk-sensitive cost criterion. The uncertainty in the system description is assumed to be linearly parameterized, where the unmeasured parameters are generated by stochastic differential equations. By employing the backstepping design technique on the estimates of the unmeasured states, provided by a simple state estimator, an output-feedback adaptive controller is constructed which maintains an arbitrarily small average value for the risk-sensitive cost. The controller designed achieves boundedness in probability for all closed-loop signals and, under certain conditions, the tracking error converges to zero almost surely.**

***Index Terms*—Adaptive backstepping, risk-sensitive control, stochastic systems, strict-feedback systems.**

I. Introduction

A recent research topic has been the design of feedback control laws to achieve stabilization or tracking for uncertain nonlinear systems where the additive uncertainty is assumed to be random. In the literature, the most common mathematical model used for such systems consists of a set of stochastic differential equations interpreted in the Itô sense; see [1] and [2]. Simply because of the extra quadratic variation terms resulting from the Itô differentiation rule (see [3]), a control law designed for a deterministic system does not lead to a satisfactory solution to the corresponding stochastic control problem. Moreover, different notions of stability and performance indexes need to be used to determine the usefulness of the feedback controllers in a stochastic setup. The most natural stochastic counterparts of the "deterministic" concepts such as boundedness and (asymptotic) stability can be found in [4], whereas the more recent concept of "noise-to-state stability," in which the word "noise" refers to the intensity of the additive random noise, can be found in [5], which has several chapters on stabilization of stochastic nonlinear systems. Again, in the context of noise-to-state stability, [6] and [7] have developed adaptive controllers, equipped with state and output information, respectively, for stochastic strict-feedback systems, where the adaptive nature of the controllers is related to the unknown intensity of the additive random disturbances. A more relevant concept to our work, however, is "risk-sensitive cost criterion" in which not only the mean value but also the variance of an integral cost is penalized; see [8] in the linear context, and [9] and [10] in the nonlinear context. The rigorous investigation of the risk-sensitive index presented in [10] revealed that for a nonlinear system the H_2 and H_∞ norms can be recovered as the small risk and the small noise limits (respectively) of the risk-sensitive index. Finally, we cite [15] as the key reference that has presented a control design for strict-feedback systems perturbed by random disturbances, which results in closed-loop signals achieving an arbitrarily small average risk-sensitive cost.

The previous work on this general topic has not addressed, however, the problem of adaptive control of strict-feedback systems for tracking in the face of additive random disturbances as well as time-varying uncertainty, which is what we do in this note, where we take the time-varying unknown parameters enter the system dynamics linearly. The note can also be viewed as one extending the results of [12] to strict-feedback systems with additional uncertainty in the form of known functions multiplied by time-varying unknown parameters and output measurements. We note that it is possible to construct identifiers, which are stochastic counterparts of those presented in [11], to estimate the unknown parameters based on the relationship between risk-sensitive stochastic designs and stochastic games. One can then attempt to use these estimates to design an adaptive controller which makes a relevant risk-sensitive cost for the closed-loop system bounded. Nevertheless, for simplicity and to avoid technical difficulties, we will not follow that path in this note; instead, we will construct simpler adaptive controllers so that a meaningful average risk-sensitive cost for the closed-loop system becomes arbitrarily small; see [12] for our preliminary results on the latter approach. Since we consider the case where only the output is available for feedback, we use a simple state estimator to estimate the unmeasured states. Then using these estimates, we design an output feedback adaptive controller that maintains an arbitrarily small (which could be zero, under certain conditions) average risk-sensitive cost; see [13] for the case where the system has no unknown parameters. The controller design technique used in the note is based on the well-known back-stepping methodology. The adaptive controller keeps the closed-loop signals bounded in probability and under certain conditions make the tracking error asymptotically approach the origin with probability one.

One can of course consider the problem addressed in this note a stochastic *nonlinear* adaptive control problem under a risk-sensitive cost criterion, and as such one can attach *a priori* fixed weights on the states and the control action. However, this is a very challenging optimal control problem because of the nonlinear nature of the dynamics and incompleteness of the dynamic information available to the controller. The approach taken here circumvents this difficulty, but leads to suboptimal controllers, which are however computable and are finite dimensional. Since no cost is imposed on the control

Manuscript received July 18, 2001; revised April 21, 2002. Recommended by Associate Editor C. Wen. This work was supported in part by the U.S. Department of Energy under Grant DOE-DEFG-02-94ER13939.

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Digital Object Identifier 10.1109/TAC.2002.803553.

action, these controllers could involve high gain, particularly when the performance specifications on state variables are tight. However, since the design procedure offers substantial flexibility in choosing the weights on the state variables, a judicious choice of these parameters would lead to an acceptable tradeoff between good transient response of the states and boundedness of the control action.

