TABLE I
METHOD OF HOOKE AND JEEVES

h	p_1	p_2	Final Vector—X x_1	x_2	x_3	$f(X)$
1	0.02	0.02	3	3	3	18
"	0.2	0.2	"	"	"	"
0.1	0.02	0.02	3.3	3.2	3.2	19.5
"	"	2	3.6	0	3.5	14.3
0.01	"	"	3.63	0.04	3.51	14.43
"	0.2	20	"	0	"	14.40
0.001	"	"	3.629	0.004	3.509	14.41
"	2	200	"	0	"	14.40
"	20	2000	"	"	"	"

TABLE II
MODIFIED PATTERN SEARCH

h	p_1	p_2	Final Vector—X x_1	x_2	x_3	$f(X)$
1	0.02	0.02	4	1	3	17
"	0.2	0.2	4	0	3	15
0.1	"	"	"	0.3	2.9	15.5
"	2	2	"	0	"	14.9
0.01	"	"	3.99	0.03	2.91	14.94
"	20	20	"	0	"	14.88
0.001	"	"	3.989	0.003	2.911	"
"	200	200	"	0	"	"
"	2000	"	"	"	"	"

objective evaluated. If the perturbation yields an improvement in the objective, it is retained; if not, the variable is set to its initial value and the next variable is perturbed. The process terminates after all variables have been considered in turn. The important point here is that no two variables are perturbed simultaneously unless one yielded an improved value of the objective by itself. Consequently, desirable moves resulting from two or more variables being perturbed together may be overlooked and the convergence terminated prematurely. An example is given

To modify this deficiency, an additional step is adjoined to the method of Hooke and Jeeves which is implemented only when the unmodified method terminates at each step size. The step is simply an exhaustive search on the corners of the hypercube of radius h centered at the final point obtained with the unmodified method. As soon as this exhaustive search locates an improvement, the exhaustive search is terminated and the algorithm returns to the original method of Hooke and Jeeves initialized at the improved point.

III. AN EXAMPLE

The constrained optimization problem

$$\begin{aligned}\text{maximize:}\quad & z=3x_1+2x_2+x_3\\ \text{subject to:}\quad & x_1^4+x_2^4+x_3^4-325=0\\ & x_1x_2=0\end{aligned}$$

attains its maximum at $x_1 \simeq 4.03084$, $x_2=0$, $x_3 \simeq 2.79483$ with $z \simeq 14.8874$. Applying both the unmodified pattern search and then the modified method to the associated unconstrained problem

$$\text{maximize:}\quad z=f(X)-p_1g_1^2(X)-p_2g_2^2(X) \tag{3}$$

with $f(X)=3x_1+2x_2+x_3$, $g_1(X)=x_1^4+x_2^4+x_3^4-325$, and $g_2(X)=x_1x_2$, we obtain Tables I and II, respectively. The superiority of the modified search is immediately apparent.

With $h=1$, the method of Hooke and Jeeves converges to $X=(3,3,3)$. Improvements in the objective of problem (3) are possible but only by perturbing more than one variable at the same time. An exhaustive search applied to $(3,3,3)$ with $h=1$ yields $(2,2,4)$ as an improved value for X. The unmodified pattern search then identifies sequentially $(3,1,4)$, $(4,0,3)$, and $(4,1,3)$ as still better estimates to the solution of problem (3). At this point, the algorithm terminates with or without the modification until either the step size or the penalty weights are changed.

IV. CONCLUSIONS

Augmenting an exhaustive search to the method of Hooke and Jeeves and implementing the modification only when the original method terminates can improve convergence. Since exhaustive searches are prohibitive when the number of variables is large, the modification does have limitations. Nonetheless, in those cases where a few exhaustive searches can be tolerated, it can be advantageous to use them.

The modification is most useful in the early stages of the search procedure when the step size is comparatively large. By restricting its use accordingly, one can shorten the computations required without adversely affecting accuracy. A point of reference is Table II in which the modification did not produce any improvements after the first run.

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Mixed Stackelberg Strategies in Continuous-Kernel Games

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***Abstract*—It is shown that continuous-kernel nonzero-sum games with compact strategy spaces could admit both pure and mixed Stackelberg equilibrium solutions, if the cost function of each player is either nonquadratic or nonconvex in his own decision variable. In such a case, the mixed Stackelberg strategy will yield a lower average cost for the leader than the pure Stackelberg strategy. It is also verified that, if the cost functions of the players are quadratic and strictly convex, then only pure Stackelberg strategies can exist.**

I. INTRODUCTION

Hitherto, research on investigation of the Stackelberg equilibrium solution in two-person nonzero-sum games has been restricted to pure strategy spaces [1]. Our aim in this paper, is to bring to bare the significance of mixed-strategy solutions in some reasonably well-structured static games, and to illustrate the fact that existence of a pure-strategy equilibrium solution in a Stackelberg game does not necessarily imply that the best the leader can do is to adopt that strategy. To this end, we produce two counterexamples which explicitly display that convex or quadratic continuous-kernel games with compact strategy spaces could admit both pure- and mixed-strategy Stackelberg solutions,

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with the latter providing the leader with a lower average equilibrium cost than the former.

