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# Simultaneous Design of Measurement and Control Strategies for Stochastic Systems with Feedback*

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*For first-order Gaussian ARMA models, the joint optimum design of measurement and control strategies under a quadratic performance index leads to linear rules in both finite and infinite horizons, while for higher order systems the optimum designs are generally nonlinear.*

**Key Words**—Stochastic control; team theory; information theory; optimal systems; non-classical information patterns.

**Abstract**—We consider stochastic dynamic decision problems where at each step two consecutive decisions must be taken, one being what information bearing signal(s) to transmit and the other what control action(s) to exert. For finite-horizon problems involving first-order ARMA models with Gaussian statistics and a quadratic cost criterion, we show that the optimal measurement strategy consists of transmitting the innovation linearly at each stage, which in turn leads to optimality of a linear control law. We then extend this result to infinite-horizon models with discounted costs, showing optimality of linear designs. Subsequently, we show that these appealing results do not necessarily carry over to higher order ARMA models, for which we first characterize the best designs within the affine class, and then derive instances of the problem for which there exist non-linear designs that outperform the optimal designs within the affine class. The paper also includes some illustrative numerical examples on the different classes of problems considered.

## 1. INTRODUCTION

### 1.1. *Motivation*

WITH ADVANCES in large scale decentralized systems, brought about by improved computing and data transmitting capabilities, problems requiring simultaneous measurement and control are becoming increasingly commonplace. Decentralized systems are replacing their centralized counterparts for reasons which range from cost effectiveness to sheer necessity due to large systems being more complex and less manageable. In such decentralized systems decisions are often taken on the basis of information generated by other members of the same system and garbled by noisy communication channels. Quite often we may identify agents with one of two kinds of roles:

(a) agents who perform the communication task of generating information bearing signals;

(b) agents who perform the control functions of forming estimates, minimizing errors and reducing costs.

For example, we may think of a space probe and an earth station together constituting a decentralized system, with the probe making measurements, and then encoding and transmitting them, thereby performing the communication task, and the earth station performing the requisite control functions.

In this paper we shall be concerned with such multi-agent stochastic decision problems (teams) requiring simultaneous design of measurement as well as control signals.

### 1.2. *Problem formulation*

As a prototype for the class of stochastic decision problems alluded to above, we first consider the stochastic system described by the following set of scalar equations:

$$x_{i+1} = \rho_i x_i + m_i - v_i \tag{1a}$$

$$y_i = u_i + w_i \tag{1b}$$

where, for some Borel measurable functions $h_i$ and $\gamma_i$

$$u_i = h_i(x_i, y^{i-1}) \tag{1c}$$

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and

$$v_i = \gamma_i(y^i) \quad (1d)$$

with

$$y^i := (y_0, \ldots, y_i).$$

Here (1a) and (1b) are the state and measurement equations, respectively, $v_i$ is the control variable and $u_i$ the information variable that carries information on the current and past values of the state. The functions $h_i$ and $\gamma_i$ are the measurement and control strategies, respectively, and are chosen such that $u_i$ and $v_i$ are second-order random variables. The initial state $x_0$, the system noises $m_i$'s and the measurement noises $w_i$'s are assumed to be independent, Gaussian, with zero mean and variances $\sigma^2_{(\cdot)}$ (the subscript being the identifier).

The problem is to jointly design the control and measurement policies so as to obtain optimal system performance. This objective is formalized as the minimization of a quadratic cost functional

$$J(h^N, \gamma^N) = E\left[\sum_{i=0}^{N} (a_{i+1}x_{i+1}^2 + b_i v_i^2 + q_i u_i^2)\right] \quad (2)$$

where the weighting parameters $a_i$'s, $b_i$'s and $q_i$'s are all taken to be positive, for all $i$. In the above we have adopted the convention that $h^i = (h_0, \ldots, h_i)$, $\gamma^i = (\gamma_0, \ldots, \gamma_i)$, and have implicitly assumed the given relationships (1c) and (1d) between the policy variables and the action variables. We now have problem $P$ below, which we state for future reference.

*Problem P.*

$$\underset{h^N, \gamma^N}{\text{Minimize}}\, J(h^N, \gamma^N)$$

subject to (1a)–(1d), where $J(h^N, \gamma^N)$ is defined by (2).

It is worth noting that if the measurement policy $h_i$ is fixed to be linear in the current value of the state, then Problem $P$ becomes equivalent to the so-called LQG problem of stochastic control, which is known to admit a unique optimal control policy that is linear in the best (minimum mean square error) estimate of the state, generated by the Kalman filter. For other (say nonlinear) choices for $h_i$, however, the optimal policy is not generally a certainty equivalence controller, and even a closed form solution may not exist. The *joint* design problem formulated above is an even "harder" problem since as a two-agent stochastic team problem it features non-classical information (Başar and Cruz, 1982; Witsenhausen, 1971). This follows from the simple observation that the agent (say $A$) who takes action $v_i$ has to use the information $y^i$ which depends on the action variable, $u^{i-1}$, of the other agent, which is dependent on the state $x^{i-1}$, which is not directly observable to agent $A$.

In this paper we obtain the complete solution to Problem $P$ above (in Section 3) by first solving a hard constraint version of the problem (in Section 2). We then study the infinite horizon version (Section 4), where $J$ in (2) is replaced by

$$J(h^\infty, \gamma^\infty) = E\left[\sum_{i=0}^{\infty} \beta^i (qu_i^2 + a\beta x_{i+1}^2 + bv_i^2)\right] \quad (3)$$

where $\beta \in (0, 1)$ is the discount factor, and $a, b, q$ are three positive scalars. For this problem we establish the existence of optimal stationary policies, and provide a convergent algorithm for the numerical evaluation of these policies.

In Section 5 the more general problem is discussed, where (1a) is replaced by a higher order ARMA model. The paper ends with the concluding remarks of Section 6.

### 1.3. *Related results*

Determination of optimal measurement strategies for linear stochastic systems has been the subject of considerable previous investigations. Athans (1972) has solved an off-line dynamic optimization problem to select alternate sensor measurement policies to optimize a combination of prediction accuracy and observation cost for a linear stochastic system. Lafortune (1985) has also considered the problem of optimally selecting among different costly observations for a linear Gaussian system with quadratic costs. Other related results are provided by Mehra (1976), Herring and Melsa (1974), and Meier *et al.* (1967).

One common feature of the above investigations is that the measurement model has been specified parametrically, thus reducing the problem to one of optimal selection of a set of parameter values. As a departure from this approach, Whittle and Rudge (1976) have considered the problem where both the measurement and control sequences are simultaneously optimized, and they have formulated it as a dynamic two-person cooperative game with imperfect information. In control theoretic terms, however, their solution is unrealizable because the actions taken at a given time depend also on observations lying in the future.

In this paper we consider simultaneous optimization of both the observation and the control sequences under the further requirement that all policies be causal. As noted in Section 1.2, in this formulation the policy $h_i$ affects the measurement to be used by the controller $\gamma_i$, but the controller cannot infer all the information available to the agent who chooses $h_i$. Such information patterns have been called *nonclassical* in the stochastic team literature, as opposed to classical or quasi-classical information patterns which include the static and partially nested structures (Ho, 1980; Başar and Cruz, 1982). Stochastic teams with non-classical information have remained quite intractable from the time they were first identified some 20 years back (Witsenhausen, 1968), where it was shown that some of the simplest LQG two-person teams with non-classical information do not admit optimal linear solutions. It is, however, not only the non-classical nature of the information pattern, but also the structure of the loss functional that contributes to the difficulty, since in the absence of the product term between the decision variables in the loss functional, the optimal solutions may readily be found (Bansal and Başar, 1987a), even for some multipath systems (Bansal and Başar, 1987b). As we will see in the sequel, both the finite and infinite horizon problems formulated in Section 1.2 also belong to the class of such tractable stochastic decision problems with non-classical information, even though from a computational complexity viewpoint they are NP-Complete (Papadimitriou and Tsitsiklis, 1986).

## 2. THE HARD CONSTRAINT VERSION

In this section we formulate and solve a stochastic dynamic team problem which is a version of Problem $P$ formulated in Section 1.2, with hard power constraints on the $u_i$'s. The solution to this problem will then be used to construct the solution to Problem $P$.

Consider Problem $P1$ below.

*Problem* $P1$.

$$\underset{h^N,\gamma^N}{\text{Minimize}}\, J'(h^N, \gamma^N)$$

subject to (1a)–(1d), where $J'(h^N, \gamma^N)$ is defined by

$$J'(h^N, \gamma^N) = E\left[\sum_{i=0}^{N} (a_{i+1}x_{i+1}^2 + b_i v_i^2)\right] \quad (4)$$

and the $u_i$'s further satisfy the power constraints

$$E[u_i^2] \le P_i^2. \quad (5)$$

### 2.1. *Construction of an equivalent problem*

The gist of this subsection is as follows. First an equivalent Problem $P2$ is constructed from $P1$ which differs only in the form of the cost function. The cost function for $P2$ is in the form of a sum of the squared differences between state and control variables. In the transformation from $P1$ to $P2$ the constraints represented by (1a)–(1d) are unaltered. In a follow-up step, Problem $P3$ is constructed from Problem $P2$ such that the structure of the cost function is left unaltered, while the state equations are redefined so as to facilitate subsequent analysis.

These two transformations are presented below as Claims 1 and 2, respectively.

*Claim* 1. Under the set of constraints represented by (1a)–(1d), the cost function for Problem $P1$, defined by (4), is identical to

$$J'(h^N, \gamma^N) = E\left[\sum_{i=0}^{N} a_i'(v_i - b_i'x_i)^2\right] + c_N \quad (6)$$

where

$$b_i' := k_{i+1}\rho_i/(b_i + k_{i+1}) \quad (7a)$$

$$a_i' := b_i + k_{i+1} \quad (7b)$$

$$c_N := k_0\sigma_{x_0}^2 + \sum_{i=0}^{N} k_{i+1}\sigma_{m_i}^2 \quad (7c)$$

and $\{k_i\}$ is a sequence defined recursively by

$$k_i = a_i + k_{i+1}b_i\rho_i^2/(b_i + k_{i+1}) \quad (8)$$
$$k_{N+1} = a_{N+1}.$$

*Proof.* This is a standard result in stochastic LQ control with perfect state information (see, e.g. Bertsekas (1987) or Kumar and Varaiya (1986), known also as "completing the squares". We note that $c_N$ is a constant (independent of the control sequence $\{v_i\}$), and (8) is the so-called discrete-time Riccati equation for this scalar problem.

