Decentralized Multicriteria Optimization of Linear Stochastic Systems

TAMER BAŞAR, MEMBER, IEEE

Abstract—By adopting a decision-theoretic approach and under the noncooperative equilibrium solution concept of game theory, decentralized multicriteria optimization of stochastic linear systems with quasiclassical information patterns is discussed. First, the static M-person quadratic decision problem is considered, and sufficiency conditions are derived for existence of a unique equilibrium solution when the primitive random variables have *a priori* known but arbitrary probability distributions with finite second-order moments. The optimal strategies are given in the form of the limit of a convergent sequence which is shown to admit a closed-form linear solution for the special case of Gaussian distributions. Then, this result is generalized to dynamic LQG problems, and a general theorem is proven, which states that under the one-step-delay observation sharing pattern this class of systems admit unique affine equilibrium solutions. This result, however, no longer holds true under the one-step-delay sharing pattern, and additional criteria have to be introduced in this case. These results are then interpreted within the context of LQG team problems, so as to generalize and unify some of the results found in the literature on team problems.

I. Introduction

By ADOPTING a decision theoretic approach and under the noncooperative (Nash) equilibrium solution concept of game theory, this paper develops a theory of multicriteria optimization of LQG systems under the one-step-delay observation sharing pattern which provides each *DM* (decision maker) with the observations (but not the actions) of all the other *DM*'s with a one-step delay.

A special version of the general problem to be treated in this paper is characterized by a single criterion and a single *DM* having access to classical information, and it is known as the standard LQG stochastic control problem for which a complete theory has been developed in the early 1960's [1]. The first decentralized result in that context has been obtained by Radner [2] who has shown among other things that a static LQG team problem admits a unique team-optimal solution linear in the observation of each *DM*. This result, however, is to be interpreted with caution when the information structure is dynamic and nonclassical. The famous counterexample of Witsenhausen is indicative of this fact, that the team-optimal solution of a dynamic 2-member team problem with 2-step-delay information will in general not be linear [3].

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The author is with the Applied Mathematics Division, Marmara Scientific and Industrial Research Institute, Gebze, Kocaeli, Turkey.

Ho and Chu have studied in [4]–[6] nonclassical but nested information structures and have applied within that context Radner's above cited result to dynamic LQG team problems. The first systematic formulation of decentralized stochastic team problems within a general framework has been given in [7] where Witsenhausen has also made several important assertions. One of these assertions was team-optimality of linear solutions in the optimization of dynamic LQG team problems under the one-step-delay information sharing pattern. This assertion was then considered almost independently in [8]–[10], where the authors adopted a dynamic programming approach to derive a set of relations for the linear solution of a 2-member LQG team problem to satisfy.

For the other special class of stochastic multicriteria optimization problems—the zero-sum dynamic games—some of these structural properties might not hold true. In particular, it has been shown by Başar and Mintz [11]–[12] that LQG zero-sum dynamic games with 2-step delay observation sharing pattern could admit unique linear saddle-point solutions.

For the LQG multicriteria optimization problems and within the context of the noncooperative equilibrium solution concept, the first proof of existence of a unique equilibrium solution was given in [13] by directly making use of Radner's result and for a special class of static problems in which the decision variables are scalar. For the most general LQG multicriteria stochastic optimization problem with two *DM*'s and static information structure, an innovative approach was taken in [14]–[15] to prove existence of unique linear equilibrium strategies by making use of the fixed point of an appropriately structured Banach space. This result can be considered as a generalization of Radner's result on LQG teams to multicriteria problems; however, the sufficiency conditions obtained under which the former result has been shown to be true imply, but are not exactly the same as, Radner's conditions of existence for the special case of team problems. In [15], existence and uniqueness of equilibrium strategies have also been established under the most general probability distributions for the primitive random variables. These results have then been extended in [16] to a dynamic 2-member 2-criteria LQG decision problem, and it has been shown that for such problems there is a great difference between the one-step-delay observation sharing pattern and the one-step-delay sharing pattern. While the former results in unique equilibrium solutions, the latter allows nonunique "representations" and consequently gives rise to essentially nonunique solutions.

