denote the table of the $i$-shortened code for each $i$. We notice that if $i \in H$ and $\Pi(k) = l$, then $\text{Table}_i(k) = \text{Table}_l(k)$. Furthermore, if there is an $m$ such that $\text{Table}_i(m) = \text{Table}_i(k)$ for every $n$, then $i \in H$ and $\text{Table}_i(k) = \text{Table}_i(l)$. With the computer we found the tables and from them we found that $\theta(H) = 1$. This was done finding an $m^0$ for which $\text{Table}_i(k) = \text{Table}_i(l)$ whenever $\text{Table}_i(k) = \text{Table}_i(l)$. In this way we found $\theta(H)$ and concluded that $\theta(H) = 1$ and therefore $G^f = H^f$. There is one exception: $H^f$ (the group for $X^4 + X + 1$) has an extra symmetry, that given by the transposition $(i, j)$ where $i, j$ are the positions corresponding to the field elements $0, 1$. This comes from having a codeword with weight two in the dual code.

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**REFERENCES**


$m$ must have the property $\text{Table}_i(m) \neq \text{Table}_i(k)$ for every $k$.

**Optimum Linear Coding in Continuous-Time Communication Systems with Noisy Side Information at the Decoder**

TANGÜL Ü. BAŞAR, MEMBER, IEEE, AND TAMER BAŞAR, SENIOR MEMBER, IEEE

**Abstract**—The problem of optimal coding in a communication system with a noiseless feedback link and with noisy side information at the decoder is treated. The message is taken as a Gaussian random variable and both the main and the side channels are assumed to be continuous and white-Gaussian, and the transmission is characterized as the sum $y_t = x_t + n_t$, where $x_t$ is a Gaussian random variable with mean zero, variance $\sigma_x^2$, and independent of the Gaussian process $n_t$. The encoder is assumed to apply a linear transformation to its input, specifically, if the output of the receiver is denoted by $r_t$ at time $t$, then the output of the encoder at time $t$ is given by $\eta_t = A(t)[x_t - r_t]$ and is then encoded appropriately and transmitted through the channel. It is further assumed that there is a noiseless feedback link from the receiver to the transmitter and that the receiver is provided with noisy side information correlated with $x_t$. This side information is also transmitted through a continuous-time additive white-Gaussian noise channel, the output of which is appropriately decoded and sent to the receiver.

As depicted in Fig. 1, the main and the side channel noise processes are denoted by $\sigma_x dw_1/dt$ and $\sigma_n dw_2/dt$, respectively, where $w_1$ and $w_2$ are independent standard Wiener processes defined in the interval $[0, T]$. The side information $y_t$ is characterized as the sum $y_t = x_t + n(t)$, where $n(t)$ is a Gaussian random variable with mean zero, variance $\sigma_n^2$, and independent of $x_t$. The encoder is assumed to apply a linear transformation to its input, specifically, if the output of the receiver is denoted by $r_t$ at time $t$, then the output of the encoder at time $t$ is given by

$$\xi_t = A(t)[x_t - r_t]$$

subject to the input power constraint to the channel $E[\xi_t^2] < P$,

$$E[\xi^2_t] < P,$$ (2a)

where $x_t \sim x: 0 < t < T$.

The two decoders, on the other hand, are considered as appropriate causal (measurable) transformations on their input, with the decoder of the main channel satisfying a linear growth property. That is, we assume that $B_1(\xi_t, s < t)$ grows at most linearly with $\sup_{s < t} |\xi_t|$, where $x_t$ is defined by

$$\xi_t = \int_0^t \xi_s ds + \sigma_n w_t = \int_0^t A(t)[x_t - r_t] ds + \sigma_n w_t.$$ (3a)

No such restriction has to be imposed on the other decoder $C_2(\cdot)$ whose input is the Gaussian process

$$z_t = \int_0^t y_s ds + \sigma_n w_t = (u + x) s + \sigma_n w_t.$$ (3b)

The output of the receiver is the stochastic process $r_t$, satisfying

$$r_t = B_1(\xi_t, s < t) + C_2(z_t, s < t).$$ (3c)
where \( r_s \) implicitly depends on \((r_t, t \leq s)\) through \((3a)\). The linear growth property imposed on \( B_t(\cdot) \) insures that \((3c)\) admits a unique solution.