*Claim* 2. The solution to Problem $P2$ may be obtained by solving the following equivalent Problem $P3$.

*Problem* $P3$.

$$\underset{h^N,\tilde{\gamma}^N}{\text{Minimize}}\, J''(h^N, \tilde{\gamma}^N)$$

where

$$\tilde{x}_{i+1} = \rho_i\tilde{x}_i + m_i, \quad \tilde{x}_0 = x_0 \quad (9a)$$

$$\tilde{v}_i = \tilde{\gamma}_i(y^i) \quad (9b)$$

$$y_i = u_i + w_i \quad (9c)$$

$$u_i = h_i(\tilde{x}_i, y^{i-1}) \quad (9d)$$

the $u_i$'s satisfy the power constraint (5), and

$$J''(h^N, \tilde{\gamma}^N) := E\left[\sum_{i=0}^{N} a_i'(\tilde{v}_i - b_i'\tilde{x}_i)^2\right] + c_N. \quad (10)$$

*Proof.* The situation is depicted in Fig. 1.

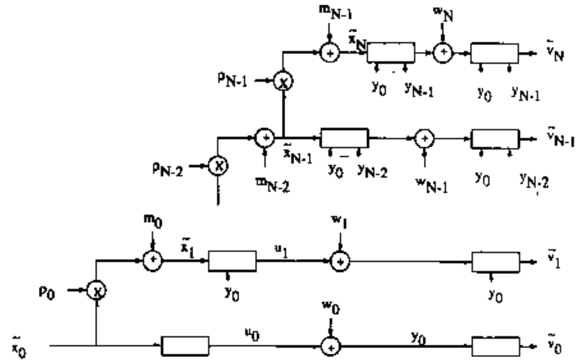

FIG. 1. Diagrammatic representation of Problem $P3$.

Substituting for $x_1$, using (1a), we get

$$v_1 - b_1'x_1 = v_1 + b_1'v_0 - b_1'(\rho_0 x_0 + m_0). \quad (11)$$

Similarly

$$\begin{aligned} v_2 - b_2'x_2 = v_2 + b_2'v_1 + b_2'\rho_1 v_0 \\ - b_2'(\rho_1(\rho_0 x_0 + m_0) + m_1) \quad (12) \end{aligned}$$

and at the $i$th stage we have

$$\begin{aligned} v_i - b_i'x_i = v_i + b_i'v_{i-1} + b_i'\rho_{i-1}v_{i-2} + \cdots \\ + b_i'\rho_{i-1}\cdots\rho_1 v_0 \\ - b_i'(\rho_{i-1}(\rho_{i-2}(\cdots(\rho_0 x_0 + m_0)\cdots) \\ + m_{i-2}) + m_{i-1}). \quad (13) \end{aligned}$$

We now define the sequence $\{\tilde{x}_i\}$ through

$$\begin{aligned} \tilde{x}_0 &= x_0 \\ \tilde{x}_{i+1} &= \rho_i \tilde{x}_i + m_i, \quad i = 0, 1, \ldots \end{aligned} \quad (14)$$

and let

$$\begin{aligned} \tilde{v}_i := v_i + b_i'v_{i-1} + b_i'\rho_{i-1}v_{i-2} \\ + \cdots + b_i'\rho_{i-1}\cdots\rho_1 v_0. \quad (15) \end{aligned}$$

Using these new variables, the cost function (6) can be rewritten as

$$\begin{aligned} J''(h^N, \tilde{\gamma}^N) = E\left[\sum_{i=0}^{N} a_i'(\tilde{v}_i - b_i'\tilde{x})^2\right] + c_N \\ \tilde{v}_i = \tilde{\gamma}_i(y^i, v^{i-1}) \end{aligned} \quad (16)$$

where the evolution of the $\tilde{x}_i$'s is determined by (14). Since this is a team problem, and for each fixed $h^N$ the resulting stochastic control problem has classical information, minimization of (16) over $(h^N, \tilde{\gamma}^N)$ is equivalent to its minimization over $(h^N, \hat{\gamma}^N)$ where $\hat{\gamma}_i$ has only $y^i$ as its argument.

The relationship between $\tilde{v}_i$'s and $v_i$'s is given by

$$\tilde{v}^N = v^N B^{\mathrm{T}} \quad (17)$$

where $B^{\mathrm{T}}$ is the transpose of the following non-singular lower triangular matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 \\ b_1' & 1 & 0 & 0 & \cdot & 0 & 0 \\ b_2'\rho_1 & b_2' & 1 & 0 & \cdot & 0 & 0 \\ b_3'\rho_2\rho_1 & b_3'\rho_2 & b_3' & 1 & \cdot & 0 & 0 \\ \vdots & & & & \cdot & 1 & 0 \\ b_N'\rho_{N-1}\cdots\rho_1 & b_N'\rho_{N-1}\cdots\rho_2 & & & & b_N' & 1 \end{pmatrix}.$$

We may thus write

$$v^N = \tilde{v}^N[B^{\mathrm{T}}]^{-1}$$

and we thus see that in order to obtain the optimal solution to $P1$, we may equivalently solve the problem in terms of $\tilde{v}_i$ and $\tilde{x}_i$ (over $\hat{\gamma}^N, h^N$), which is precisely Problem $P3$. This then leads to the following equivalence between the solutions of Problems $P1$ and $P3$.

*Lemma* 1. (i) Problem $P1$ admits a solution if, and only if, Problem $P3$ does.

(ii) If $(h^N, \gamma^N)$ is a solution for $P1$, then $(h^N, \gamma^N B^{\mathrm{T}})$ solves $P3$; conversely, if $(h^N, \hat{\gamma}^N)$ solves $P3$, then $(h^N, \hat{\gamma}^N[B^{\mathrm{T}}]^{-1})$ is a solution for $P1$. □

### 2.2. *An auxiliary problem*

In this subsection we formulate and solve an auxiliary problem which will play an important role in the solution to Problem $P3$.

Consider the situation depicted in Fig. 2, the problem being one of finding the signals $u_0, \ldots, u_N$ subject to the power constraints (5) so as to minimize the mean square error in the estimation of $z_N$ using $y^N$. It is given that $z_0$, $n_i$, $w_i$, $i = 0, 1, \ldots,$ are mutually independent Gaussian random variables, each with mean zero and variance indicated by $\sigma^2_{(\cdot)}$, the subscript being the identifier. Following the convention of information theory, we let $I(z_N; y^N)$ denote the mutual information between $z_N$ and $y^N$, and call the supremum of this quantity the capacity of the corresponding system.

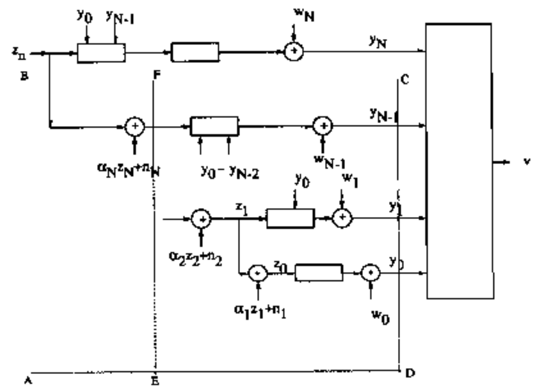

FIG. 2. Diagrammatic representation of the auxiliary problem.

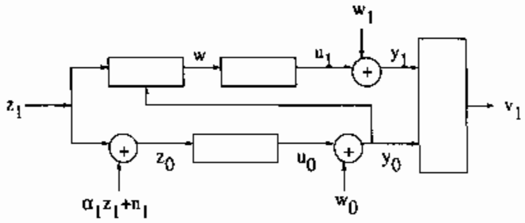


FIG. 3. The auxiliary problem with $N = 1$.

We shall first solve this problem for the case $N = 1$ which is illustrated in Fig. 3. The proof for arbitrary finite $N$ will then be shown to follow a similar line of reasoning.

It is a known result in information theory [see, for example, Shannon (1959), Berger (1971)], that minimum possible distortion $D^*$ that can result from the use of a channel is related to its capacity by

$$R(D^*) = C.$$

First consider the lower branch of Fig. 3. We find $D^*$ for this system to be (Bansal, 1988)

$$D^* = \frac{\sigma_{z_1}^2 \sigma_{n_1}^2}{(1+\alpha_1)^2 \sigma_{z_1}^2 + \sigma_{n_1}^2} + \frac{(1+\alpha_1)^2 \sigma_{z_1}^4}{((1+\alpha_1)^2 \sigma_{z_1}^2 + \sigma_{n_1}^2)} \frac{\sigma_{w_0}^2}{(P_0^2 + \sigma_{w_0}^2)}. \quad (18)$$

Now using the expression

$$R(D) = \mathrm{Max}\left(0, \tfrac{1}{2}\log\left(\frac{\sigma_{z_1}^2}{D}\right)\right) \quad (19)$$

we may compute $R(D^*)$, to arrive at the following result.