In this paper, we first present nontrivial generalizations of the results given in [14] and [15]. In particular, for the 2-person case we obtain sufficiency conditions less restrictive than those given in [15], which coincide with Radner's convexity condition for the special case of a team problem. For the non-Gaussian probability structure, we develop a convergent sequence that yields the unique equilibrium strategy of each *DM*. We then generalize these results to the M-person M-criteria ($M>2$) case to obtain similar results (i.e., we establish uniqueness under general probabilistic structure and prove existence of unique linear equilibrium strategies for the LQG case). Then, we consider the general M-criteria LQG problem with one-step-delay observation sharing pattern. In this context, we prove that the equilibrium strategies are unique and affine under certain nonvoid sufficiency conditions, but because of limitations on space we do not give the exact expressions for the equilibrium strategies. Finally, we establish nonuniqueness of the solution under the one-step-delay sharing pattern and provide an illustrative example. As a by product, we also obtain an existence and uniqueness result for the LQG M-member team problem under either of the aforementioned two information structures.

A precise mathematical formulation is given in Section II. Section III is devoted to the static M-criteria decision problem, and Section IV is concerned with the dynamic multistage problem.

II. Problem Formulation

Denoting the index set $\{0,1,\cdots,N-1\}$ by θ_N, the evolution of the state $x(\cdot)$ over θ_N is described by the difference equation

$$x(n+1)=F(n)x(n)+\sum_{j=1}^{M} G_j(n)u_j(n)+w(n);\quad x(0)=x_o, \tag{1}$$

where $x(n)$ denotes the p-dimensional state vector at stage n, $u_j(n)$ denotes the r_j-dimensional decision (control) variable of DMj (jth DM) at stage n, x_o is a Gaussian random vector with mean $\bar{x}_0$ and covariance Q, and $\{w(n),n\in\theta_N\}$ is a zero-mean Gaussian white noise process statistically independent of x_0 and with a covariance function

$$E\left[w(n)w^T(k)\right]=\phi(n)\delta_{nk}\qquad \text{for } n,k\in\theta_N.$$

At every stage, each of the M decision makers (*DMs*) will make an independent linear observation of the state in additive Gaussian noise. This observation will be denoted by the m_j-dimensional vector $z_j(n)$ for DMj at stage n and will be given by

$$z_j(n)=H_j(n)x(n)+v_j(n);\qquad j\in\hat{\theta}_M,\ n\in\theta_N, \tag{2}$$

where $\hat{\theta}_M$ denotes the index set $\{1,\cdots,M\}$, $H_j(\cdot)$ is an appropriate dimensional observation matrix, and $\{v_j(n),n\in\theta_N\}$, $j\in\hat{\theta}_M$, are statistically independent zero-mean Gaussian white noise processes with covariance functions $E[v_j(n)v_j^T(k)]=R_j(n)\delta_{nk}$ for $n,k\in\theta_N$, $j\in\hat{\theta}_M$, and with $R_j(\cdot)>0$. Furthermore, these random processes are also statistically independent of $\{w(\cdot)\}$ and x_o.

In a general situation, the information that is known to (and can be utilized by) DMj at stage n will be comprised of $z_j(n)$ as well as part or all of: i) his past observations; ii) other *DM*'s' past observations; iii) other *DM*'s' past actions. Unfortunately, at the present, it is not possible to develop a general theory of multicriteria optimization for the most general information structure, since it is not known yet how to handle even the single-criterion single-*DM* LQG problems with nonclassical information patterns [17]. Hence, our concern in this paper will be primarily with *quasiclassical* information patterns in Witsenhausen's [17] terminology as to be described below.