The problem considered in this correspondence is now to minimize the quadratic distortion measure

\[
D_T = E[(r_T - x_T)^2]
\]

over all admissible \( A(t), B_t(\cdot), \) and \( C_t(\cdot), \) and subject to \((2b)\). It will be shown below that this problem does not admit a unique solution, with the optimal forms of \( B_t(\cdot) \) and \( C_t(\cdot) \) being linear integral transformations.

III. DERIVATION OF OPTIMAL ENCODER AND DECODER TRANSFORMATIONS

Let \( \bar{S}_t \) denote the collection of observations \((\xi_s, z_s; s < t)\) where \( \xi_s \) and \( z_s \) are defined by \((3a) \) and \((3b)\). Introduce the stochastic process

\[
\xi_t = \int_0^t A(\tau)x_\tau \, d\tau + \sigma_v \xi_0,
\]

and let \( \bar{S}_t \) denote the collection of observations \((\xi_s, z_s; s < t)\). It should be noted that \( \bar{S}_t \) does not depend on the decoder transformations \((B_t(\cdot), C_t(\cdot); s < t)\), whereas \( \bar{S}_t \) does. It has, however, been shown in the Appendix that \( \bar{S}_t \) and \( \bar{S}_t \) generate the same sigma-field for every admissible pair of decoder transformations. This implies that, for each fixed \( A(t), \) \((4)\) is minimized uniquely by the conditional mean \( \hat{x}_t = E[x|\bar{S}_t] \), i.e., \( r_t = \hat{x}_t \). But since \( \xi_t \) and \( z_t \) are linear functions of \( u_t, x_t, v_t, \) and \( w_t \), and the underlying statistics are Gaussian, \( x_t \) can be determined as the output of an appropriate Kalman–Bucy filter, which in turn implies that the output of the receiver \( r_t \) has to be equal to \( r_t = \hat{x}_t = E[x|\bar{S}_t] = E[x|\bar{S}_t], \) where the last equality follows from the equivalence of the sigma-fields generated by \( \bar{S}_t \) and \( \bar{S}_t \). Hence to determine the optimal decoder transformations for each fixed \( A(\cdot), \) we simply substitute \( r_t = \hat{x}_t \) in \((3a)\) and seek to obtain a stochastic differential equation for \( \hat{x}_t = E[x|\bar{S}_t]. \)

To this end, let

\[
q_t = (x(t), u(t))' \quad \text{and} \quad \eta_t = (\xi_t, z_t)' \quad 0 < t < T.
\]

Then \( q_t \) and \( \eta_t \) satisfy the stochastic differential equations

\[
\begin{align*}
dq_t &= 0, \quad q_0 = (x, u)', \\
d\eta_t &= H(t)q_t \, dt - (A(t), 0)'\dot{\xi}_t \, dt + G(t) \, d\xi_t, \quad \eta_0 = 0,
\end{align*}
\]

with

\[
H(t) = \begin{pmatrix} A(t) & 0 \\ 1 & 1 \end{pmatrix} \quad G = \begin{pmatrix} \sigma_v & 0 \\ 0 & \sigma_w \end{pmatrix}
\]

Using this notation \( \hat{x}_t \) can be written as

\[
\hat{x}_t = (1, 0)E[q|\bar{S}_t],
\]

where \( \hat{x}_t = E[q|\bar{S}_t] \) is given recursively (from the Kalman–Bucy filter theory [3]) by

\[
\begin{align*}
\hat{x}_t &= K(t)dh_t + (A(t), 0)'\hat{x}_t \, dt - H(t)\hat{x}_t \, dt, \quad \hat{x}_0 = 0, \\
K(t) &= \Sigma(t)H'(t)[GG']^{-1}, \\
\frac{d\Sigma(t)}{dt} &= -\Sigma(t)H'(t)[GG']^{-1}H(t)\Sigma(t), \quad \Sigma(0) = \text{diag} (\sigma_v^2, \sigma_w^2).
\end{align*}
\]