II. Problem Formulation

A nonlinear system described by the following Itô differential equations is considered:

$$
\begin{aligned}
dx_i &= \left[x_{i+1} + f_i(y) + \phi_i^T(y)\theta\right] dt + h_i^T(y)\, dw \\
&\qquad i = 1, \ldots, r-1 \\
dx_r &= \left[b(y)u + f_r(y) + \phi_r^T(y)\theta\right] dt + h_r^T(y)\, dw \\
d\theta &= G(t)\, dw - m_N\left(|\theta|^2\right)\theta\, dt.
\end{aligned} \tag{1}
$$

Here

$x := [x_1, \ldots, x_r]^T$	state vector;
u	scalar control input;
$y := x_1$	scalar output;
θ	d-dimensional stochastic process generated by the previous third equation;
w	$\mathcal{R}^q$ valued standard Wiener process;

and the initial condition $(x(0), \theta(0))$ is fixed. All the functions involved in the plant model (1), and the function $1/b$, are known and are sufficiently smooth. The time-varying parameter vector θ is not measured, and the model describing the dynamics of θ is a generalization of the constant unknown parameter model. One of the most interesting cases captured by this dynamic uncertainty structure is the case where the unknown time-varying parameters experience small fluctuations around their mean values. The second term m_N on the right-hand side of the dynamics of θ guarantees the stochastic boundedness of θ (a relaxation of the usual assumption in the literature that the unknown parameters are bounded by some known bounds), and is defined as $m_N(|\theta|^2) = m(|\theta|^2/N - 1)$, where $N > 0$ is a known constant, and the increasing switching function $m(\rho)$: $\mathcal{R} \mapsto [0, 1]$ satisfies $m(\rho) = 0$ for $\rho \le 0$, and $m(\rho) = 1$ for $\rho \ge 1$.

The objective is to design a feedback controller such that the states of the plant (1) remain bounded in probability, and the following risk-sensitive performance inequality holds:

$$
\limsup_{T\to\infty} \frac{2}{\mu T} \ln E \exp\left[\frac{\mu}{2}\int_0^T \left[(y - y_d)^2 + l(t, x)\right] dt\right] \le R \tag{2}
$$

where y_d is the reference signal to be tracked, $\mu > 0$ is an arbitrary risk-sensitivity parameter,[1] $R > 0$ is an arbitrary constant representing an upper bound on the desired average risk-sensitive cost, and $l(t, x) \ge 0$ is some weight function. We further want to achieve a zero average risk-sensitive cost, i.e., $R = 0$, and almost sure convergence of the tracking error to zero, provided that the following conditions are all met:

- C1) the reference signal $y_d(t)$ is a constant;
- C2) $G(t) = 0_{d\times q}$ for all $t \ge 0$;
- C3) the functions $f_i(y)$, $\phi_i(y)$, $h_i(y)$ vanish at $y = y_d$, $\forall i \in \{1, \ldots, r\}$.

Although we will not be able to pick the weight function $l(t, x)$ arbitrarily, the design procedure will still give us a considerable degree of freedom in shaping this function. We finally make the following two assumptions.

[1]Although the symbol "θ" is commonly used to denote the risk-sensitivity parameter in the literature, we prefer to use instead "μ" here to eliminate any source of confusion with the notation used for unknown parameters.

Assumption 1: The reference trajectory $y_d(t)$ is r times continuously differentiable, where $y_d(t)$ and its derivatives $y_d^{(1)}(t), \ldots, y_d^{(r)}(t)$ are known and uniformly bounded by some constant, say, $M_d > 0$. We also use the notation $y_d^{[i]}$ to denote $[y_d, y_d^{(1)}, \ldots, y_d^{(i)}]^T$.

Assumption 2: $\sup_{t\ge 0} \mathrm{Tr}[G(t)G^T(t)] \le M_G$, for some known constant $M_G > 0$.

Assumption 3: There exists a known $M_H > 0$ such that $\sup_{y\in\mathcal{R}} \sigma_{\max}^2(H(y)) \le M_H$, where $H(y) := [h_1(y), \ldots, h_r(y)]^T$. This technical assumption is necessary to bound the variance of the filtering error by appropriate terms, which will be clear later.