Let Z_n denote the complete set of observations made by all *DMs* up to (and including) stage n, i.e.,

$$Z_n=\{z_1(0),\cdots,z_M(0),\cdots,z_1(n),\cdots,z_M(n)\}, \tag{3a}$$

and furthermore let η_j^n, information available to DMj at stage n, be given by

$$\eta_j^n=\{Z_{n-1},z_j(n)\}. \tag{3b}$$

We will call this kind of information pattern for each *DM* the "*one-step-delay observation sharing pattern*" to differentiate it from the "one-step-delay sharing pattern" of Witsenhausen [7] which also includes past actions of all the *DMs*. Even though these two information patterns can be shown to be essentially equivalent in the case of team problems, this equivalence ceases to hold true in multicriteria optimization problems as it will be more elaborated on in Section IV. Actually, the former information pattern will allow us to obtain *unique* equilibrium strategies whereas the latter provides so much richness that the result is a plethora of essentially nonunique equilibrium solutions.

Letting $\mathcal{B}^m$ denote the Borel field generated by subsets of $\mathcal{R}^m$, we assume the decision space of DMj to be $(\mathcal{R}^{r_j},\mathcal{B}^{r_j})$ and a permissible decision law γ_j^n of DMj at stage n to be a real vector-valued Borel-measurable function mapping $(\mathcal{R}^{m_j^n},\mathcal{B}^{m_j^n})$ into $(\mathcal{R}^{r_j},\mathcal{B}^{r_j})$, where $m_j^n\triangleq m_j+n\Sigma m_i$. This, however, does not characterize the possible strategy set of *DM* completely, since as it will be clear in subsequent sections, we also have to require each decision function to be a second-order random vector well-defined on a certain measurable space. This is done by basically constructing a Banach space as the permissible strategy set of each *DM*. For the zero'th stage a precise construction has been given in [15], and we then basically require $\gamma_j^o(\eta_j^o)$ to possess a finite second-order moment under the probability measure $\mathcal{P}_{\eta_j^o}$ induced by η_j^o, and denote the equivalence classes of all such strategies by $\mathcal{L}_{r_j}^{\eta_j^o}$ for DMj, which is a Banach space. At subsequent stages, however, this construction is not as straightforward, since the probability measure $\mathcal{P}_{\eta_j^n}^{\gamma^{n-1}}$ induced by η_j^n on $\mathcal{B}^{m_j^n}$ will be dependent on the previous decision laws for $n \geq 1$, which allows only a recursive definition. The construction at stage n will then be as follows:

Assuming η_j^n to be a well-defined random vector on $(\mathcal{R}^{m_j^n}, \mathcal{B}^{m_j^n})$ and to possess a finite second-order moment for each permissible strategy sequence $\{\gamma_1^0, \cdots, \gamma_M^o, \cdots, \gamma_1^{n-1}, \cdots, \gamma_M^{n-1}\} \triangleq \gamma^{n-1}$, we denote the probability measure induced by η_j^n on $\mathcal{B}^{m_j^n}$ by $\mathcal{P}_{\eta_j^n}^{\gamma^{n-1}}$ (which is not necessarily the Gaussian measure). Then the permissible decision law γ_j^n for DMj at stage n is required to satisfy the finite norm restriction

$$\|\gamma_j^n\|_{r_j}^{\eta_j^n} \triangleq \left[\int_{\mathcal{R}^{m_j^n}} \gamma_j^{n^T}(\xi)\gamma_j^n(\xi)\,dP_{\eta_j^n}^{\gamma^{n-1}}(\xi) \right]^{1/2} < \infty. \quad (4)$$

We denote the space of equivalence classes of all such Borel functions by $\mathcal{L}_{r_j}^{\eta_j^n}(\gamma^{n-1})$ for each permissible sequence of strategies γ^{n-1}. It is now not difficult to see that under (4), $\mathcal{L}_{r_j}^{\eta_j^n}(\gamma^{n-1})$ is a Banach space for every fixed sequence of strategies γ^{n-1}. Furthermore, since the observation equations are linear, it readily follows that η_i^{n+1} is a well-defined random vector and it possesses a finite second-order moment, for every $i \in \hat{\theta}_M$. This then implies that the assumption made concerning η_j^n at the beginning of the construction is valid, and, hence, this concludes the recursive definition of the permissible strategy space of every DM at every stage of the decision process.