*Lemma* 2. For the system depicted in the lower branch of Fig. 3, the mutual information $I(z_1; y_0)$ is bounded above by

$$C_{\mathrm{eq}} = \tfrac{1}{2}\log\left[\frac{(P_0^2 + \sigma_{w_0}^2)((1+\alpha_1)^2\sigma_{z_1}^2 + \sigma_{n_1}^2)}{\sigma_{n_1}^2(P_0^2 + \sigma_{w_0}^2) + (1+\alpha_1)^2 \sigma_{z_1}^2 \sigma_{w_0}^2}\right]. \quad (20)$$

□

Now, consider the general system depicted in Fig. 3. We first note the fundamental inequality

$$I(z_1; y_0, y_1) = I(z_1; y_0) + I(z_1; y_1 \mid y_0). \quad (21)$$

Now

$$\begin{aligned} I(z_1; y_1 \mid y_0) &= I(z_1, w; y_1 \mid y_0) \\ &\quad - I(w; y_1 \mid z_1, y_0) && (22a) \\ &\le I(z_1, w; y_1 \mid y_0) && (22b) \\ &= I(w; y_1 \mid y_0) && (22c) \\ &= H(y_1 \mid y_0) - H(y_1 \mid w, y_0) && (22d) \\ &= H(y_1 \mid y_0) - H(y_1 \mid w) && (22e) \\ &\le H(y_1) - H(y_1 \mid w) && (22f) \\ &= I(y_1; w) && (22g) \\ &\le \tfrac{1}{2}\log\left(\frac{P_1^2 + \sigma_{w_1}^2}{\sigma_{w_1}^2}\right) && (22h) \end{aligned}$$

where $H(\cdot)$ is the entropy and $H(\cdot \mid \cdot)$ the conditional entropy. Here steps (a), (d) and (g) follow from the definition of mutual information, steps (c) and (e) are due to the Markov property, step (b) follows because information is always positive, step (f) is valid because conditioning cannot increase entropy, and the last step holds because, for a fixed variance, the Gaussian random variable has the maximum entropy (Kagan *et al.*, 1973).

Using (20) and (22), along with (21), we arrive at the next result.

*Lemma* 3. For the system depicted in Fig. 3

$$I(z_1; y_0, y_1) \le \tfrac{1}{2}\log\left[\frac{(P_0^2 + \sigma_{w_0}^2)((1+\alpha_1)^2\sigma_{z_1}^2 + \sigma_{n_1}^2)}{(\sigma_{n_1}^2(P_0^2 + \sigma_{w_0}^2) + (1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_0}^2)} \frac{P_1^2 + \sigma_{w_1}^2}{\sigma_{w_1}^2}\right]. \quad (23)$$

□

Using this upper bound on $I(z_1, y_0, y_1)$ we can find a lower bound on the minimum mean square error achievable when the problem is to estimate $z_1$ from the observations $y_0$ and $y_1$.

We have

$$I(z_1; y_0, y_1) \ge I(z_1; v_1) \ge \tfrac{1}{2}\log \frac{\sigma_{z_1}^2}{E[(z_1 - v_1)^2]} \quad (24)$$

which implies (using (23))

$$\tfrac{1}{2}\log \frac{(P_0^2 + \sigma_{w_0}^2)((1+\alpha_1)^2\sigma_{z_1}^2 + \sigma_{n_1}^2)}{(\sigma_{n_1}^2(P_0^2 + \sigma_{w_0}^2) + (1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_0}^2)} \frac{P_1^2 + \sigma_{w_1}^2}{\sigma_{w_1}^2} \ge \tfrac{1}{2}\log \frac{\sigma_{z_1}^2}{E[(z_1 - v_1)^2]} \quad (25)$$

i.e.

$$E[(z_1 - v_1)^2] \ge \frac{\sigma_{w_1}^2(\sigma_{n_1}^2(P_0^2 + \sigma_{w_0}^2) + (1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_0}^2)}{(P_0^2 + \sigma_{w_0}^2)(P_1^2 + \sigma_{w_1}^2)\sigma_{z_0}^2}. \quad (26)$$

We next note that if we use the policy

$$\begin{aligned} u_0 &= \lambda_0 z_0 \\ u_1 &= \lambda_1[z_1 - E(z_1 \mid y_0)] \end{aligned} \quad (27)$$

with $\lambda_0$ and $\lambda_1$ chosen so as to satisfy the power constraints, the minimum mean square error is indeed achieved. Hence, we have the following lemma.

*Lemma* 4. The policies given by (27) are the policies which minimize the mean square error in estimating $z_1$ from the pair $(y_0, y_1)$. □

We now consider the case with arbitrary $N$, depicted in Fig. 2. First, consider the problem in the absence of the most recent observation, i.e.

with the uppermost channel of Fig. 2 removed. Assuming that the version of the problem with $(N-1)$ channels has already been solved, the capacity $C_{N-1}$ (i.e. maximum mutual information between input and output) for the portion of the system within the rectangular box CDEF is known. We can therefore find the minimum achievable distortion for a memoryless Gaussian source with variance $\sigma^2_{Z_N}$ when only the portion within the box ABCD is in use, by computing the conditional estimate of $z_N$ given $z_{N-1}$, and transmitting this optimally. Using this minimum achievable distortion we can compute an upper bound for $I(z_N; y^{N-1})$ as follows (where $z^j_N$ is the sequence $z_{N_1}, \ldots, z_{N_j}$)

$$D^* = \lim_{j\to\infty} \frac{1}{j} E\,\|z^j_N - v^j_N\|^2 = \frac{\sigma^2_{z_N}\sigma^2_{n_N}}{(1+\alpha_N)^2\sigma^2_{z_N} + \sigma^2_{z_N}} + \frac{(1+\alpha_N)^2\sigma^4_{z_N}}{(1+\alpha_N)^2\sigma^2_{z_N} + \sigma^2_{z_N}} e^{-2C_{N-1}} \quad (28)$$

i.e.

$$I(z_N; y^{N-1}) \le \tfrac{1}{2}\log\left[\frac{(1+\alpha_N)^2\sigma^2_{z_N} + \sigma^2_{n_N}}{\sigma^2_{n_N} + (1+\alpha_N)^2\sigma^2_{z_N} e^{-2C_{N-1}}}\right]. \quad (29)$$

Furthermore, we can use a series of inequalities as in (22) to show that

$$I(z_N; y_N \mid y^{N-1}) \le \tfrac{1}{2}\log\frac{P^2_N + \sigma^2_{w_N}}{\sigma^2_{w_N}}. \quad (30)$$

Now since

$$I(z_N; y^N) = I(z_N; y^{N-1}) + I(z_N; y_N \mid y^{N-1}) \quad (31)$$

we get

$$I(z_N; y^N) \le \tfrac{1}{2}\log\left[\frac{(1+\alpha_N)^2\sigma^2_{z_N} + \sigma^2_{n_N}}{\sigma^2_{n_N} + (1+\alpha_N)^2\sigma^2_{z_N} e^{-2C_{N-1}}}\,\frac{P^2_N + \sigma^2_{w_N}}{\sigma^2_{w_N}}\right] := C_N \quad (32)$$

and we have the following lemma.

*Lemma* 5. The mutual information $I(z_N; y^N)$ is bounded above by $C_N$, which is the last step of the recursion:

$$C_0 = \tfrac{1}{2}\log\left(\frac{P^2_0 + \sigma^2_{w_0}}{\sigma^2_{w_0}}\right) \quad (33a)$$

and for $i = 1, \ldots, N$

$$C_i = \tfrac{1}{2}\log\left(\frac{(1+\alpha_i)^2\sigma^2_{z_i} + \sigma^2_{n_i}}{\sigma^2_{n_i} + (1+\alpha_i)^2\sigma^2_{z_i} e^{-2C_{i-1}}}\,\frac{P^2_i + \sigma^2_{w_i}}{\sigma^2_{w_i}}\right). \quad (33b)$$

□

Next, let $\Delta_i$ denote the minimum achievable mean square error when $z_i$ is estimated using $y^i$, i.e. (using (24))

$$\Delta_i := \sigma^2_{z_i} e^{-2C_i}. \quad (34)$$

Then, we have

$$\Delta_0 = \frac{\sigma^2_{z_0}\sigma^2_{w_0}}{P^2_0 + \sigma^2_{w_0}} \quad (35a)$$

and for $i = 1, \ldots, N$

$$\Delta_i = \frac{\sigma^2_{w_i}}{P^2_i + \sigma^2_{w_i}}\left(\frac{(1+\alpha_i)^2\sigma^4_{z_i}}{\sigma^4_{z_{i-1}}}\Delta_{i-1} + \frac{\sigma^2_{z_i}}{\sigma^2_{z_{i-1}}}\sigma^2_{n_i}\right). \quad (35b)$$

We shall next show that this lower bound is tight and may be achieved by using the policies

$$u_0 = \lambda_0 z_0 \quad (36a)$$

and for $i = 1, \ldots, N$

$$u_i = \lambda_i(z_i - E(z_i \mid y^{i-1})). \quad (36b)$$

Here the $\lambda_i$'s are chosen so as to meet the given power constraints with equality. Since

$$z_{i-1} = (1+\alpha_i)z_i + n_i \quad (37)$$

we may equivalently write

$$z_i = \tilde{\rho}_{i-1} z_{i-1} + m_{i-1} \quad (38)$$

where

$$\tilde{\rho}_{i-1} := (1+\alpha_i)\frac{\sigma^2_{z_i}}{\sigma^2_{z_{i-1}}} \quad (39)$$

and $m_{i-1}$ is a Gaussian random variable which is independent of $z_{i-1}$ and has variance

$$\sigma^2_{m_{i-1}} = \frac{\sigma^2_{z_i}}{\sigma^2_{z_{i-1}}}\sigma^2_{n_i}. \quad (40)$$

With the policies chosen as in (36), we have

$$y_0 = \lambda_0 z_0 + w_0 \quad (41a)$$

and for $i = 1, \ldots, N$

$$y_i = \lambda_i(z_i - E(z_i \mid y^{i-1})) + w_i. \quad (41b)$$

Let $\Sigma_i$ denote the mean square error in the estimation of $z_i$ from $y^i$ when the communication strategies are chosen as in (36), i.e.

$$\Sigma_i := E[(z_i - E(z_i \mid y^i))^2].$$

We then have, using (35)

$$\lambda^2_0 = \frac{P^2_0}{\sigma^2_{z_0}} \quad (42a)$$

and for $i = 1, \ldots, N$

$$\lambda^2_i = \frac{P^2_i}{(1+\alpha_i)^2\dfrac{\sigma^4_{z_i}}{\sigma^4_{z_{i-1}}}\Sigma_{i-1} + \dfrac{\sigma^2_{z_i}}{\sigma^2_{z_{i-1}}}\sigma^2_{n_i}}. \quad (42b)$$