The covariance equation \((8)\) admits the unique solution

\[
\Sigma(t) = \frac{1}{\Theta(t)} \begin{pmatrix}
t & 0 \\ 0 & t
\end{pmatrix} - \frac{t}{\sigma_v^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sigma_v^2} \int_0^t A^2(s) \, ds
\]

where

\[
\Theta(t) = \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} + \frac{1}{\sigma_v^2} \int_0^t A^2(s) \, ds + \frac{1}{\sigma_v^2 \sigma_w^2}.
\]

Hence, for each fixed \( A(\cdot), \) the minimum of \((4)\) over admissible decoder transformations is given by

\[
D_T = \Sigma_{11}(T) = \begin{pmatrix} T & 1 \\ 1 & T \end{pmatrix} \bigg/ \Theta(T),
\]

and the power constraint \((2b)\) can be written as

\[
E[\hat{x}_t^2] = A^2(t)\left[ E(x_t - x_t)^2 \right] = A^2(t)D_t
\]

\[
= A^2(t) \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} + \frac{1}{\sigma_v^2} \leq P.
\]

We now have to minimize \( D_T \) over \( A(\cdot) \) under the constraint \((11)\), which is equivalent to solving the equation

\[
A^2(t) \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} + \frac{1}{\sigma_v^2} = P.
\]

This integral equation can be written in the form of a differential equation

\[
\frac{d}{dt} A^2(t) = \frac{P}{\sigma_v^2} A^2(t) + \frac{P \sigma_v^2}{(\sigma_v^2 + \sigma_w^2)^2}, \quad A^2(0) = \frac{P}{\sigma_v^2}.
\]
whose solution is
\[
[A*(t)]^2 = \frac{P}{\sigma_2^2} \exp \left( \frac{Pt}{\sigma_2^2} \right) + \int_0^t \exp \left( \frac{P(t-\tau)}{\sigma_2^2} \right) \frac{P_0 \sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} d\tau.
\]
(14)

By evaluating the second term of (14) from integral tables, we can also express \(A*(t)\) as the convergent limit of an infinite sequence—a form which we will have occasion to use in Section IV.

In particular, if \(A(t)\) is given by (14) (the optimal value), we have
\[
L*(t) = \frac{P}{\sigma_2^2} [A*(t)]^2,
\]
and substitution of this expression, together with (19a)–(19b), into (17c) yields
\[
\dot{x}_r = \left( \frac{P}{\sigma_2^2} [A*(t)]^2 \right) \int_0^t A(s) d\phi_s + \frac{P [A*(t)]^2}{\sigma_2^2 + \tau \sigma_2^2} \int_0^t dz,
\]
Furthermore, if we make use of (17a), we obtain the integral equation
\[
\dot{x}_r = \frac{P}{\sigma_2^2} \left[ \int_0^t A^*(s) ds + \int_0^t [A^*(s)]^2 d\dot{x}_r ds \right]
+ \frac{\sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} \int_0^t dz,
\]
which can also be written as a stochastic differential equation
\[
d([A^*(t)]^2) = \left( \frac{P}{\sigma_2^2} [A^*(t)]^2 \right) d\dot{x}_r + \frac{P [A^*(t)]^2}{\sigma_2^2 + \tau \sigma_2^2} dz + \frac{\sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} d\tau,
\]

admitting the unique solution
\[
\dot{x}_r = \left( \frac{P}{\sigma_2^2} [A^*(t)]^2 \right) \left[ \int_0^t e^{-\tau} \frac{P \sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} \int_0^t e^{-\tau} A^*(s) ds d\phi_s \right]
+ \int_0^t \left[ \frac{P \sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} F(t,s) + \frac{\sigma_2^2}{[A^*(t)]^2} d\dot{x}_r \right] dz,
\]
where
\[
F(t,s) = \int_s^t \left[ e^{-\tau} \frac{P \sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} \int_0^t e^{-\tau} A^*(s) ds d\phi_s \right] d\tau.
\]

We have thus completed the proof of our main theorem.