III. The Estimator

Since $[x_2, \ldots, x_r]^T$ is not measured, it has to be estimated using the available online information, which is y. For this purpose, we rewrite the plant dynamics (1) as

$$
\begin{aligned}
dx &= [Ax + F(y) + \Phi(y)\theta + B(y)u]\, dt + H(y)\, dw \\
d\theta &= G(t)\, dw - m_N\left(|\theta|^2\right)|\theta|^2\, dt.
\end{aligned} \tag{3}
$$

Let us assume for the moment that θ is known. Then, to construct the state x in (3), we introduce the following two deterministic filters from [14], which equally apply to the stochastic system here

$$
\begin{aligned}
d\chi &= [A_o\chi + ky + F(y) + B(y)u]\, dt, \qquad \chi(0) = \chi_0 \\
d\xi &= [A_o\xi + \Phi(y)]\, dt, \qquad \xi(0) = \xi_0
\end{aligned} \tag{4}
$$

where $C := [1, 0_{1\times r-1}]^T$, $k := [k_1, \ldots, k_r]^T$ is some gain vector $A_o := A - kC^T$, and the initial conditions χ_0 and ξ_0 are arbitrarily picked. Since Φ and H are smooth functions, we can find constants $\psi_\Phi^0 \ge 0$, $\psi_H^0 \ge 0$ and nonnegative-valued functions $\psi_\Phi(z_1, y_d)$, $\psi_H(z_1, y_d)$ such that $\mathrm{Tr}[\Phi\Phi^T] \le \psi_\Phi^0 + z_1^2\psi_\Phi(z_1, y_d)$, and $\mathrm{Tr}[HH^T] \le \psi_H^0 + z_1^2\psi_H(z_1, y_d)$, where z_1 is the tracking error, i.e., $z_1 := y - y_d$. The gain vector k is picked in such a way that there exists a positive–definite matrix P that satisfies

$$
PA_o + A_o^T P + 2\,\mu M_H PP + P \le 0
$$

with

$$
\lambda_{\max}(P) = \begin{cases} 1, & \text{if } \psi_H^0 = 0 \\ R/(R + 2\psi_H^0), & \text{otherwise.} \end{cases}
$$

The state x of (3) is now reconstructed as

$$
\hat{x} =: \chi + \xi\theta \tag{5}
$$

which contains the unknown parameter θ. Again, assuming knowledge of θ, the estimation error $\tilde{x} := x - \hat{x}$ satisfies

$$
d\tilde{x} = A_o\tilde{x}\, dt + H\, dw \tag{6}
$$

which indicates that $\tilde{x}$ converges to zero exponentially in the absence of random disturbances and when θ is known; see [14]. For future reference, we introduce $W := \mathrm{Tr}[\xi^T P\xi]$ and $\tilde{W} := |\tilde{x}|_P^2$, whose Itô differentials satisfy

$$
\begin{aligned}
dW &\le \left[-W/2 + 2\lambda_{\max}(P)\left[\psi_\Phi^0 + z_1^2\psi_\Phi(z_1, y_d)\right]\right] dt \\
d\tilde{W} &\le \left[-\tilde{W} + (R/2)I_{\{\psi_H^0 > 0\}} + z_1^2\psi_H(z_1, y_d)\right] dt \\
&\quad + \tilde{\sigma}^T\, dw - \frac{\mu}{2}\,|\tilde{\sigma}|^2\, dt
\end{aligned} \tag{7}
$$

where $\tilde{\sigma} := 2H^T P\tilde{x}$. We note that the estimator introduced above is not the only possible choice to achieve our objective (2). In fact, we can use a state estimator which itself is risk-sensitive optimal [but more complex than (5)] to achieve the same objective (2). Nonetheless, we will prefer to use (5) because of its simplicity, and construct, in the next subsection, an output feedback controller which will use the state estimate generated by (5). Our controller design will again be based on the backstepping design technique which will be applied on the signals $x_1, \chi_2, \ldots, \chi_r$. This way, we will effectively convert the problem to one with perfect state feedback with parametric uncertainty as will be clear in the following section.

IV. Controller Design

In this section, we present design steps of an output-feedback controller that achieves an arbitrarily small positive average risk-sensitive cost for the plant (1), by employing the backstepping design methodology.

Step 1: We start by computing the Itô differential of the tracking error z_1 as follows:

$$dz_1 = \left(x_2 + \overline{f}_1 + \overline{\phi}_1^T\theta\right) dt + \overline{h}_1^T dw - y_d^{(1)} dt \tag{8}$$

where the functions with the overline symbol denote the equivalents of those without the overline symbol in terms of the new variable z_1. This notation will be used throughout to denote the equivalent forms of the functions in terms of $z_1, \ldots, z_r$. We decompose ξ as $\xi =: [\xi_1, \ldots, \xi_r]^T$, where the rows of ξ satisfy

$$d\xi_i = \left[-k_i\xi_1 + \xi_{i+1} + \overline{\phi}_i\right] dt =: \eta_i(z_1, y_d, \xi_1, \xi_{i+1})\, dt$$
$$i = 1, \ldots, r,\ \xi_{r+1} := 0.$$