Further, by our specific choice of the policy, $(z_i - E(z_i \mid y^{i-1}))$ is a zero mean Gaussian random variable, and therefore

$$E[(z_i - E(z_i \mid y^{i-1}) - E[(z_i - E(z_i \mid y^{i-1})) \mid y_i])^2] = \frac{s_i^2 \sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} \quad (43)$$

where

$$s_i^2 := E[(z_i - E(z_i \mid y^{i-1}))^2] = (1+\alpha_i)^2 \frac{\sigma_{z_i}^4}{\sigma_{z_{i-1}}^4} \Sigma_{i-1} + \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \sigma_{n_i}^2. \quad (44)$$

Also, $E(E(z_i \mid y^{i-1}) \mid y_i) = 0$ by our choice of policy, since $y_i$ is independent of $y^{i-1}$. Therefore the expression on the left-hand side of (43) becomes

$$E[(z_i - E(z_i \mid y^{i-1}) - E(z_i \mid y_i))^2] = E[(z_i - E(z_i \mid y^i))^2] = \Sigma_i$$

and we obtain the recursion (for $i = 1, \ldots, N$)

$$\Sigma_i = \frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} \left[ \frac{(1+\alpha_i)^2 \sigma_{z_i}^4}{\sigma_{z_{i-1}}^4} \Sigma_{i-1} + \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \sigma_{n_i}^2 \right] \quad (45a)$$

with the initial condition

$$\Sigma_0 = \frac{\sigma_{w_0}^2 \sigma_{z_0}^2}{P_0^2 + \sigma_{w_0}^2}. \quad (45b)$$

The recursion for the $\Sigma_i$'s is therefore identical to the recursion for the $\Delta_i$'s (given by (35)), which denote the minimum mean square error achievable. This shows that the lower bound on error is indeed tight, which then leads to the following theorem.

*Theorem* 1.

(a) The policies given by (36) minimize the mean square error incurred in estimating $z_N$ from $y^N$ for the system depicted in Fig. 2, where the $\lambda_i$'s are defined by (42) using the $\Sigma_i$'s defined by (45).

(b) The minimum mean square error is given by the last step of the recursion (45) or equivalently by the last step of the recursion (35). □

### 2.3. *Solutions to Problems P3 and P1*

We now return to Problem $P3$ defined in Section 2.1, where the policies

$$u_i = h_i(\tilde{x}_i, y^{i-1})$$

and

$$\tilde{v}_i = \hat{\gamma}_i(y^i)$$

are to be chosen in order to minimize

$$J'' := E\left[ \sum_{i=0}^{N} a_i'(\tilde{v}_i - b_i'\tilde{x}_i)^2 \right] + c_N \quad (46)$$

under the constraints depicted in Fig. 1.

We first consider the minimization of the $N$th term in the expression for $J$, which is

$$E[a_N'(\tilde{v}_N - b_N'\tilde{x}_N)^2]$$

the optimization problem being equivalent to minimizing

$$E[a_N' b_N'^2 (\tilde{v}_N' - \tilde{x}_N)^2]$$

where

$$\tilde{v}_N' \equiv \frac{\tilde{v}_N}{b_N'}$$

i.e. the problem is one of forming the best estimate of $x_N$ under the mean square distortion criterion. We now show that the situations depicted in Figs 1 and 2 are identical except for nomenclature. To show this equivalence we note that for $i = 1, \ldots, N$

$$z_{i-1} = (1+\alpha_i) z_i + n_i \quad (47)$$

which implies

$$z_i = (1+\alpha_i) \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} z_{i-1} + m_{i-1} \quad (48)$$

where the $m_i$'s are zero mean Gaussian random variables each with variance

$$\sigma_{m_i}^2 = \frac{\sigma_{z_{i+1}}^2}{\sigma_{z_i}^2} \sigma_{n_{i+1}}^2. \quad (49)$$

We therefore have

$$\rho_{i-1} = (1+\alpha_i) \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \quad \text{for } i = 1, \ldots, N$$

and by defining $\sigma_{x_0}^2 = \sigma_{z_0}^2$ we can complete the correspondence between the variables $\tilde{x}_i$'s and $z_i$'s for $i = 0, 1, \ldots, N$.

The solution of the problem of minimizing the mean square error in estimating $\tilde{x}_N$ from $y^N$ may therefore be obtained as in Section 2.2.

(a) The minimum mean square error in estimating $x_i$ using $y^i$ is given by $\Delta_i$, where $\Delta_i$'s satisfy the recursion (for $i = 1, \ldots, N$)

$$\Delta_i = \frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} (\rho_{i-1}^2 \Delta_{i-1} + \sigma_{m_{i-1}}^2) \quad (50a)$$

with the initial condition

$$\Delta_0 = \frac{\sigma_{x_0}^2 \sigma_{w_0}^2}{P_0^2 + \sigma_{w_0}^2}. \quad (50b)$$

(b) The optimal communication strategies are

$$u_0 = h_0^*(\tilde{x}_0) = \lambda_0 \tilde{x}_0 \quad (51a)$$

and for $i = 1, \ldots, N$

$$u_i = h_i^*(\tilde{x}^i, y^{i-1}) = \lambda_i(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1})) \quad (51b)$$

where the $\lambda_i$'s satisfy the recursion (for

$i = 1, \ldots, N$)

$$\lambda_i^2 = \frac{P_i^2}{\rho_{i-1}^2 \Delta_{i-1} + \sigma_{m_{i-1}}^2} \quad (52a)$$

with the initial condition

$$\lambda_0^2 = \frac{P_0^2}{\sigma_{\bar{x}_0}^2} \quad (52b)$$

(the $\Delta_i$'s being as defined by (54)).

(c) The optimal choice for the $\bar{\gamma}_i$'s is

$$\bar{v}_i = \hat{\gamma}_i^*(y^i) = b_i' E(\bar{x}_i \mid y^i) \quad (53)$$

where the $E(\bar{x}_i \mid y^i)$'s satisfy the recursion (for $i = 1, \ldots, N$)

$$E(\bar{x}_i \mid y^i) = \rho_{i-1} E(\bar{x}_{i-1} \mid y^{i-1}) + \frac{P_i}{P_i^2 + \sigma_{w_i}^2} (\rho_{i-1}^2 \Delta_{i-1} + \sigma_{m_{i-1}}^2)^{1/2} y_i \quad (54a)$$

with the initial condition

$$E(\bar{x}_0 \mid y_0) = \frac{P_0 \sigma_{\bar{x}_0} y_0}{P_0^2 + \sigma_{\bar{x}_0}^2} \quad (54b)$$

(the $\Delta_i$'s being as defined in (50)).

We finally note that the policies which minimize the mean square error in the estimation of $\bar{x}_i$ given $y^i$ for $i = 0, \ldots, N-1$, are identical to the corresponding policies used in the estimation of $\bar{x}_N$ given $y^N$, and we therefore have the following theorem.

*Theorem* 2.

(a) The optimal policies $h_i$ and $\hat{\gamma}_i$ for Problem $P3$ are given by (51) and (53), respectively, using the $\lambda_i$'s and $\Delta_i$'s as defined by (52) and (50), respectively.

(b) The minimum value of the cost function for Problem $P3$ is

$$J^* = \sum_{i=0}^{N} a_i' b_i'^2 \Delta_i + c_N. \qquad \square$$

We now turn to the original Problem $P1$ formulated in Section 2.1. Taking the difference

$$\frac{\bar{v}_i}{b_i'} - \rho_{i-1} \frac{\bar{v}_{i-1}}{b_{i-1}'} \quad (55)$$

and using (53), (54) and (15), we find that

$$\frac{v_i}{b_i'} = (\rho_{i-1} - b_{i-1}') \frac{v_{i-1}}{b_{i-1}'} + \frac{P_i}{P_i^2 + \sigma_{w_i}^2} (\rho_{i-1}^2 \Delta_{i-1} + \sigma_{m_{i-1}}^2)^{1/2} y_i \quad (56)$$

which implies that the optimal control policies for the original problem are

$$v_i^* = \gamma_i^*(y^i) = b_i' E(x_i \mid y^i) \triangleq b_i' \hat{x}_i \quad (57)$$

where $\hat{x}_i := E(x_i \mid y^i)$ satisfies the recursion (for $i = 1, \ldots, N$)

$$\hat{x}_i = (\rho_{i-1} - b_{i-1}') \hat{x}_{i-1} + \frac{P_i}{P_i^2 + \sigma_{w_i}^2} (\rho_{i-1}^2 \Delta_{i-1} + \sigma_{m_{i-1}}^2)^{1/2} y_i \quad (58a)$$

with the initial condition

$$\hat{x}_0 = \frac{P_0 \sigma_{x_0}}{P_0^2 + \sigma_{x_0}^2} y_0 \quad (58b)$$

the $\Delta_i$'s being as defined in (50).

We therefore have the following theorem.

*Theorem* 3.

(a) The optimal policies $\{h_i^*\}$ and $\{\gamma_i^*\}$ for Problem $P1$ are given by (51) and (57), respectively, using the $\lambda_i$'s and $\Delta_i$'s as defined by (52) and (50), respectively.