In this paper, we also prove an extended version of a result on existence of a pure-strategy Stackelberg solution to continuous-kernel games with compact strategy spaces, which also accounts for possible nonunique reactions of the follower. Finally, we verify that, for convex quadratic games, mixed-strategy Stackelberg solutions do not exist. For the proofs of the theorems the reader is referred to [2].

II. Formulation of the Stackelberg Game

Consider the two-player static nonzero-sum Stackelberg game in which player $P1$ is the leader and $P2$ is the follower. The cost function of Pi is continuous and is given by $J_i(u_1,u_2)$, where u_j is the decision variable of Pj taking values in a Hilbert space U_j. Then, a mixed strategy for Pi is a probability distribution μ_i on U_i. Let us denote the space of all such probability distributions for Pi by M_i. Note that M_i also includes all one-point distributions, and hence (by an abuse of notation) we can also consider $u_i \in U_i$ as an element of M_i. The average cost of P_i, corresponding to a pair ($\mu_1 \in M_1$, $\mu_2 \in M_2$), is given by

$$\bar{J}_i(\mu_1,\mu_2)=\int_{U_1 X U_2} J_i(u_1,u_2)\, d\mu_1(u_1)\, d\mu_2(u_2), \qquad i=1,2.$$

Definition 1: A subset of M_2 defined by

$$\bar{R}_2(\mu_1) \triangleq \left(\mu_2^0 \in M_2 : \bar{J}_2(\mu_1,\mu_2^0) \leqslant \bar{J}_2(\mu_1,\mu_2), \forall \mu_2 \in M_2 \right)$$

is called the *rational reaction set* of the follower in *mixed strategies*. If both players are confined to pure strategies, the set

$$R_2(u_1) \triangleq \{ u_2^0 \in U_2 : J_2(u_1,u_2^0) \leqslant J_2(u_1,u_2), \forall u_2 \in U_2 \}$$

is known as the *rational reaction set* of the follower in *pure strategies.* □

Definition 2: A strategy $\mu_1^* \in M_1$ is called a *mixed Stackelberg strategy* for the leader, if

$$\bar{J}_1^* = \inf_{M_1} \sup_{\bar{R}_2(\mu_1)} \bar{J}_1(\mu_1,\mu_2) = \sup_{\bar{R}_2(\mu_1^*)} \bar{J}_1(\mu_1^*,\mu_2)$$

where $\bar{J}_1^*$ is known as the *average Stackelberg* cost of the leader in *mixed strategies.* □

Definition 3: If M_1 is replaced by U_1 and $\bar{R}_2$ by R_2 in the preceding definition, the resulting equilibrium strategy, denoted by $u_1^* \in U_1$, is known as a *pure Stackelberg strategy* for the leader. The corresponding cost-function value of the leader, which is

$$J_1^* \triangleq \inf_{U_1} \sup_{R_2(u_1)} J_1(u_1,u_2) = \sup_{R_2(u_1^*)} J_1(u_1^*,u_2),$$

is called the *Stackelberg cost* of the leader in *pure strategies.* □

Definition 4: Let $\epsilon > 0$ be a given number. Then, a strategy $\mu_1^\epsilon \in M_1$ is called a *mixed ϵ-Stackelberg strategy* for the leader if

$$\sup_{\mu_2 \in \bar{R}_2(\mu_1^\epsilon)} \bar{J}_1(\mu_1^\epsilon,\mu_2) \leqslant \bar{J}_1^* + \epsilon.$$

A pure *ϵ-Stackelberg strategy* can be defined analogously by obvious modifications. □

III. Main Results

We first note the following property of a mixed ϵ-Stackelberg strategy (for the leader), which follows readily from Definitions 1–4.