We then write (8) as

$$dz_1 = \left[\chi_2 + r_1 + p_1\left(\overline{\phi}_1^T\theta + \xi_2^T\theta + \tilde{x}_2\right)\right] dt + s_1^T\, dw$$

where $r_1(z_1, y_d^{[1]}) := \overline{f}_1 - y_d^{(1)}$, $p_1 := 1$, $s_1(z_1, y_d) := \overline{h}_1$. From the fact that s_1 is smooth, it follows that we can find smooth nonnegative-valued functions $\psi_{10}(y_d)$ and $\psi_{11}(z_1, y_d)$ such that $|s_1|^2 \le \psi_{10} + z_1^2\psi_{11}$. Next, the smooth function $V_1 := \Xi_1 z_1^2 + \Gamma|\tilde{\theta}_1|^2$ is introduced, where $\tilde{\theta}_1 := \theta - \hat{\theta}_1$

$$\Xi_1(y_d) := \begin{cases} 1, & \text{if } \psi_{10} = 0 \\ R/\left(R + 4r\psi_{10}\right), & \text{otherwise} \end{cases}$$

$$\Gamma := \begin{cases} \min\left\{\dfrac{1}{8r\,\mu M_G}, \dfrac{R/(4r)}{M_G + 4N}\right\}, & \text{if } M_G > 0 \\ 1, & \text{if } M_G = 0 \end{cases}$$

$$d\hat{\theta}_1 := \left[\Xi_1 z_1 p_1\left(\overline{\phi}_1 + \xi_2\right)/\Gamma - m_N\left(|\hat{\theta}_1|^2\right)\hat{\theta}_1\right] dt$$
$$=: \delta_1\left(z_1, y_d, \hat{\theta}_1, \xi_2\right) dt$$

where $\hat{\theta}_1(0) = \theta_0$ for some arbitrary θ_0. Its Itô differential satisfies

$$\begin{aligned} dV_1 \le\ & 2\Xi_1 z_1\left[\chi_2 + m_1\right] dt + \sigma_1^T\, dw - \frac{r\mu}{2}|\sigma_1|^2\, dt \\ & - \frac{\Gamma}{2} m_{4N}\left(|\tilde{\theta}_1|^2\right)|\tilde{\theta}_1|^2\, dt + \left(|\tilde{x}|_P^2/(2r)\right) dt \\ & + \left[\Xi_1\psi_{10} + (R/(4r))I_{\{M_G>0\}}\right] dt \end{aligned} \tag{9}$$

where

$$\sigma_1 := 2\Xi_1 z_1 s_1 + 2\Gamma G^T\tilde{\theta}_1$$
$$\begin{aligned} m_1 :=\ & r_1 + p_1\left(\overline{\phi}_1 + \xi_2\right)^T\hat{\theta}_1 + \frac{r\Xi_1 z_1}{\lambda_{\min}(P)} + \frac{z_1\psi_{11}}{2} \\ & + 2r\,\mu\Xi_1|s_1|^2 z_1 + \frac{z_1}{2\Xi_1}\frac{\partial\Xi_1}{\partial y_d}y_d^{(1)} \end{aligned}$$

and, in the first inequality, we used the following facts:

$$2\Gamma\tilde{\theta}_1^T\left[-m_N\left(|\theta|^2\right)\theta + m_N\left(|\hat{\theta}_1|^2\right)\hat{\theta}_1\right] \le -\Gamma m_{4N}\left(|\tilde{\theta}_1|^2\right)|\tilde{\theta}_1|^2$$
$$2\Xi_1 z_1\tilde{x}_2 \le |\tilde{x}|_P^2/(2r) + 2r\Xi_1^2 z_1^2/\left(\lambda_{\min}(P)\right).$$

Now, the desired value of the virtual control input χ_2 is picked as

$$\begin{aligned} \alpha_1\left(z_1, y_d^{[1]}, \hat{\theta}_1, \xi_2\right) :=\ & -m_1 - z_1 \\ & \cdot\left[\beta_1 + r + 2\lambda_{\max}(P)\psi_\Phi + \psi_H\right]/(2\Xi_1) \end{aligned}$$

where $\beta_1(z_1, y_d^{[1]}, \hat{\theta}_1, \xi_2) > 0$ is some design function. By defining the second error term z_2 as $z_2 = \chi_2 - \alpha_1$, (9) can be written as

$$\begin{aligned} dV_1 \le & \Big[2\Xi_1 z_1 z_2 - (\beta_1 + r + 2\lambda_{\max}(P)\psi_\Phi + \psi_H)\, z_1^2 \\ & - \frac{\Gamma}{2} m_{4N}\left(|\tilde{\theta}_1|^2\right)|\tilde{\theta}_1|^2\Big]\, dt, + \sigma_1^T\, dw - \left(r\mu|\sigma_1|^2/2\right) dt \\ & + \left[|\tilde{x}|_P^2/(2r) + \Xi_1\psi_{10} + (R/(4r))I_{\{M_G>0\}}\right] dt. \end{aligned}$$

This completes Step 1.