(b) The minimum value of the cost function for Problem $P1$ is

$$J^* = \sum_{i=0}^{N} a_i' b_i'^2 \Delta_i + c_N. \qquad \square$$

*An illustration*

Consider the case with $N = 2$, $\rho_i = \sigma_{x_0}^2 = \sigma_{w_i}^2 = \sigma_{m_i}^2 = P_i^2 = 1.0$ and the objective being to minimize the cost functional

$$J(h^2, \gamma^2) = \sum_{i=0}^{2} (x_{i+1}^2 + v_i^2).$$

Using Claim 1, we get the equivalent cost functional

$$\sum_{i=0}^{2} a_i'(v_i - b_i' x_i)^2 + c_2$$

where

$$a_0' = 13/5, \quad a_1' = 5/2, \quad a_2' = 2$$
$$b_0' = 8/13, \quad b_1' = 3/5, \quad b_2' = 1/2$$

and

$$c_2 = 613/130.$$

Using Theorem 3, we obtain the optimal designs for the problem to be

$$u_0^* = x_0; \quad u_1^* = (\sqrt{(2/3)})(x_1 - E(x_1 \mid y^0)); \quad u_2^* = (\sqrt{(4/7)})(x_2 - R(x_2 \mid y^1))$$

and

$$v_0^* = \tfrac{8}{3}\hat{x}_0; \quad v_1^* = \tfrac{3}{5}\hat{x}_1; \quad v_2^* = \tfrac{1}{2}\hat{x}_2$$

where

$$\hat{x}_0 = \tfrac{1}{2} y_0;$$
$$\hat{x}_1 = \tfrac{5}{26} y_0 + \tfrac{3}{34} y_1;$$
$$\hat{x}_2 = \tfrac{1}{3} y_0 + \tfrac{3}{2} y_1 + \tfrac{7}{16} y_2.$$

The minimum cost is 6.1852.

### 3. THE SOFT CONSTRAINT VERSION—FINITE HORIZON CASE

In this section we use the solution to the hard constraint version (obtained in Section 2) to construct the solution to the original problem $P$. Let $J_P$ denote the infimum of $J$ under the hard power constraints

$$J_P := \operatorname*{Inf}_{h^N;\gamma^N;E[h_i^2]=P_i^2,\forall i} J(h^N, \gamma^N). \quad (59)$$

We then have the following series of equalities and inequalities:

$$\begin{aligned} J_P &= \sum_{i=0}^N q_i P_i^2 + \operatorname*{Inf}_{h^N;\gamma^N;E[h_i^2]=P_i^2,\forall_i} E\left[\sum_{i=0}^N (a_i'(v_i - b_i'x_i)^2)\right] \\ &\geq \sum_{i=0}^N q_i P_i^2 + \operatorname*{Inf}_{h^N;\gamma^N;E[h_i^2]\leq P_i^2,\forall_i} E\left[\sum_{i=0}^N (a_i'(v_i - b_i'x_i)^2)\right] \\ &= \sum_{i=0}^N q_i P_i^2 + \sum_{i=0}^N a_i' b_i'^2 \Delta_i \qquad (60) \\ &\geq \operatorname*{Min}_{P_i^2\geq 0} \left[\sum_{i=0}^N q_i P_i^2 + a_i' b_i'^2 \Delta_i\right] \\ &= \sum_{i=0}^N q_i P_i^{*2} + a' b_i'^2 \Delta_i^* \end{aligned}$$

where the $\Delta_i$'s are defined in (50) and $\Delta_i^*$'s are defined recursively likewise, with $P_i$ replaced by $P_i^*$. In order to find the optimal power levels ($P_i^{*2}$'s), we can solve the following problem:

$$\operatorname*{Min}_{P_0,\ldots,P_N} \sum_{i=0}^N q_i P_i^2 + a_i' b_i'^2 \Delta_i \quad (61)$$

which is a non-linear optimal control problem, the solution to which is given by the following dynamic program (with $\rho_{-1} := 1$ and $\sigma_{m-1}^2 := 0$). Here $W_i(\Delta)$ is the "optimum cost to go" given that the system is at state $\Delta$ at stage $i$

$$W_{N+1} = 0$$

$$\begin{aligned} W_i(\Delta) = \operatorname*{Min}_{P_i^2} \Bigg[ q_i P_i^2 + a_i' b_i'^2 \frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} (\rho_{i-1}^2 \Delta \\ + \sigma_{m_{i-1}}^2) \\ + W_{i+1}\left[\frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2}(\rho_{i-1}^2\Delta + \sigma_{m_{i-1}}^2)\right]\Bigg]. \quad (62) \end{aligned}$$

The optimal value of the cost is

$$\operatorname*{Min}_{P_0^2,\ldots,P_N^2} J(P_0^2, \ldots, P_N^2) = W_0(\sigma_{x_0}^2). \quad (63)$$

We next show that a solution to the above problem always exists. If we define

$$\begin{aligned} f(\Delta, P_i^2) := q_i P_i^2 + \frac{a_i' b_i'^2 \sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2}(\rho_{i-1}^2 \Delta + \sigma_{m_{i-1}}^2) \\ + W_{i+1}\left(\frac{\sigma_{w_i}^2(\rho_{i-1}^2\Delta + \sigma_{m_{i-1}}^2)}{P_i^2 + \sigma_{w_i}^2}\right) \quad (64) \end{aligned}$$

then

$$W_i(\Delta) = \min_{P_i^2} f(\Delta, P_i^2).$$

Note that $W_i$ is a continuous function of its argument if $W_{i+1}$ is, since the continuity of $W_{i+1}$ implies continuity of $f$. From the continuity of $W_{N+1}$ (which was defined to be zero), the continuity of $W_i$ follows for all $i$. We also note that as $P_i^2 \to \infty$, $f(\Delta, P_i^2) \to \infty$ also, and since $P_i^2 \geq 0$, the search for $P_i^{*2}$ can be confined to a compact set over which a continuous function always admits a minimum.

The dynamic program (62) can therefore be solved, yielding values for

$$P_0^*, P_1^*, \ldots, P_N^*$$

and we have the following theorem.

*Theorem* 4. Consider the problem

$$\operatorname*{Minimize}_{h^N,\gamma^N} J(h^N, \gamma^N)$$

subject to (1a)–(1d), where $J(h^N, \gamma^N)$ is defined by (2).

(a) The optimal policies $\{h_i^*\}$ and $\{\gamma_i^*\}$ are given by (51) and (57), respectively, using $\lambda_i$'s and $\Delta_i$'s as defined by (50) and (52), with the solution to the dynamic program (62) providing the optimum power levels, i.e.

$$P_i^2 = P_i^{*2} \quad \text{for} \quad i = 0, \ldots, N.$$

(b) The optimal cost is given by

$$W_0(\sigma_{x_0}^2) = \sum_{i=0}^N q_i P_i^{*2} + a_i' b_i'^2 \Delta_i^*. \qquad \square$$

*An illustration*

The optimal power levels depend critically on the power penalties ($q_i$'s). If we assume $N = 1$ and the following parameter values:

$$\begin{array}{llll} \sigma_{x_0}^2 = 1.0, & \sigma_{w_0}^2 = 1.0, & \sigma_{w_1}^2 = 1.0, & \sigma_{m_0}^2 = 1.0 \\ q_0 = 2.0, & q_1 = 4.0, & \rho_0 = 0.5, & a_0' = 1.0 \\ b_0' = 1.0, & a_1' = 2.0, & & b_1' = 1.0 \end{array}$$

then the optimal value of the cost is 3.5 which is attained by $P_0^2 = P_1^2 = 0.0$. If the power penalty $q_0$ is changed to 0.25 with all other parameters remaining the same, we can achieve an optimal cost of 2.9747, which is attained by using $P_0^2 = 1.4495$ and $P_1^2 = 0.0$. If the power penalty $q_1$ is also changed to 0.25, then the optimal cost is further reduced to 1.9968 which is attained by $P_0^2 = 1.1609$ and $P_1^2 = 1.9876$. It is notable that the optimal solution satisfies a threshold property, and the number of channels in use depends on the relative magnitudes of the weighting terms.

## 4. THE INFINITE HORIZON PROBLEM

We now turn to the analysis of the infinite horizon problem. For this case the cost may be rewritten as

$$J=\sum_{i=0}^{\infty} q\beta^i u_i^2+\sum_{i=0}^{\infty} a_i'(\bar{v}_i-b_i'\bar{x}_i)^2$$

where

$$a_i' \to \beta^i(b+k\beta)\equiv a'\beta^i$$

$$b_i' \to \frac{k\beta\rho}{a'}\equiv b'$$

and $k$ is determined as the positive root of the equation

$$(k-a)(k\beta+b)=kb\beta\rho^2 \quad (65)$$

i.e.

$$k=\frac{1}{2\beta}(\sqrt{((b-b\beta\rho^2-a\beta)^2+4ab\beta)} -(b-b\beta\rho^2-a\beta)). \quad (66)$$

Thus, an infinite horizon version of the originally formulated problem, with discounted cost, may be solved by solving a problem of the form $P^\infty$, which is given next.

*Problem $P^\infty$.*

$$\underset{h^\infty,\gamma^\infty}{\text{Minimize}}\, J(h^\infty,\gamma^\infty)=E\left[\sum_{i=0}^{\infty}(qu_i^2+a(v_i-x_i)^2)\beta^i\right]$$

where $a$, $q$ are given positive constants, $\beta$ is the given discount factor $(0<\beta<1)$ and

$$u_i=h_i(x_i,y^{i-1}) \quad (67a)$$

$$v_i=\gamma_i(y^i). \quad (67b)$$

□

We treat the infinite horizon problem as a limit of the finite horizon case with horizon length $N$, as $N\to\infty$. Each such finite horizon problem is identical with the one considered in Section 3, with the only exception that now all parameter values are constants and there is an additional discount factor $\beta$. The counterpart of the Dynamic Programming recursion (62) is

$$W_{N+1}(\Delta)=0 \quad (68a)$$

$$W_k(\Delta)=\inf_{P^2}\left[qP^2+\frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2) +\beta W_{k+1}\left(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right)\right] \quad \text{for } k\le N. \quad (68b)$$

We now study, in a series of lemmata, properties of $W_k(\Delta)$, and in particular its asymptotic behavior.

*Lemma* 6. $W_k(\Delta)$ is strictly increasing for decreasing $k$, for all $\Delta>0$, i.e. $W_k(\Delta)> W_{k+1}(\Delta)$ for all $k\le N$.