Property 1: Let $\{\mu_1^{\epsilon_i}\}$ be a given sequence of mixed ϵ-Stackelberg strategies in M_1, with $\epsilon_i > \epsilon_j$ for $i<j$, and $\lim_{j\to\infty}\epsilon_j = 0$. Then, if there exists a convergent subsequence $\{\mu_1^{\epsilon_j}\}$ in M_1 with its limit denoted as μ_1^*, and further if $\sup_{\mu_2 \in \bar{R}_2(\mu_1)} \bar{J}_1(\mu_1,\mu_2)$ is a continuous function of μ_1 in an open neighborhood of $\mu_1^* \in M_1, \mu_1^*$ is a mixed Stackelberg strategy (for the leader). □

Remark 1: It should be apparent that the above property remains valid if the players are confined to pure strategies. □

From the pure-strategy version of Property 1, we now obtain the following result which also clarifies and extends [1, Proposition 3.1].

Theorem 1: Let U_1 and U_2 be compact metric spaces and J_i be continuous on $U_1 X U_2$, $i=1,2$. Furthermore, let there exist a finite family of continuous mappings T^i: $U_1 \to U_2$, indexed by a parameter $i \in I \triangleq \{1,\cdots,N\}$, so that

$$R_2(u_1)=\{u_2 \in U_2 : u_2 = T^i u_1, i \in I\}.$$

Then, there exists a pure Stackelberg strategy for the leader. □

Remark 2: The assumption of Theorem 1 concerning the structure of $R_2(\cdot)$ imposes some severe restrictions on J_2; but such an assumption is inevitable as the following example demonstrates. Take $U_1=U_2=[0,1]$, $J_1=-u_1u_2$, and $J_2=(u_1-1/2)u_2$. Here, $R_2(\cdot)$ is determined by a mapping T which is continuous on the half-open intervals $[0,1/2)$, $(1/2,1]$, but is multivalued at $u_1=1/2$. The Stackelberg cost of the leader is clearly $J_1^*=-1/2$, but a pure Stackelberg strategy does not exist because of the "multivalued" nature of T. □

We now note that, even if a pure-strategy Stackelberg equilibrium exists, the statement of Theorem 1 does not rule out the possibility of better mixed-strategy solutions for the leader. The following proposition, in fact, even strengthens such a possibility. (The statement of the proposition is not trivial, as it might at first seem, since the strategy spaces of *both* players are enlarged by inclusion of mixed strategies.)

Proposition 1:

$$\bar{J}_1^* \leqslant J_1^*. \tag{1}$$

□

It is generally not true that $\bar{J}_1^*=J_1^*$, as it will be explicitly demonstrated in the sequel; hence there is, in general, a gap between $\bar{J}_1^*$ and J_1^*. One set of conditions under which this gap is zero is furnished by the following theorem.

Theorem 2: Let J_1 and J_2 be quadratic functions of u_1 and u_2, with J_i being strictly convex in u_i, $i=1,2$. Then, $\bar{J}_1^*=J_1^*$, and moreover, only pure Stackelberg equilibrium strategies can exist (for the leader). □

The next natural question to ask is whether the sufficiency conditions of Theorem 2 can be relaxed. The following two examples indicate that neither the convexity requirement nor the quadratic nature of the cost function can be dispensed with. The first example illustrates existence of a mixed Stackelberg strategy which provides the leader with better performance than any pure strategy, in a game with a nonconvex quadratic cost functional. In the second example, however, a similar feature is displayed for a game with convex nonquadratic cost functional. In both problems, the decision spaces of the players are closed bounded intervals of the real line.

Example 1: Let $U_1=U_2=[-b,b]$, $b>0$, and

$$J_1=(u_1)^2-2(u_2-u_1)^2; \quad J_2=(u_2)^2-2u_1u_2.$$

This game problem admits a unique Stackelberg solution in pure strategies which is $u_1^*=0$, $u_2^*=u_1$, and the Stackelberg cost for the leader (in pure strategies) is $J_1^*=0$. If mixed strategies are also allowed, then $\bar{R}_2(\mu_1)$ is uniquely characterized by the relation

$$u_2=\int_{U_1} u_1\, d\mu_1(u_1) \triangleq E[u_1].$$

Using this in $\bar{J}_1$, we obtain

$$\begin{aligned}\bar{J}_1 &= -2E\left[(u_1-E[u_1])^2\right]+E\left[(u_1)^2\right]\\ &= -E\left[(u_1-E[u_1])^2\right]+(E[u_1])^2 \equiv -\mathrm{var}(u_1)+(E[u_1])^2.\end{aligned}$$

Hence, in minimizing $\bar{J}^1$ over all probability distributions defined on $U_1=[-b,b]$, we can restrict ourselves to those distributions whose mean value is closest to zero and whose variance is as large as possible. But these requirements are uniquely satisfied by the two-point atomic probability distribution μ_1 such that

$$u_1=\begin{cases} -b & \text{w.p. } \frac{1}{2}, \\ +b & \text{w.p. } \frac{1}{2}, \end{cases}$$

in which case $E[u_1]=0$, $\text{var}(u_1)=b^2$. Thus we arrive at the conclusion

$$\bar{J}_1^*=-b^2<0=J_1^*. \qquad \square$$

Example 2: Let $U_1=U_2=[0,-1]$, $J_1=\epsilon(u_1)^2+u_1\sqrt{u_2}-u_2$, $\epsilon>0$; $J_2=[u_2-(u_1)^2]^2$, where $\epsilon>0$ is a sufficiently small parameter.