Step k ($k = 2, \ldots, r-1$): Assume the following structure from the previous step:

$$z_1 = x_1 - y_d \quad z_i = \chi_i - \alpha_{i-1}\left(z_{[i-1]}, y_d^{[i-1]}, \hat{\theta}_{[i-1]}, \xi_{[i]}\right)$$
$$i = 2, \ldots, k$$
$$\begin{aligned} dz_i = & \Big[z_{i+1} + \alpha_i + r_i\left(z_{[i]}, y_d^{[i]}, \hat{\theta}_{[i-1]}, \xi_{[i+1]}\right) \\ & + p_i\left(z_{[i-1]}, y_d^{[i-1]}, \hat{\theta}_{[i-1]}, \xi_{[i]}\right)\left(\overline{\phi}_1^T\theta + \xi_2^T\theta + \tilde{x}_2\right)\Big] dt \\ & + s_i^T\left(z_{[i-1\vee 1]}, y_d^{[i-1]}, \hat{\theta}_{[i-1]}, \xi_{[i]}\right) dw \\ & i = 1, \ldots, k-1 \end{aligned}$$
$$\tilde{\theta}_i = \theta - \hat{\theta}_i, \quad d\hat{\theta}_i = \delta_i\left(z_{[i]}, y_d^{[i-1]}, \hat{\theta}_{[i]}, \xi_{[i\vee 2]}\right) dt$$
$$i = 1, \ldots, k-1$$
$$V_{k-1} = \sum_{i=1}^{k-1}\left[\Xi_i\left(z_{[i-1]}, y_d^{[i-1]}, \hat{\theta}_{[i-1]}, \xi_{[i]}\right) z_i^2 + \Gamma|\tilde{\theta}_i|^2\right]$$
$$\begin{aligned} dV_{k-1} \le & -z_1^2\left(1 + 2\lambda_{\max}(P)\psi_\Phi + \psi_H\right) dt + 2\Xi_{k-1}z_{k-1}z_k\, dt \\ & - \sum_{i=1}^{k-1}(\beta_i + r - k + 1)\, z_i^2\, dt \\ & - \frac{\Gamma}{2}\sum_{i=1}^{k-1} m_{4N}\left(|\tilde{\theta}_i|^2\right)|\tilde{\theta}_i|^2\, dt + \sum_{i=1}^{k-1}\left[\sigma_i^T\, dw - \frac{r\mu}{2}|\sigma_i|^2\, dt\right] \\ & + \sum_{i=1}^{k-1}\left[|\tilde{x}|_P^2/(2r) + \Xi_i\psi_{i0} + (R/(4r))I_{\{M_G>0\}}\right] dt \end{aligned}$$

where $z_{[i]} = [z_1, \ldots, z_i]^T$. From this, the Itô differential of z_k is obtained as

$$dz_k = \left(\chi_{k+1} + r_k + p_k\left(\overline{\phi}_1^T\theta + \xi_2^T\theta + \tilde{x}_2\right)\right) dt + s_k^T\, dw \tag{10}$$

where

$$\begin{aligned} & r_k\left(z_{[k]}, y_d^{[k]}, \hat{\theta}_{[k-1]}, \xi_{[k+1]}\right) \\ & = k_k(z_1 + y_d) + \overline{f}_k - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial z_i}[z_{i+1} + \alpha_i + r_i] \\ & - \sum_{i=1}^{k}\frac{\partial\alpha_{k-1}}{\partial y_d^{(i-1)}}y_d^{(i)} - \sum_{i=1}^{k}\frac{\partial\alpha_{k-1}}{\partial\xi_i^T}\eta_i \\ & - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}_i^T}\delta_i - \frac{1}{2}\sum_{i,j\in\{1,\ldots,k-1\}}\frac{\partial^2\alpha_{k-1}}{\partial z_i\partial z_j}s_i^T s_j \end{aligned}$$

$$\begin{aligned} p_k\left(z_{[k-1]},\, y_d^{[k-1]},\, \hat{\theta}_{[k-1]},\, \xi_{[k]}\right) &= -\sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial z_i}\, p_i \\ s_k\left(z_{[k-1]},\, y_d^{[k-1]},\, \hat{\theta}_{[k-1]},\, \xi_{[k]}\right) &= -\sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial z_i}\, s_i. \end{aligned}$$