*Proof.* Clearly the lemma is true for $k=N$, since $W_{N+1}\equiv 0$ and $W_N(\Delta)$ is necessarily larger than zero for all $\Delta$. We now note the following sequence of equalities and inequalities:

$$\begin{aligned}
&W_k(\Delta)-W_{k+1}(\Delta)\\
&=\underset{P^2>0}{\text{Min}}\left[qP^2+\frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)+\beta W_{k+1}\left(\frac{\sigma_2^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right)\right]\\
&\quad-\underset{P^2>0}{\text{Min}}\left[qP^2+\frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)+\beta W_{k+2}\left(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right)\right]\\
&\ge q\hat{P}^2+\frac{a\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)+\beta W_{k+1}\left(\frac{\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right)\\
&\quad-q\hat{P}^2+\frac{a\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)-\beta W_{k+2}\left(\frac{\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right)\\
&=\beta(W_{k+1}-W_{k+2})\left(\frac{\sigma_w^2(\rho^2\Delta+\sigma_m^2)}{\hat{P}^2+\sigma_w^2}\right)
\end{aligned}$$

where $\hat{P}^2$ is chosen as the argument of the first minimization. (In case of non-unique solutions, any one of the minimizing solutions may be chosen.)

Thus if $W_{k+1}(\Delta)$ is larger than $W_{k+2}(\Delta)$, then $W_k(\Delta)$ is larger than $W_{k+1}(\Delta)$. Since $W_N(\Delta)$ is known to be larger than $W_{N+1}(\Delta)$, the proof is complete. □

*Lemma* 7. $W_k(\Delta)$ is an increasing function of $\Delta$ $\{W_k(\Delta)\uparrow\Delta\}$ for all $k\le N$.

*Proof.* We prove this by induction. First consider the case with $k=N$. We have

$$W_N(\Delta)=\underset{P^2\ge 0}{\text{Min}}\left[qP^2+\frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right].$$

There are only two possible cases, either $P^{2*}=0$ or $P^{2*}>0$. If $P^{2*}=0$, then $W_N(\Delta)=a(\rho^2\Delta+$

$\sigma_m^2$). If $P^{2^*} > 0$, which requires

$$q - \frac{a\sigma_w^2(\rho^2\Delta + \sigma_m^2)}{(P^{2^*} + \sigma_w^2)^2} = 0$$

i.e.

$$P^{2^*} = \left[\frac{a\sigma_w^2(\rho^2\Delta + \sigma_m^2)}{q}\right]^{1/2} - \sigma_w^2 \qquad (69)$$

and we get

$$W_N(\Delta) = 2a^{1/2}q^{1/2}\sigma_w(\rho^2\Delta + \sigma_m^2)^{1/2} - q\sigma_w^2. \quad (70)$$

Thus the lemma is true for $k = N$.

Now, if $W_{k+1}(\Delta) \uparrow \Delta$, then for each $P^2$

$$W_{k+1}\left(\frac{a\sigma_w^2}{P^2 + \sigma_w^2}(\rho^2\Delta + \sigma_m^2)\right) \uparrow \Delta$$

since

$$\frac{a\sigma_w^2}{P^2 + \sigma_w^2}(\rho^2\Delta + \sigma_m^2) \uparrow \Delta.$$

Also

$$\left(qP^2 + \frac{a\sigma_w^2(\rho^2\Delta + \sigma_m^2)}{P^2 + \sigma_w^2}\right) \uparrow \Delta$$

and thus both terms in the expression to be minimized to obtain $W_k(\Delta)$ are increasing in $\Delta$ for all $P^2$. Therefore, $W_k(\Delta) \uparrow \Delta$. □

*Lemma* 8. For each $\Delta > 0$, $W_k(\Delta)$ is bounded above for all $k$, by an affine function of $\Delta$, i.e.

$$0 < W_k(\Delta) \leq \Omega_{1,k}\Delta + \Omega_{2,k}. \qquad (71)$$

*Proof.* The proof is by induction, using the observation that since $W_k(\Delta)$ is given by the minimum over $P^2$, an upper bound is given by the value that the expression to be minimized attains when $P^2$ is fixed arbitrarily at zero.

Thus $W_N(\Delta) \leq a(\rho^2\Delta + \sigma_m^2)$ and we may choose $\Omega_{1,N} = a\rho^2$, $\Omega_{2,N} = a\sigma_m^2$.

Now consider the following sequence of equalities and inequalities:

$$\begin{aligned}
W_k(\Delta) &= \operatorname*{Min}_{P^2}\left[qP^2 + \frac{a\sigma_w^2}{P^2 + \sigma_w^2}(\rho^2\Delta + \sigma_m^2)\right.\\
&\qquad \left. + \beta W_{k+1}\left(\frac{\sigma_w^2}{P^2 + \sigma_w^2}(\rho^2\Delta + \sigma_m^2)\right)\right]\\
&\leq \operatorname*{Min}_{P^2}\left[qP^2 + \frac{a\sigma_w^2}{P^2 + \sigma_w^2}(\rho^2\Delta + \sigma_m^2)\right.\\
&\qquad \left. + \beta W_{k+1}(\rho^2\Delta + \sigma_m^2)\right]\\
&\leq a(\rho^2\Delta + \sigma_m^2) + \beta W_{k+1}(\rho^2\Delta + \sigma_m^2)\\
&\leq a(\rho^2\Delta + \sigma_m^2) + \beta(\Omega_{1,k+1}(\rho^2\Delta + \sigma_m^2)\\
&\qquad + \Omega_{2,k+1})\\
&= (a\rho^2 + \beta\Omega_{1,k+1}\rho^2)\Delta + a\sigma_m^2 + \beta\Omega_{1,k+1}\sigma_m^2\\
&\qquad + \beta\Omega_{2,k+1}\\
&\triangleq \Omega_{1,k}\Delta + \Omega_{2,k}.
\end{aligned}$$

Therefore, the lemma is proved with the sequences $\{\Omega_{1,k}\}$ and $\{\Omega_{2,k}\}$ defined recursively by

$$\begin{aligned}
&\Omega_{1,N} = a\rho^2, \quad \Omega_{2,N} = a\sigma_m^2\\
&\Omega_{1,k} = a\rho^2 + \beta\rho^2\Omega_{1,k+1}\\
&\Omega_{2,k} = a\sigma_m^2 + \beta\sigma_m^2\Omega_{1,k+1} + \beta\Omega_{2,k+1}.
\end{aligned} \quad □$$

Note that since the optimal measurement policy for the finite horizon problem is linear, the stationary limiting policy is given by

$$h_n^*(x_n, y^{n-1}) = \lambda^*(x_n - E(x_n \mid y^{n-1})) \qquad (72)$$

where

$$\lambda^{*2} = \frac{P^{2^*}}{\rho^2\Delta^* + \sigma_m^2} \qquad (73)$$

with $P^{2^*}$ and $\Delta^*$ being obtained through the stationary solution of the optimum control problem as $N \to \infty$.

For each $N$, denote the solution by $\{P_k^{2^*}\}^N$, $k < N$. We then expect that

$$P^{2^*} = \lim_{N\to\infty} \{P_k^{2^*}\}^N$$

for every finite $k$.

To establish the existence of this limit, we recall that $W_k(\Delta)$ is strictly increasing for decreasing $k < N$ (Lemma 6) and further that it is bounded above by an affine function (Lemma 8). This last property follows since both $\Omega_{1,k}$ and $\Omega_{2,k}$ are bounded in retrograde time

$$\Omega_{1,k} < \left(\frac{a\rho^2}{1 - \beta\rho^2}\right) \quad \text{and} \quad \Omega_{2,k} < \frac{a\sigma_m^2}{(1-\beta)(1-\rho^2\beta)}. \qquad (74)$$

Hence

$$\lim_{k\to\infty} W_k(\Delta) = W(\Delta)$$

where the limiting function satisfies

$$W(\Delta) = \operatorname*{Min}_{P^2}\left[qP^2 + \frac{a\sigma_w^2(\rho^2\Delta + \sigma_m^2)}{P^2 + \sigma_w^2} + \beta W\left(\frac{\sigma_w^2}{P^2 + \sigma_w^2}(\rho^2\Delta + \sigma_m^2)\right)\right]. \quad (75)$$

Denote the minimizing solution here by $P^2(\Delta)$. Let $P_k^2 = P_k^2(\Delta)$ be a minimizing solution of the right-hand side of (68b) (which always exists, as shown in Section 3), and let $\{\Delta_k^*\}_{k=0}^N$ be the trajectory sequence defined recursively by

$$\Delta_0^* = \left(\frac{\sigma_w^2\sigma_x^2}{P_0^2(\sigma_x^2) + \sigma_w^2}\right) \qquad (76a)$$

$$\Delta_k^* = \frac{\sigma_w^2}{P_k^2(\Delta_{k-1}^*) + \sigma_w^2}(\rho^2\Delta_{k-1}^* + \sigma_m^2). \quad (76b)$$

Finally, let

$$P_0^{2^*} = P_0^2(\sigma_x^2) \quad (77a)$$

$$P_k^{2^*} = P_k^2(\Delta_{k-1}^*). \quad (77b)$$

Now note that since $\rho^2 < 1$, equation (76b) describes a stable system with $P_k^2$ replaced by $P^2$, and hence $\Delta_k \to \Delta^*$ where $\Delta^*$ solves

$$\Delta^* = \frac{\sigma_w^2}{P^2(\Delta^*) + \sigma_w^2}(\rho^2\Delta^* + \sigma_m^2). \quad (78)$$

Let

$$P^{2^*} := P^2(\Delta^*). \quad (79)$$

Then we have the following solution to the infinite horizon problem.