The objective functional for DMj is defined by

$$J_j^l(\{u_i(n)\}) = x^T(N)C_j(N)x(N) + \sum_{n=l}^{N-1} \left\{ x^T(n)C_j(n)x(n) + \sum_{i=1}^{M} u_i^T(n)D_{ij}(n)u_i(n) \right\}, \quad (5a)$$

with $C_j(\cdot) \geq 0$, $D_{ij}(\cdot) \geq 0$, $D_{ii}(\cdot) > 0$. For every fixed $\{\gamma_i^0 \in \mathcal{L}_{r_i}^{\eta_i^0}, \gamma_i^n \in \mathcal{L}_{r_i}^{\eta_i^n}(\gamma^{n-1}),\ n = 1, \cdots, N-1, i \in \hat{\theta}_M\}$, we define the expected cost to DMj by

$$\bar{J}_j(\{\gamma_i^n\}) = E\left[J_j^0(\{u_i(n)\}) | \{u_i(n) = \gamma_i^n(\cdot)\}\right], \quad (5b)$$

where expectation is taken with respect to the statistics of all the primitive random variables. It can actually be shown that $\bar{J}_j$ is well-defined and finite for every permissible sequence $\{\gamma_i^n\}$.

In order to introduce the (Nash) equilibrium concept within the context of the stochastic optimization problem formulated above, we let

$$\mathcal{G}_k^n \triangleq \{\gamma_1^n, \gamma_2^n, \cdots, \gamma_{k-1}^n, \gamma_{k+1}^n, \cdots, \gamma_M^n\}, \quad (6a)$$

that is, $\mathcal{G}_k^n$ denotes the entire sequence $\{\gamma_j^n\}$ for fixed $n \in \theta_N$ and with γ_k^n missing. Without any subscript, $\mathcal{G}^n$ will denote the entire sequence $\{\gamma_j^n\}$ for fixed $n \in \theta_N$. Furthermore, we let

$$\mathcal{G}_{nk} \triangleq \{\mathcal{G}^0, \cdots, \mathcal{G}^{n-1}, \mathcal{G}_k^n, \cdots, \mathcal{G}^{N-1}\}, \quad (6b)$$

and use the notation $\bar{\gamma}_j \in \mathcal{L}_{r_j}^{\eta_j}$ to imply the sequence

$$\left\{\gamma_j^0 \in \mathcal{L}_{r_j}^{\eta_j^0}, \gamma_j^n \in \mathcal{L}_{r_j}^{\eta_j^n}(\gamma^{n-1}),\ n = 1, \cdots, N-1\right\}$$

for every $j \in \hat{\theta}_M$. Then, we have

Definition 1: A set $\{^*\bar{\gamma}_j \in \mathcal{L}_{r_j}^{\eta_j}\}$ constitutes a *(Nash) equilibrium solution* for the M-person M-criteria optimization problem (nonzero-sum dynamic game) formulated above if for every $j \in \hat{\theta}_M$

$$\bar{J}_j(\{^*\bar{\gamma}_i\}) \leq \bar{J}_j(\{^*\mathcal{G}_j^n\}, \bar{\gamma}_j) \quad (7)$$

for all $\{\bar{\gamma}_j \in \mathcal{L}_{r_j}^{\eta_j}\}$.

A solution concept that is weaker than the one given above is the *recursive equilibrium solution* which we now define:

Definition 2: A set $\{^*\bar{\gamma}_j \in \mathcal{L}_{r_j}^{\eta_j}\}$ is said to constitute a *recursive (stagewise) equilibrium solution* if for every $j \in \hat{\theta}_M$, $m \in \theta_N$,

$$\bar{J}_j(\{^*\bar{\gamma}_i\}) \leq \bar{J}_j(^*\mathcal{G}_{mj}, \gamma_j^m) \quad (8)$$

for all $\gamma_j^m \in \mathcal{L}_{r_j}^{\eta_j^0}$ if $m = 0$ and $\gamma_j^m \in \mathcal{L}_{r_j}^{\eta_j^m}(^*\gamma^{m-1})$ if $m \geq 1$.