**Theorem 1:** For the communication system with side information formulated in Section II and depicted in Fig. 1, there exists a unique encoder-decoder triple with the optimal encoder transformation \(A*(t)\) being given by (14) (or (15)). The corresponding optimal decoder transformations are given by
\[
I^*(z,t) = \frac{P}{\sigma_2^2} \left[ A^*(t) \right]^2 e^{-\tau} \int_0^t e^{-\tau} A^*(s) ds d\phi_s,
\]
(21a)

\[
C^*(z,t) = \frac{P}{[A^*(t)]^2} \left[ \int_0^t F(t,s) d\phi_s \right],
\]
(21b)

where \(F(\cdot, \cdot)\) is defined by (21b). The minimum distortion at the receiver (after \(T<s\)) is given by
\[
D^*_r = \Sigma^*_r(T) = P/A^*(T).
\]
(21c)

**Remark 1:** From an implementation point of view, it might sometimes be simpler to generate the solution of the stochastic differential equation (7b) at the output of the decoders, as compared to designing the two decoders whose mathematical input–output relations are given by (21a) and (21b). In order to achieve this objective, we replace the two decoders (the region inside the dotted lines in Fig. 1) by the two-dimensional system depicted in Fig. 2, whose inputs are \((z, \dot{z}, 0 < t < T)\) and output is \(\dot{x}_r\). The quantities \(K^*_r\) used in the figure are the entries of \(K(\cdot)\) defined by (7c) and (9a), but with \(A(\cdot)\) replaced by \(A^*(\cdot)\). The relevant expressions for these entries are
\[
K^*_r(t) = P/\left[ A^2\left[t\right] \right],
\]
\[
K^*_r(t) = P/\left[ A^2\left[t\right] \right],
\]
\[
K^*_r(t) = \frac{\sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} [A^2(t)]^2,
\]
\[
K^*_r(t) = \frac{\sigma_2^2}{\sigma_2^2 + \tau \sigma_2^2} [A^2(t)]^2.
\]
IV. ASYMPTOTIC PERFORMANCE COMPARISON FOR SMALL AND LARGE NOISE INTENSITY ON THE SIDE CHANNEL

When the transmission of the side information is through a perfect noiseless channel (i.e., when \( a = 0 \)), then the optimum achievable performance of the communication system of Fig. 1 can be determined by making use of a recent result of Wyner and Ziv [4]. The rate-distortion function for the source \( x \) with the side information \( y = x + u \) at the decoder and under the mean-squared error criterion is shown in [4] to be

\[
R(D) = \max \left[ 0, \frac{1}{2} \log_2 \frac{\sigma_x^2 \sigma_y^2}{(\sigma_x^2 + \sigma_u^2)D} \right].
\]

On the other hand, the channel capacity of the Gaussian white-noise channel with feedback and under the power constraint

\[
\int_0^T E[\xi_s^2] \, ds < PT
\]

is (from [5])

\[
C(P) = \frac{PT}{2\sigma_u^2} \text{ nats.}
\]

Equating (22a) to (22b) we obtain the optimum performance relation

\[
D = \frac{\sigma_x^2 \sigma_u^2}{\sigma_x^2 + \sigma_u^2} \exp \left(-\frac{PT}{\sigma_u^2}\right).
\]

This is of course also an upper bound on the performance of our system with additive noise in the side channel. This upper bound will in general not be achievable if \( \sigma_u^2 > 0 \). However, we can compare the performance of our encoder–decoder design with that given by (28) when \( 0 < \sigma_u^2 < T \), that is, when the noise intensity is sufficiently low but not exactly zero. The reason why we cannot let \( \sigma_u^2 = 0 \) is because then the Kalman–Bucy filtering theory is not valid (since in deriving the results of Section III we had to assume \( GG' \) to be invertible, which is violated if \( \sigma_u^2 = 0 \)). It is possible to circumvent this difficulty by employing generalized Kalman–Bucy filter equations which involve observers, but this will not be pursued here. We will instead show that the bound (23) is asymptotically achieved by our encoder–decoder design as the noise intensity approaches zero.