*Theorem* 5. With $N \to \infty$, the joint design problem under consideration admits the optimal stationary measurement policies

$$u_n = h_n^*(x_n, y^{n-1}) = \lambda^*(x_n - E(x_n \mid y^{n-1})),$$
$$n = 0, 1, \ldots \quad (80)$$

where

$$\lambda^{*2} = \frac{P^{2^*}}{\rho^2\Delta^* + \sigma_m^2}$$

with $\Delta^*$ and $P^{2^*}$ given by (78) and (79). The optimal stationary control policies are also linear, and are given by

$$v_n = \gamma^*(\hat{x}_n) = \frac{k\beta\rho}{b + k\beta}\hat{x}_n, \quad n = 0, 1, \ldots \quad (81)$$

where $k$ is defined by (66) and $\hat{x}_n := E[x_n \mid y^n]$ by (58) with all quantities replaced by their time-invariant counterparts, also in view of (78) and (79). □

To numerically compute the optimal stationary policies, we start with $\Delta_0 = 0$ and run the following algorithm:

*Algorithm A.*

(1) Compute

$$(P_k^2)^* = \arg\min_{P^2 \geq 0}\left[qP^2 + \frac{a\sigma_w^2(\rho^2\Delta_k + \sigma_m^2)}{P^2 + \sigma_w^2} + \frac{\beta(qP^2 + a\Delta_k)}{(1-\beta)}\right].$$

(2) Compute the new value, $\Delta_{k+1}$, by

$$\Delta_{k+1} = \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2).$$

(3) Go to step (1), and iterate.

The optimal cost is then given by

$$W(\Delta^*) = \left(q(P^2)^* + \frac{a\sigma_w^2(\rho^2\Delta^* + \sigma_m^2)}{(P^2)^* + \sigma_w^2}\right)\Big/(1-\beta).$$

*Claim* 3. Algorithm A always converges. □

*Proof.* First note that $(P_k^2)^*$, found from step (1) of the algorithm, satisfies

$$(P_k^2)^* = \mathrm{Max}\left\{0, \left(\frac{a^{1/2}(1-\beta)^{1/2}}{q^{1/2}} \times \sigma_w(\rho^2\Delta_k + \sigma_m^2)^{1/2} - \sigma_w^2\right)\right\}$$

which implies that if $\Delta_{k+1} \geq \Delta_k$, then $(P_{k+1}^2)^* \geq (P_k^2)^*$.

Now, given $\Delta_k > \Delta_{k-1}$, we have

$$\begin{aligned}
\Delta_{k+1} - \Delta_k &= \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2) - \frac{\sigma_w^2}{(P_{k-1}^2)^* + \sigma_w^2}(\rho^2\Delta_{k-1} + \sigma_m^2) \\
&\geq \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2) - \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_{k-1} + \sigma_m^2) \\
&= \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}\rho^2(\Delta_k - \Delta_{k-1}) \\
&> 0.
\end{aligned}$$

Therefore, since $\Delta_1 > \Delta_0$, it follows that the $\Delta_k$'s form a monotone increasing sequence. Further, to show that the $\Delta_k$'s are bounded above, we consider the sequence

$$\Gamma_0 = 0$$
$$\Gamma_{k+1} = \rho^2\Gamma_k + \sigma_m^2$$

and note that if $\Gamma_k \geq \Delta_k$, we have

$$\begin{aligned}
\Gamma_{k+1} &= \rho^2\Gamma_k + \sigma_m^2 \geq \rho^2\Delta_k + \sigma_m^2 \\
&\geq \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2) = \Delta_{k+1}
\end{aligned}$$

i.e. if $\Gamma_k \geq \Delta_k$, then $\Gamma_{k+1} \geq \Delta_{k+1}$. But $\Gamma_0 = \Delta_0 (\equiv 0)$, and the sequence $\Gamma_k$ is bounded above by

$$\frac{\sigma_m^2}{(1-\rho^2)}.$$

Therefore, the monotone sequence $\Delta_k$ is also bounded above, and the convergence of the algorithm follows.

## 5. HIGHER ORDER ARMA MODELS

In Section 3 we studied a stochastic dynamic system involving a first-order ARMA model, with the current state directly correlated only with the immediately preceding state. In case we allow this correlation to extend to $j$ previous stages, we obtain a $j$th order ARMA model. Accordingly, let us suppose that the stochastic system is specified by the following equation:

$$x_{i+1} = \sum_{k=0}^{j-1} \rho_{i+1,i-k} x_{i-k} + m_i - v_i \qquad (82)$$

along with (1b)–(1d), where we have the same statistical description for the random quantities $x_0$, $m_i$, and $w_i$, and take $\rho_{kj} = 0$ for $j < 0$.

As seen in Sections 2 and 3, we may formulate both hard and soft constraint versions of the above problem, by adopting appropriate cost criteria. The soft constraint version is given below as Problem $PS^0$.

*Problem $PS^0$.*

$$\underset{h^N, \gamma^N}{\text{Minimize}}\, E\Big[\sum_{i=0}^{N} q_i u_i^2 + a_{i+1} x_{i+1}^2 + b_i v_i^2\Big]$$

subject to (82), and (1b)–(1d).

Using completion of squares and redefinition of the $v_i$'s as $\bar{v}_i$'s (as in the case of the first-order ARMA model) we can obtain the equivalent problem $PS$ below.

*Problem PS.*

$$\underset{h^N, \gamma^N}{\text{Minimize}}\, E\Big[\sum_{i=0}^{N} q_i u_i^2 + a_i\Big(\bar{v}_i - \sum_{k=0}^{j-1} b_{i,i-k}\bar{x}_{i-k}\Big)^2\Big]$$

subject to (1b)–(1d), with $x_i$ replaced by $\bar{x}_i$ where the latter is generated by

$$\bar{x}_{i+1} = \sum_{k=0}^{j-1} \rho_{i+1,i-k}\bar{x}_{i-k} + m_i$$

and $v_i$ replaced by $\bar{v}_i$. The precise expressions for the $b_{i,i-k}$'s and $\bar{v}_i$'s are given in Bansal (1988) where the details of the justification for this reformulation may also be found.

We now turn to analyzing these reformulated stochastic team problems.

## 5.1. *Optimality over the affine class*

In this subsection we show that if we confine the design to the affine class, then the optimal measurement strategies use a linear transformation on the innovation process.

*Theorem* 6. Consider the general formulation of Problem $PS$ with $h_i$ restricted to the class

$$u_i = h_i(\bar{x}_i, y^{i-1}) = L_i(\bar{x}_i, y^{i-1})$$

where $L_i$ is a general affine mapping. Then one may, without loss of generality, confine to optimizing over the class of measurement policies which satisfy the structural restriction

$$u_i = \lambda_i[x_i - E(\bar{x}_i \mid y^{i-1})]. \qquad (83)$$

Thus it is sufficient to optimize over the class of policies which use a *linear transformation on the innovation in $\bar{x}_i$.*

*Proof.* Note that over the affine class we may write

$$u_i = L_i(\bar{x}_i, y^{i-1}) = \bar{u}_i + p_i$$

with

$$\bar{u}_i = \lambda_i(\bar{x}_i - E(\bar{x}_i \mid y^{i-1}))$$

and

$$p_i = L_i'(y^{i-1})$$

where $L_i'$ is an arbitrary affine mapping, and $\bar{u}_i$ and $p_i$ are uncorrelated. Thus we have

$$\begin{aligned}
\min_{\gamma_N} J(h^N, \gamma^N) &= E\Big[\sum_{i=0}^{N}\Big(q_i u_i^2 + a_i\Big(E\Big(\sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k} \mid y^i\Big) - \sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k}\Big)^2\Big)\Big] \\
&= E\Big[\sum_{i=0}^{N}\Big(q_i\bar{u}_i^2 + q_i p_i^2 + a_i\Big(E\Big(\sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k} \mid y^i\Big) - \sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k}\Big)^2\Big)\Big] \\
&\geq E\Big[\sum_{i=0}^{N} q_i\bar{u}_i^2 + a_i\Big(E\Big(\sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k} \mid y^i\Big) - \sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k}\Big)^2\Big)\Big].
\end{aligned}$$

We now note that the sigma field generated by $\bar{y}^i$ is the same as the sigma field generated by $y^i$ where

$$\bar{y}_i = \bar{u}_i + w_i = y_i - p_i$$

since $p_i$ is $y^{i-1}$ measurable.

Hence

$$\min_{\gamma_N} J(h^N, \gamma^N) \geq E\Big[\sum_{i=0}^{N}\Big(q_i\bar{u}_i^2 + a_i\Big(E\Big(\sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k} \mid \bar{y}^i\Big) - \sum_{k=0}^{j-1} c_{i,i-k}\bar{x}_{i-k}\Big)^2\Big)\Big]$$

and therefore the cost functional may be optimized under the structural constraint (83). □

The optimal strategies within the affine class for problems involving higher order ARMA models may now be found as for the first-order problem, which involves the solution of a sequential

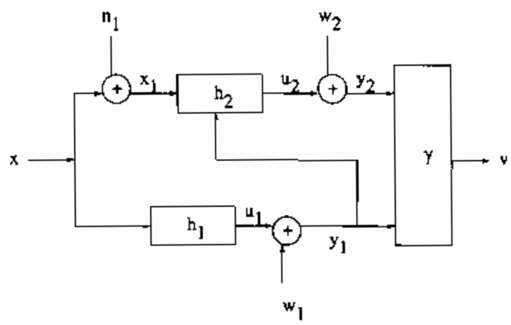


FIG. 4. Schematics for Problem $P_a$.

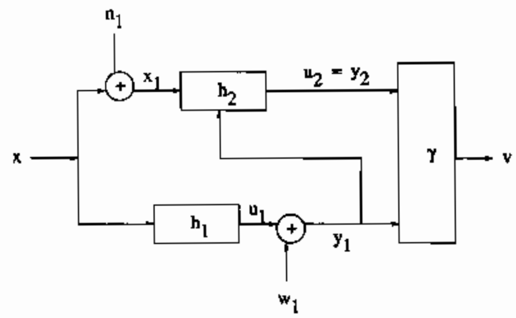


FIG. 5. Schematics for Problem $P'_a$.

optimization problem. Since the procedure is the same as in the first-order problem, we do not pursue it any further here.

### 5.2. *Nonoptimality of linear laws*

Here we show that for one of the simplest team problems of the type above, involving an ARMA model of order 2, the optimum linear solution may be outperformed by an appropriately chosen non-linear policy.

We first restrict our attention to the following stochastic team Problem $P_a$, a schematic representation of which is provided in Fig. 4.