Since s_k is a smooth function, we can find nonnegative valued functions $\psi_{k0}(y_d^{[k-1]},\, \hat{\theta}_{[k-1]},\, \xi_{[k]})$ and $\psi_{k1}(z_{[k-1]},\, y_d^{[k-1]},\, \hat{\theta}_{[k-1]},\, \xi_{[k]})$, which are also smooth, such that $|s_k|^2 \le \psi_{k0} + |z_{[k-1]}|^2 \psi_{k1}$. Now, the smooth function $V_k = V_{k-1} + \Xi_k z_k^2 + \Gamma|\tilde{\theta}_k|^2$ is introduced, where $\tilde{\theta}_k := \theta - \hat{\theta}_k$

$$\Xi_k\left(z_{[k-1]},\, y_d^{[k-1]},\, \hat{\theta}_{[k-1]},\, \xi_{[k]}\right) := \begin{cases} 1/(1+\psi_{k1}), & \text{if } \psi_{k0} = 0 \\ R/(R + R\psi_{k1} + 4r\psi_{k0}), & \text{otherwise} \end{cases} \tag{11}$$

$$\begin{aligned} d\hat{\theta}_k &:= \left[\left(2\Xi_k z_k p_k + z_k^2 \sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}\, p_i\right)\left(\overline{\phi}_1 + \xi_2\right)/(2\Gamma) - m_N\left(|\hat{\theta}_k|^2\right)\hat{\theta}_k\right] dt \\ &=: \delta_k\left(z_{[k]},\, y_d^{[k-1]},\, \hat{\theta}_{[k]},\, \xi_{[k]}\right) dt \end{aligned} \tag{12}$$

where $\hat{\theta}_k(0) = \theta_0$. The Itô differential of V_k satisfies

$$\begin{aligned} dV_k \le\, & dV_{k-1} + 2\Xi_k z_k[x_{k+1} + m_k]\, dt + \sigma_k^T\, dw \\ & - \frac{r\mu}{2}|\sigma_k|^2\, dt - \frac{\Gamma}{2} m_{4N}\left(|\tilde{\theta}_k|^2\right)|\tilde{\theta}_k|^2\, dt \\ & \cdot \left[|\tilde{x}|_P^2/(2r) + |z_{[k-1]}|^2\, dt + \Xi_k\psi_{k0} \right. \\ & \left. + (R/(4r)) I_{\{M_G>0\}}\right] dt \end{aligned} \tag{13}$$

where

$$\begin{aligned} \sigma_k :=\, & 2\Xi_k z_k s_k + z_k^2 \sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}\, s_i + 2\Gamma G^T \tilde{\theta}_k \\ m_k :=\, & r_k + \left(p_k + \frac{z_k}{2\Xi_k} \sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}\, p_i\right)(\overline{\phi}_1^T + \xi_2^T)\hat{\theta}_k \\ & + \frac{z_k}{2\Xi_k}\sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}(z_{i+1} + \alpha_i + r_i) \\ & + \frac{z_k}{2\Xi_k}\sum_{i=1}^{k} \frac{\partial \Xi_k}{\partial y_d^{(i-1)}}\, y_d^{(i)} + \frac{z_k}{2\Xi_k}\sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial \hat{\theta}_i^T}\, \delta_i \\ & + \frac{z_k}{2\Xi_k}\sum_{i=1}^{k} \frac{\partial \Xi_k}{\partial \xi_i^T}\, \eta_i + \frac{1}{\Xi_k}\sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}\, s_k^T s_i \\ & + \frac{z_k}{4\Xi_k}\sum_{i,\, j\in\{1,\, \ldots,\, k-1\}} \frac{\partial^2 \Xi_k}{\partial z_i \partial z_j}\, s_i^T s_j \\ & + \frac{r\mu z_k}{2\Xi_k}\left|2\Xi_k s_k + z_k \sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}\, s_i\right|^2 \\ & + \frac{r z_k}{4\Xi_k \lambda_{\min}(P)}\left[2\Xi_k p_k + z_k \sum_{i=1}^{k-1} \frac{\partial \Xi_k}{\partial z_i}\, p_i\right]^2. \end{aligned}$$

The desired value of the virtual control input x_{k+1} is picked as

$$\begin{aligned} \alpha_k&\left(z_{[k]},\, y_d^{[k]},\, \hat{\theta}_{[k]},\, \xi_{[k+1]}\right) \\ &:= -m_k - (\beta_k + r - k)z_k/(2\Xi_k) - \Xi_{k-1}z_{k-1}/\Xi_k \end{aligned} \tag{14}$$

where $\beta_k(z_{[k]},\, y_d^{[k]},\, \hat{\theta}_{[k]},\, \xi_{[k+1]}) > 0$ is some design function. We can now rewrite (13) in terms of the $(k+1)$st error term z_{k+1}, defined as $z_{k+1} := \chi_{k+1} - \alpha_k$, as

$$\begin{aligned} dV_k \le\, & -z_1^2\left(1 + 2\lambda_{\max}(P)\psi_\Phi + \psi_H\right) dt + 2\Xi_k z_k z_{k+1}\, dt \\ & - \sum_{i=1}^{k}(\beta_i + r - k) z_i^2\, dt + \sum_{i=1}^{k} \frac{|\tilde{x}|_P^2}{2r}\, dt \\ & + \sum_{i=1}^{k}\left[\sigma_i^T\, dw - \frac{r\mu}{2}|\sigma_i|^2\, dt\right] + \sum_{i=1}^{k} \\ & \cdot \left[\Xi_i\psi_{i0} + \frac{R}{4r} I_{\{M_G>0\}} - \frac{\Gamma}{2} m_{4N}\left(|\tilde{\theta}_i|^2\right)|\tilde{\theta}_i|^2\right] dt. \end{aligned} \tag{15}$$