This completes the formulation of the stochastic multicriteria optimization problem which will be the main concern of subsequent sections. Our aim in this paper is first to obtain the complete solution to the single stage (static) version of this problem as an extension and generalization of the results given in [14] and [15], and then to prove a general existence and uniqueness result concerning the equilibrium solution of the N-stage problem formulated above.

III. Single-Stage Problem

The single-stage problem is important in its own right, and the theory to be developed in this section can be considered as an extension of the theory of static teams developed by Radner [2]. We now take the objective functional of DMj to be quadratic and strictly convex in u_j, and redefine it in the most general context as (see Lemma A1 of [15])

$$J_j(\{u_i\}) = u_j^T C_j x + \tfrac{1}{2} u_j^T D_{jj} u_j + \sum_{i \neq j} u_j^T D_{ji} u_i \quad (9)$$

with $D_{jj} > 0$, and x a second-order random vector with known (but not necessarily Gaussian) statistics. We also assume the observation vector z_j of DMj to be functionally related to x but not necessarily linear. Since this is a static problem, we have $\eta_j \equiv z_j$. We now quote the following two results as direct extensions of those given in [15] for $M = 2$:

Lemma 1: The linear operator $E_{j|i}$, defined by

$$E_{j|i}\gamma_j \triangleq E[\gamma_j(z_j)|z_i] \quad (10a)$$

is a nonexpansive transformation from $\mathfrak{L}_{r_j}^{\eta_j}$ into $\mathfrak{L}_{r_j}^{\eta_i}$, i.e., for any $\gamma,\xi \in \mathfrak{L}_{r_j}^{\eta_j}$,

$$\|E_{j|i}\gamma - E_{j|i}\xi\|_{r_j}^{z_i} \leq \|\gamma-\xi\|_{r_j}^{z_j} \tag{10b}$$

where

$$\|\gamma\|_r^z \triangleq \left\{\int_{\mathfrak{R}^m} \gamma^T(z)\gamma(z)\,d\mathfrak{P}_z\right\}^{1/2} \tag{10c}$$

with r identifying the dimension of γ, and m the dimension of z.

Lemma 2: If $\{\gamma_j \in \mathfrak{L}_{r_j}^{\eta_j};\ j=1,\cdots,M\}$ is an equilibrium solution, then the following M relations should be satisfied:

$$\gamma_j(z_j) = -D_{jj}^{-1}C_jE[x|z_j] - D_{jj}^{-1}\sum_{i\neq j} D_{ji}E[\gamma_i(z_i)|z_j], \qquad j\in\hat{\theta}_M. \tag{11}$$

Hence, proving existence of a unique equilibrium solution amounts to verifying existence of a unique set $\{{}^*\gamma_j \in \mathfrak{L}_{r_j}^{\eta_j};\ j=1,\cdots,M\}$ satisfying (11). Let us first consider the case $M=2$, and substitute γ_2 obtained from (11) into (11) with $j=1$ to yield

$$\gamma_1(z_1) = k_1(z_1) + K_1\gamma_1(z_1) \tag{12a}$$

where

$$k_1(z_1) = -D_{11}^{-1}C_1E[x|z_1] + D_{11}^{-1}D_{12}D_{22}^{-1}C_2E[E[x|z_2]|z_1] \tag{12b}$$

and K_1 represents the linear operator

$$K_1(\cdot) = D_{11}^{-1}D_{12}D_{22}^{-1}D_{21}E[E[\cdot|z_2]|z_1]. \tag{12c}$$

Concerning the solution of (12a), we can now prove the following result (which is a generalization of Theorem 2 of [15]):

Theorem 1: Let $\mathcal{C}_i$ denote for each $i=1,2$ the class of $r_i\times r_i$ real matrices *similar* to $D_{ii}^{-1}D_{ij}D_{jj}^{-1}D_{ji}$, $j\neq i$, $j=1,2$. Then if there exists at least one member Λ of either $\mathcal{C}_1$ or $\mathcal{C}_2$ with the property

$$\Lambda^T\Lambda < I, \tag{13}$$

there exists a unique pair of $\gamma_i \in \mathfrak{L}_{r_i}^{\eta_i}$, $i=1,2$, that solves (11) with $M=2$. Equivalently, under the above given condition, the two-person version of the decision problem of this section, with x, z_1, z_2 arbitrary second-order random variables, admits a unique equilibrium solution.