*Problem $P_a$.*

$$\min_{h_1,h_2,\gamma}\text{imize } E[(x-v)^2]$$

where

$$x_1 = x + n_1 \tag{84}$$

$$u_1 = h_1(x) \tag{85}$$

$$y_1 = u_1 + w_1 \tag{86}$$

$$u_2 = h_2(x_1, y_1) \tag{87}$$

$$y_2 = u_2 + w_2 \tag{88}$$

$$v = \gamma(y_1, y_2) \tag{89}$$

subject to the hard power constraints

$$E[u_1^2] \le P_1^2 \tag{90a}$$

$$E[u_2^2] \le P_2^2. \tag{90b}$$

□

Note that if the problem involved estimating $x_1$ at the decoder (instead of $x$), then we would have had the two-stage version of a problem involving a first-order ARMA model as studied in Section 2, for which the optimal solutions have been shown to be linear.

We now show that Problem $P_a$ above does not, in general, admit an optimal linear solution. This is done by constructing an instance of the problem where the optimal strategies in the linear class are outperformed by appropriately chosen non-linear strategies.

In order to see why one might suspect nonoptimality of affine laws, consider the above problem with $\sigma_{w_2}^2 = 0$. We then have Problem $P'_a$ below which is represented schematically in Fig. 5, and for which the hard power constraint on $u_2$ is immaterial since there is no noise to combat.

*Problem $P'_a$.*

$$\text{Minimize } E[(x-v)^2]$$

subject to (84)–(87), and (89), along with the restriction:

$$y_2 = u_2. \tag{91}$$

□

Note that since

$$v = \gamma(y_1, y_2)$$

where

$$y_2 = u_2 = h_2(x_1, y_1)$$

we may equivalently write

$$v = \gamma(y_1, h_2(x_1, y_1)) \tag{92a}$$

$$= \gamma'(y_1, x_1) \tag{92b}$$

and thus Problem $P'_a$ may equivalently be represented as in Fig. 6.

We thus obtain a problem of simultaneously designing encoding and decoding policies *with side information at the decoder*, for which non-linear strategies which outperform the optimal linear strategies do exist.

Since linear policies are not optimal for Problem $P_a$ with $\sigma_{w_2}^2 = 0$, they may continue to be suboptimal for small enough values of $\sigma_{w_2}^2$. We show next that this is precisely the case. In particular, if we consider the optimal linear

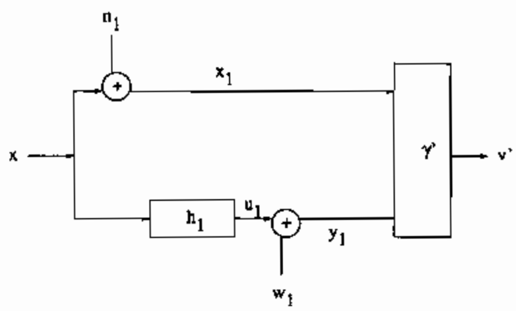


FIG. 6. Equivalent representation for Problem $P'_a$.

design for Problem $P_a$, using

$$u_1 = h_1(x) = \lambda_1 x$$
$$u_2 = h_2(x_1, y_1) = \lambda_2[x_1 - E(x_1 \mid y_1)]$$

where $\lambda_1$ and $\lambda_2$ are chosen to meet the hard power constraints with equality (this being the optimal choice in the affine class, as shown in Section 5.1), we have

$$E(x_1 \mid y_1) = \frac{\lambda_1 \sigma_x^2}{P_1^2 + \sigma_{w_1}^2} y_1 \tag{93}$$

and

$$x_1 - E(x_1 \mid y_1) = n_1 + \frac{\sigma_{w_1}^2}{P_1^2 + \sigma_{w_1}^2} x - \frac{\lambda_1 \sigma_x^2}{P_1^2 + \sigma_{w_1}^2} w_1 \tag{94}$$

which implies that

$$u_2 = \lambda_2 \left[ \frac{\sigma_{w_1}^2}{P_1^2 + \sigma_{w_1}^2} x - \frac{\lambda_1 \sigma_x^2}{P_1^2 + \sigma_{w_1}^2} w + n_1 \right]$$

with

$$\lambda_2^2 = \frac{P_2^2 (P_1^2 + \sigma_{w_1}^2)}{\sigma_x^2 \sigma_{w_1}^2 + \sigma_{n_1}^2 (P_1^2 + \sigma_{w_1}^2)}.$$

The mean square error in estimating $x$ from the simultaneous observation of $y_1$ and $y_2$ then is

$$E[(x - E(x \mid y_1, y_2))^2] = \frac{\sigma_x^2 \sigma_{w_1}^2}{(P_1^2 + \sigma_{w_1}^2)(P_2^2 + \sigma_{w_2}^2)} \times \left[ \sigma_{w_2}^2 + \frac{P_2^2 \sigma_{n_1}^2}{(\sigma_{n_1}^2 + \sigma_x^2 \sigma_{w_1}^2/(P_1^2 + \sigma_{w_1}^2))} \right]. \tag{95}$$

Considering the situation with $\sigma_x^2 = 100.0$, $\sigma_{n_1}^2 = 0.99$, $\sigma_{w_1}^2 = 1.0$, $\sigma_{w_2}^2 = 0.01$, $P_1^2 = 85.0423$ and $P_2^2 = 100.99$, we find that the optimal linear policy yields a cost of 0.53467.

We next consider the design

$$h_1(x) = x + \varepsilon \operatorname{sgn} x; \quad h_2(x_1) = x_1$$

and

$$\gamma(y_1, y_2) = \begin{cases} (y_1 + y_2 - \varepsilon)/2 & \text{if } y_2 \geq 0 \\ (y_1 + y_2 + \varepsilon)/2 & \text{if } y_2 < 0 \end{cases}$$

(letting $\varepsilon = -1.0$ we obtain $E[u_1^2] = 85.0423$ and $E(u_2^2) = 100.99$).

Now

$$y_2 = x + n_1 + w_2 \equiv x + w_3$$

where

$$w_3 \sim N(0, 1).$$

If we calculate the mean square error under the above policy, we find that the non-linear policy yields a cost of 0.53172, and hence is superior to the linear optimal policy.

We now return to the problem of showing nonoptimality of linear laws for at least some instances of higher order ARMA models. Consider the following second-order model with feedback, illustrated in Fig. 7. We have

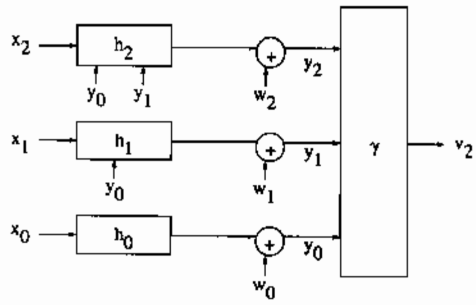


FIG. 7. Schematics for a second-order model with feedback.

$$x_2 = \rho_{21} x_1 + \rho_{20} x_0 + m_1$$
$$x_1 = \rho_{10} x_0 + m_0$$

and $x_0$, $m_0$ and $m_1$ are given independent, zero mean, Gaussian random variables.

The problem is to minimize $E[(v_2 - x_2)^2]$ under the schematics of Fig. 7, with

$$v_2 = \gamma(y_0, y_1, y_2).$$

Let us now suppose that $\sigma_{w_2}^2$ is arbitrarily large, essentially making the third channel redundant, and therefore

$$E(x_2 \mid y_0, y_1, y_2) = E(x_2 \mid y_0, y_1).$$

Further suppose that

$$\rho_{21} = 0, \quad \rho_{20} = 1, \quad \sigma_{m_2}^2 = 0, \quad \rho_{10} = 1$$

which imply

$$x_2 = x_0$$
$$x_1 = x_0 + m_1$$

and we obtain the problem depicted in Fig. 8. We thus obtain a problem of the type $P_a$, discussed earlier in this section, for which there are instances when linear strategies are not optimal. Hence we see that there are instances of the general problem (described by second-order ARMA processes) for which designs that are optimal in the affine class are nonoptimal in the general class of policies.

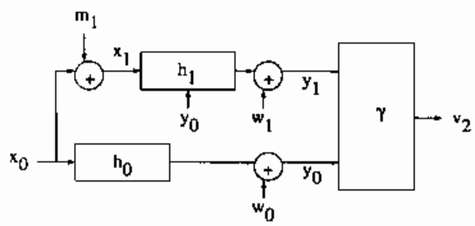


FIG. 8. The second-order ARMA model under the given restrictions.

## 6. CONCLUSION

In this paper we have considered stochastic dynamic team problems where at each step two consecutive decisions were to be taken, one being what information-bearing signal to transmit, and the other being what control action to exert. Such problems arise in the simultaneous optimization of both the observation and the control sequences in stochastic systems. We solved the problem completely for first-order systems under quadratic cost criteria. This was done by first constructing an equivalent problem having a cost function consisting of a sum of squared differences, and then solving this equivalent problem using some bounds from information theory. For cases with hard power constraints, it was shown that the optimal measurement strategy is to linearly amplify the innovation at each stage to the maximum permissible power level. For cases with soft power constraints the structure of the solution was found to be similar, with the optimal power levels being found via solving a non-linear optimal control problem, this in turn being done by using a dynamic program. The results were then extended to cases with an infinite time horizon and a discounted cost functional, and the existence of optimal stationary policies for these problems was established.

We then considered stochastic dynamic decision problems requiring simultaneous optimization of both the observation and the control sequences for second- and higher order systems under quadratic cost criteria. We considered optimality over the affine class for problems involving a general $j$th order model, and showed that within this class the optimal measurement strategy for the hard constraint version consists of transmitting the innovation linearly at each stage, leading to linear control laws. We then showed that for some of the simplest classes of such problems, involving second-order ARMA models, strategies which are optimal in the linear class may be outperformed by appropriately chosen non-linear strategies.

For related work in the continuous time, we should mention that the hard constraint version formulated in Section 2 was studied earlier in Liptser and Shiryayev (1976), where the optimality of linear encoding strategies was established. It seems that, using that result one may be able to show the optimality of innovation strategies for the continuous time counterpart of Problem $P$. However, extensions of this result to the infinite horizon case in the continuous time, and to higher order dynamics, are not immediate and are currently under study.

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