Since all of the relevant definitions and results of Step k are consistent with the induction hypothesis, it is concluded that the induction hypothesis holds true for $k \in \{1,\, \ldots,\, r-1\}$.

Step r: It is easy to see that the results of Step k hold true also for $k = r$ if χ_{r+1} is defined as $\chi_{r+1} := \overline{b}u$, where u is the actual control input. Thus, the $(r+1)$th error term z_{r+1} can be made zero by picking the control input as

$$u = \alpha_r\left(z_{[r]},\, y_d^{[r]},\, \hat{\theta}_{[r]},\, \xi_{[r]}\right)\Big/\overline{b}(z_1,\, y_d) \tag{16}$$

where α_r is obtained by setting $k = r$ in (14). With the control input (16), the Itô differential of the smooth function $V := V_r = \sum_{i=1}^{r}[\Xi_i(z_{[i-1]},\, y_d^{[i-1]},\, \hat{\theta}_{[i-1]},\, \xi_{[i]})z_i^2 + \Gamma|\tilde{\theta}_i|^2]$, where Ξ_r is obtained by setting $k = r$ in (11), satisfies (15) with index k set to r. We now note that, from (7) and (15), we have

$$\begin{aligned} dV + d\tilde{W} \le\, & -z_1^2\left(1 + 2\lambda_{\max}(P)\psi_\Phi\right) dt - \sum_{i=1}^{r}\beta_i z_i^2\, dt \\ & - \frac{1}{2}|\tilde{x}|_P^2\, dt - \frac{\Gamma}{2}\sum_{i=1}^{r} m_{4N}\left(|\tilde{\theta}_i|^2\right)|\tilde{\theta}_i|^2\, dt \\ & + \sigma^T\, dw - \frac{\mu}{4}|\sigma|^2\, dt \\ & + \left[\sum_{i=1}^{r}(\Xi_i\psi_{i0}) + (R/4)I_{\{M_G>0\}} + (R/2)I_{\{\psi_H^0>0\}}\right] dt \end{aligned}$$

where $\sigma = \sum_{i=1}^{r} \sigma_i + \tilde{\sigma}$. If conditions C1), C2), and C3) are satisfied, then we can pick $\psi_{10} = \cdots = \psi_{r0} = \psi_H^0 = M_G = 0$. If this is the case, then the last term in the aforementioned inequality drops, and R could be chosen exactly equal to zero to achieve a zero average risk-sensitive cost. This now leads to the following theorem.

Theorem 1: Consider the nonlinear system (3) under Assumptions 1, 2, and 3. If the design functions β_i are picked to satisfy $\beta_i \ge k_\beta$ for some $k_\beta > 0$, $\forall\, i \in \{1,\, \ldots,\, r\}$, then for any given risk-sensitivity parameter $\mu > 0$ and desired average risk-sensitive cost $R > 0$ (which could be chosen as zero, i.e., $R = 0$, if conditions C1), C2), and C3) are satisfied), the controller (16) achieves

1)

$$\limsup_{T\to\infty} \frac{2}{\mu T} \ln E \exp\frac{\mu}{2}\left[\int_0^T \left[z_1^2 + \sum_{i=1}^{r} \cdot \left[\beta_i z_i^2 + \frac{\Gamma}{2} m_{4N}\left(|\tilde{\theta}_i|^2\right)|\tilde{\theta}_i|^2\right] + \frac{|\tilde{x}|_P^2}{2}\right] dt\right] \le R.$$

2) The closed-loop signals are stochastically bounded in probability, i.e.,

$$\lim_{c\to\infty}\sup_{t\geq 0}\mathcal{P}\{|\zeta(t)|>c\}=0$$

where $\zeta(t) := [z_1(t), \ldots, z_r(t), \tilde{\theta}_1^T(t), \ldots, \tilde{\theta}_r^T(t), \tilde{x}^T(t), \xi_1^T(t), \ldots, \xi_r^T(t), \theta^T(t)]^T$.

3) If conditions C1), C2), and C3) are satisfied, then

a) $\mathcal{P}\{\sup_{t\geq 0}|\zeta(t)|<\infty\}=1$;
b) $\lim_{|\zeta(0)|\to 0}\mathcal{P}\{\sup_{t\geq 0}|\zeta(t)|>\epsilon\}=0$, for any $\epsilon>0$;
c) $\mathcal{P}\left\{\lim_{t\to\infty}|z_{[r]}(t)|^2+|\tilde{x}(t)|^2+|\xi_{[r]}(t)|^2=0\right\}=1$.