To this end, we start with expression (21c) and the optimal choice for \( A(T) \) which is given by (15). For sufficiently small \( \sigma_u^2 \), (15) yields

\[
[A^*(T)]^2 \sim \left( \frac{1}{\sigma_x^2} + \frac{1}{\sigma_u^2} \right) P \exp \left( \frac{PT}{\sigma_u^2} \right),
\]

and if this is substituted into (21c) we obtain

\[
(D^*_T)^2 \rightarrow \frac{\sigma_x^2 \sigma_u^2}{\sigma_x^2 + \sigma_u^2} \exp \left( -\frac{PT}{\sigma_u^2} \right),
\]

which verifies that for sufficiently small \( \sigma_u^2 \) the overall performance of our system is optimal within the class of all causal encoder–decoder triples.

Another extreme case is when there is no side information at the decoder, i.e., when \( \sigma_u^2 \rightarrow \infty \). Going back to expression (14), we note that in this case

\[
[A^*(T)]^2 \rightarrow \frac{P}{\sigma_u^2} \exp \left( \frac{PT}{\sigma_u^2} \right),
\]

and thus the distortion becomes

\[
(D^*_T)^2 \rightarrow \sigma_u^2 \exp \left( -\frac{PT}{\sigma_u^2} \right),
\]

which is identical with the overall optimal performance achievable in this case (see [6]).

We see that the encoder–decoder triple given in Theorem 1 achieves the well-known bounds for the two extreme cases. It is conceivable that the given designs are also optimal in the mid-ranges of the noise intensity. But since the rate-distortion function has not yet been obtained for this case, a justification of this conjecture is not readily available.

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APPENDIX

In this Appendix we prove that the sigma-fields generated by \( \mathcal{B}_s = (\xi_s, x_s; s < t) \) and \( \mathcal{B}_s = (\xi_s, x_s; s < t) \) are the same for all permissible decoder transformations \( B(\cdot) \), \( C(\cdot) \), and for a fixed encoder transformation \( A(\cdot) \). This result will be achieved by verifying the existence of a causal and causally invertable transformation between \( \mathcal{B}_s \) and \( \mathcal{B}_s \). Let us first note from (3a) and (5) the relation

\[
\dot{\xi}_s = \xi_s + \int_0^t A(\tau) r_s \, d\tau
\]

which together with (3c) proves that \( \mathcal{B}_s \) can be expressed in terms of \( \mathcal{B}_s \), \( s < t \) by a measurable transformation. To verify the reverse transformation let us start with (3a), writing it in the form

\[
\frac{d\xi_s}{dt} = A(s) [B_s(\xi_s, \tau < s)] \, ds - A(s) [C_s(\xi_s, \tau < s)] \, ds + d\xi_s
\]

where

\[
\frac{d\xi_s}{ds} - A(s) [B_s(\xi_s, \tau < s)] \, ds - A(s) [C_s(\xi_s, \tau < s)] \, ds + d\xi_s
\]

is a stochastic differential equation for \( \xi_s \). Under the linear growth condition on \( B_s(\cdot) \), it admits a unique solution \( \xi_s \), which clearly depends on \( \mathcal{B}_s \), \( s < t \) through a measurable causal transformation.
The Average Number of Weighings to Locate a Counterfeit Coin

D. G. Mead

Abstract—A solution is obtained to the problem of finding the minimum average number of weighings, using a balance, needed to locate one light coin from among n coins of which n-1 are genuine and of the same weight. Optimal strategies are determined, and the anomaly that it may be better to work with n+1 coins rather than n is examined and explained.