Proof: Let us assume, without any loss of generality, that (13) is satisfied by an element of $\mathcal{C}_1$ and with the corresponding real similarity transformation matrix being Π. Premultiplying (12a) by Π, we obtain the equation

$$\tilde{\gamma}_1(z_1) = \tilde{k}_1(z_1) + \tilde{K}_1\tilde{\gamma}_1(z_1) \triangleq \hat{K}\tilde{\gamma}_1(z_1) \tag{14a}$$

where

$$\tilde{\gamma}_1(\cdot) \triangleq \Pi\gamma_1(\cdot) \tag{14b}$$

$$\tilde{k}_1(z_1) \triangleq \Pi k_1(z_1) \tag{14c}$$

$$\tilde{K}_1(\cdot) \triangleq \Pi K_1\Pi^{-1}(\cdot) = \Lambda E[E[\cdot|z_2]|z_1]. \tag{14d}$$

We now note that to prove existence of a unique solution $\gamma_1 \in \mathfrak{L}_{r_1}^{\eta_1}$ to (12a) is equivalent to proving existence of a unique solution $\tilde{\gamma}_1 \in \mathfrak{L}_{r_1}^{\eta_1}$ to (14a), and thus we can consider the latter problem. Since $\tilde{k}_1(\cdot) \in \mathfrak{L}_{r_1}^{\eta_1}$, and $\tilde{K}_1$ is a linear operator mapping $\mathfrak{L}_{r_1}^{\eta_1}$ into $\mathfrak{L}_{r_1}^{\eta_1}$, $\hat{K}$ (whose definition follows from (14a)) maps the linear space $\mathfrak{L}_{r_1}^{\eta_1}$ into itself. Defining a metric $d(\gamma,\xi)$ on this space by the norm $\|\gamma-\xi\|_{r_1}^{z_1}$, we now show that $\hat{K}$ is a contraction mapping under (13):

For every $\gamma,\xi \in \mathfrak{L}_{r_1}^{\eta_1}$,

$$d(\hat{K}\gamma,\hat{K}\xi) = \|\hat{K}\gamma - \hat{K}\xi\|_{r_1}^{z_1} = \|\tilde{K}_1(\gamma-\xi)\|_{r_1}^{z_1} < \|E[E[(\gamma-\xi)|z_2]|z_1]\|_{r_1}^{z_1},$$

where the last inequality follows from (13). Now, using the notation and result of Lemma 1, we can further bound the last expression from above by

$$\|E_{2|1}E_{1|2}(\gamma-\xi)\|_{r_1}^{z_1} \leq \|E_{1|2}(\gamma-\xi)\|_{r_1}^{z_2} \leq \|\gamma-\xi\|_{r_1}^{z_1} = d(\gamma,\xi).$$

Hence,

$$d(\hat{K}\gamma,\hat{K}\xi) < d(\gamma,\xi),$$

and this implies that $\hat{K}$ is a contraction mapping, which in turn establishes existence of a unique fixed point of $\hat{K}$ by direct application of Banach's contraction mapping principle, since $\mathfrak{L}_{r_1}^{z_1}$ is a Banach space [19]. Existence of a unique solution to (14a) naturally also implies existence of a unique solution to (12a), since the transformation matrix Π is nonsingular. Moreover, since there exists a unique one-to-one relation between γ_1 and γ_2 through (11), this in turn implies existence of a unique solution pair to (11). □

Corollary 1.1: Let $\lambda_m(A)$ denote a (possibly complex) characteristic root of A with maximum absolute value. Then, the condition

$$\bar{\lambda} \triangleq |\lambda_m(D_{11}^{-1}D_{12}D_{22}^{-1}D_{21})| < 1 \tag{15}$$

is sufficient for the static two-person decision problem to admit a unique solution.[1]