Proof: From (17), it follows that $\forall T\geq 0$

$$\begin{aligned}\frac{2}{\mu T}\ln E\exp\frac{\mu}{2}\Bigg[\int_0^T\Bigg[z_1^2+\sum_{i=1}^n\beta_i z_i^2+\frac{\Gamma}{2}\sum_{i=1}^r m_{4N}\\ \cdot\left(|\tilde{\theta}_i|^2\right)|\tilde{\theta}_i|^2+\tfrac{1}{2}|\tilde{x}|_P^2\Bigg]dt\Bigg]\\ \leq\frac{V(0)+\tilde{W}(0)}{T}+R+\frac{2}{\mu T}\ln E\exp\frac{\mu}{2}\int_0^T\\ \cdot\left[\sigma^T dw-\frac{\mu}{4}|\sigma|^2 dt\right].\end{aligned}$$

Since, from [15, Th. 11], the last term on the right-hand side of the aforementioned inequality is nonpositive, letting $T\uparrow\infty$ proves the first part. To prove the second part, we write the Itô differential of ζ as $d\zeta = A(t,\zeta)\,dt + B(t,\zeta)\,dw$, where the functions $A(t,\zeta)$ and $B(t,\zeta)$ are defined by virtue of (3), (4), (6), (10), and (12). We now define $U(t,\zeta) := V+W+\tilde{W}+|\theta|^2$, which satisfies $\inf_{t\geq 0}U(t,\zeta)\to\infty$ as $|\zeta|\to\infty$, hence $\inf_{t\geq 0}U(t,\zeta)\geq\alpha(|\zeta|)$, and

$$\begin{aligned}\mathcal{L}U &:= \frac{\partial U}{\partial t}+\frac{\partial U}{\partial\zeta^T}A(t,\zeta)+\frac{1}{2}\mathrm{Tr}\left[\frac{\partial^2 U}{\partial\zeta^2}B(t,\zeta)B^T(t,\zeta)\right]\\ &\leq -c_1U+c_2\end{aligned}$$

where $\alpha(.)$ is a class K_∞ function, and c_1, c_2 are some positive constants. This implies (see [16, Th. 2, p. 330])

$$\begin{aligned}\sup_{t\geq 0}\mathcal{P}\{|\zeta(t)|>c\} &\leq \sup_{t\geq 0}\mathcal{P}\{U(t,\zeta(t))>\alpha(c)\}\\ &\leq\frac{U(0,\zeta(0))+c_2/c_1}{\alpha(c)}.\end{aligned}\tag{17}$$

Letting $c\uparrow\infty$ in (17) proves the second part. To complete the proof, we first note that if the conditions C1), C2), and C3) are satisfied, then the function U becomes independent of t and it satisfies $\mathcal{L}U\leq -c_1[|z_{[r]}(t)|^2+|\tilde{x}(t)|^2+|\xi_{[r]}(t)|^2]$. With this, part 3(a) follows from [16, Th. 1, p. 330], whereas parts 3(b) and 3(c) follow from [17, Ths. 1 and 2, pp. 38 and 39], respectively. ■

V. Conclusion

In this note, a risk-sensitive adaptive control problem for strict-feedback systems is studied in a stochastic setup with output measurements. The system model includes linearly parameterized uncertainty where the parameters are generated by a stochastic differential equation. The controller design involves the backstepping design methodology, and the use of radially unbounded functions. The resulting controllers maintain an arbitrarily small average risk-sensitive cost. Also, the closed-loop signals remain bounded in probability, and the tracking error asymptotically converges to zero with probability one under certain conditions.

We end this section by simply listing several possible extensions of the results presented in this note.

1) Generalization to partial state-feedback case: In this case, the signals $x_1, \ldots, x_M$, where M is an integer satisfying $1<M<r$, are assumed to be available for feedback, and the nonlinear functions in the system dynamics can be allowed to depend on $x_1, \ldots, x_M$.
2) Risk-sensitive asymptotic tracking of arbitrary reference signals.
3) Risk-sensitive control with structurally unknown dynamics (see [18] for the deterministic case).
4) Risk-sensitive control with unstable zero dynamics (see [19] for the deterministic case.
5) Risk-sensitive controller design with measurement noise.
6) Risk-sensitive control of pure-feedback systems, that is those described by

$$\begin{gathered}dx_i=f_i(x_1,\ldots,x_{i+1})\,dt+h_i^T(x_1,\ldots,x_i)\,dw\\ i=1,\ldots,r,\ x_{r+1}=u.\end{gathered}$$

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