The unique Stackelberg solution of this game in pure strategies is $u_1^*=0$, $u_2^*=(u_1)^2$, and the Stackelberg cost for the leader (in pure strategies) is $J_1^*=0$. We now show that the leader can actually do better by employing an appropriate mixed strategy.

First note that the follower's unique reaction to a mixed strategy of the leader is $u_2=E[(u_1)^2]$ which, when substituted into $\bar{J}_1$, yields the expression

$$\bar{J}_1=\epsilon E\left[(u_1)^2\right]+E[u_1]\sqrt{E\left[(u_1)^2\right]}-E\left[(u_1)^2\right].$$

If the leader uses the uniform probability distribution on $[0,1]$, his average cost becomes

$$\bar{J}_1=\frac{\epsilon-1}{3}+\frac{1}{2}\sqrt{\frac{1}{3}}$$

which clearly indicates that, for ϵ sufficiently small,

$$\bar{J}_1^*<0=J_1^*. \qquad \square$$

IV. Conclusion

It has been almost traditional in the game theory literature to seek for mixed-strategy equilibrium solutions only when a pure-strategy equilibrium solution does not exist [3]. This seems to be a valid approach in the cases of saddle-point and Nash equilibrium solutions. For Stackelberg games, however, a mixed strategy could be preferred by the leader even though the problem admits a Stackelberg solution in pure strategies.

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On Minimal Canonical Realization from Input–Output Data Sequences

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***Abstract*—An algorithm is presented for determining the state model of a discrete system from input–output data in canonical form. The proposed method is based on a canonical input–output representation from which the input matrix and the nondynamic part can be found by a simple recursive relation.**

I. Introduction

Methods [1]–[3] of realization from input–output data are applicable for a strictly dynamical system only. However, in some cases, e.g., socioeconomic systems, electrical networks, etc., the identification of the memoryless part is important. Recently, Liu *et al.* [5] have proposed a method of determining the state model when the nondynamic part is also present. The contribution of [5] is mainly in formulating the identifiability conditions. Through this note, a method is presented in which the realization is obtained in the canonical form given in [3] with the additional advantages that matrix inversion is not required for finding the input matrix and the memoryless part is determined by a simple method.

II. Statement of the Problem

Consider a linear time-invariant completely observable discrete system. It is well known [3], [4] that by a suitable transformation, the state equations of the system assume the following canonical form:

$$x(k+1)=Ax(k)+Bu(k)$$
$$y(k)=Cx(k)+Du(k)$$

where $x\in R^n$ is the state, $u\in R^r$ is the input, $y\in R^m$ is the output, and the matrices A,B,C,D are of compatible order. The pair (A,C) assumes the following ordered structure:

$$A=A_{ij}, \qquad i,j=1,2,\cdots,m \tag{1}$$

$$A_{ii}=\begin{bmatrix} 0 & & \\ \vdots & I_{v_i-1} & \\ 0 & & \\ a_{ii}(1) & \cdots & a_{ii}(v_i) \end{bmatrix}_{v_i\times v_i},$$

$$A_{ij}=\begin{bmatrix} 0 & & \cdots & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & & & 0 \\ a_{ij}(1) & \cdots & a_{ij}(v_{ij}) & 0 & \cdots & 0 \end{bmatrix}_{v_i\times v_j} \tag{2}$$

$$C=\begin{bmatrix} 1 & 0 & & & & \cdots & & & & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & & & & 0 \\ \hline 0 & 0 & \cdots & & & & 1 & 0 & \cdots & 0 \end{bmatrix} \tag{3}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ (v_1+1)\text{th} & (v_1+v_2+\cdots+v_{m-1}+1)\text{th} \\ \text{column} & \text{column} \end{array}$$

where $v_1,v_2,\cdots,v_m$ are the Kronecker indices associated with the pair (A,C), I_{v_i-1} is the identity matrix of order (v_i-1), and $\sum_{i=1}^{m}v_i=n$.

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