The problem of minimizing the maximum number of weighings required, using a balance, to find one off-weight counterfeit coin (which we assume is lighter) from among n genuine coins of equal weight has been solved several times (e.g., [4] and [5]) and has also appeared in several introductory texts on information theory (I2, pp. 349-50), [3, p. 517]). Other problems of this type have appeared recently ([1]). However, no one appears to have considered the problem of minimizing the average number of weighings required, using a balance, to find the one counterfeit light coin from among n coins, n-1 of which are genuine coins of equal weight. The solution to the latter problem has some surprising consequences. For example, letting f(n) be this minimum number, we shall see that f(n) is not a monotonic function of n, and in particular f(7)<f(6). This situation is not unusual and in fact occurs approximately one-fourth of the time. Also, as will be seen, n, there are often many possible optimum strategies.

A strategy, at any stage, consists of placing a and b coins on each pan of the balance, and leaving c coins alone, where 2a+b=n. We might suspect that given, let us say, sixty-two coins, one of which is the lighter coin, it may be better to work with sixty-three coins rather than sixty-two itself. However, in this case, c(62+1)-c(62)=2+1, while c(63+1)-c(63)=2+3. Thus 6q+2t+g(r)+h(s)=3k-3+4q+6t+0=3k+4t+3+n. This completes the proof that c(62)=c(63).

Lemma 2: i) If the minimum of m(a) occurs with b < 3k then the corresponding a < 3'+1. ii) If the minimum of m(a) occurs with a < 3k then the corresponding b < 3k+3.

Proof: a) If a > 3k, then c(a+2)-c(a) > 2k+2. Similarly, if b > 3k, then c(b)-c(b-4) < 4k+2. Thus m(a+2)-m(a)=6q+2t+g(r)+h(s)<3k-3+4q+6t+0=3k+4t+3+n. This completes the proof that c(62)=c(63).

Lemma 3: There exists an l so that the minimum of m(a) occurs with 3'-1 < a < b < 3'.

Proof: If the minimum occurs with h < 3', then, by lemma 2, a < 3'+1. Assume the minimum occurs with a = 3'+1. Since for 3'-1 < x < 3', c(x+1)-c(x) < t-2, and c(1)+c(2)-c(3)=l+2, it follows that m(3'+1)>m(3'), or the minimum occurs with a = 3'.

Now we assume the minimum occurs with a < 3'+1. First note that c(3'+3)-c(3'+1)=2t+4+2c(x+1)-c(x). Hence b > 3'+2. Now c(3'+2)-c(3'+1)=2l+3 and c(3'+1)-c(3'+2)=2t+2, while for 3'-1 < x < 3', c(x+1)-c(x)=2l+2 only if c(x+1)-c(x)=1+2; this latter is true only if x = 3'-4q. However, in this case, c(3'+2)-c(3'+1)=4l+4 and c(3'+1)-c(3'-3)=4l+2. That is, m(3'+2)>m(3'+3) and m(3'+1)>m(3'+3). This completes the proof of the lemma.

It is now easy to see that the minimum value of m(a) occurs for one of two adjacent values of a. Note that if 3'-1 < x < 3'+1 and x-3'=y (mod 4) then

\[ c(x+1)-c(x) = \begin{cases} i+2, & \text{if } y = 0 \\ i+1, & \text{if } y = 1 \\ i+3, & \text{if } y = 2 \\ i, & \text{if } y = 3. \end{cases} \]

Hence

\[ c(x+2)-c(x) = \begin{cases} 2l+3, & \text{if } y \text{ is even} \\ 2l+4, & \text{if } y = 0 \\ 2l+2, & \text{if } y = 1 \\ 2l+3, & \text{if } y = 3. \end{cases} \]

But since m(a)=2c(a)+c(b), it follows that as a increases, with 3'-1 < a < 3'+1, then m(a)+m(a-1)-m(a+1) runs through the sequence 1, -1, 3, -3, 1, -1, -3, -3, ···, or 0, 2, 0, -2, 0, 2, -2, ···, or 2, -2, 4, -4, 2, -2, 4, -4, ···. From this we see that either m(a) or m(a+1) must be the