Proof: If the matrix in question has real eigenvalues and is diagonalizable, then under (15) it is possible to find a particular element Λ in $\mathcal{C}_1$, where Λ is diagonal with eigenvalues of magnitude less than 1. This definitely also implies condition (13). If $D_{11}^{-1}D_{12}D_{22}^{-1}D_{21}$ has some complex eigenvalues and/or if it is not diagonalizable, then given an $\epsilon>0$ it is possible to find an element $\Lambda(\epsilon)$ in $\mathcal{C}_1$, which is in the canonical form

[1]The idea of the proof given in the sequel occurred to me after a conversation with Prof. Jan Willems.

$$\Lambda(\epsilon)=\begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Lambda_i & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \Lambda_{i+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \Lambda_{i+2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda_r \end{bmatrix}$$

with Λ_1 through Λ_i taking the form

$$\Lambda_j=\begin{bmatrix} A_j & \epsilon I & 0 & \cdots & 0 \\ 0 & A_j & \epsilon I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_j \end{bmatrix}, \qquad A_j=\begin{pmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{pmatrix}$$

and Λ_{i+1} through Λ_r taking the form

$$\Lambda_k=\begin{bmatrix} \lambda_k & \epsilon & 0 & \cdots & 0 \\ 0 & \lambda_k & \epsilon & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix}$$

where λ_k denotes a real eigenvalue, and σ_j (ω_j) denotes the real (imaginary) part of a complex eigenvalue of $D_{11}^{-1}D_{12}D_{22}^{-1}D_{21}$ [20]. Now let $\{\epsilon_n\}$ be a strictly decreasing positive sequence in $\Re$ converging to zero, and consider the sequence $\{\Lambda_n \triangleq \Lambda(\epsilon_n)\}$ in $\mathcal{C}_1$. It converges to a unique matrix Λ_* which is not in $\mathcal{C}_1$, since there is no real bounded similarity transformation matrix that yields Λ_*. We now note that the eigenvalues of $\Lambda_*^T\Lambda_*$ are $\{\lambda_k^2, \sigma_j^2+\omega_j^2\}$, and thus $\Lambda_*^T\Lambda_*<I$ because of (15). Moreover, since eigenvalues of a symmetric matrix are continuous functions of its entries, given an arbitrary $\delta>0$, we can find an n_o such that

$$|\lambda_m(\Lambda_*^T\Lambda_*)-\lambda_m(\Lambda_{n_o}^T\Lambda_{n_o})|<\delta.$$

Picking $\delta=1-\lambda_m(\Lambda_*^T\Lambda_*)\equiv 1-\bar{\lambda}^2>0$, this then implies that under (15) there exists an element of $\mathcal{C}_1$, in particular Λ_{n_o}, for which condition (13) is satisfied. We note before concluding the proof that in the statement of the corollary there is no loss in generality by looking at the characteristic roots of only $D_{11}^{-1}D_{12}D_{22}^{-1}D_{21}$, since its nonzero eigenvalues are the same as the nonzero eigenvalues of $D_{22}^{-1}D_{21}D_{11}^{-1}D_{12}$. □

Remark 1: One special version of the two-person static decision problem is the two-member team problem with the objective functional

$$J=u_1^TC_1x+u_2^TC_2x+u_1^TD_{12}u_2+\tfrac{1}{2}u_1^TD_{11}u_1+\tfrac{1}{2}u_2^TD_{22}u_2, \quad (16)$$

if $D_{21}=D_{12}^T$ in the general formulation (see Lemma A1 of [15]). For this special case, the nonnegative definite matrix $\Lambda=(D_{11}^{1/2})^{-1}D_{12}D_{22}^{-1}D_{12}^T(D_{11}^{1/2})^{-1}$ can be shown to be in $\mathcal{C}_1$ [by picking the similarity transformation $\Pi=(D_{11}^{1/2})^{-1}$], and, hence, condition (15) implies $\Lambda<I$ is sufficient for existence and uniqueness. But this is always satisfied if J is strictly convex in the pair $\{u_1,u_2\}$ [see [15], proof of Theorem 4]. This implies that the condition of Theorem 1 is *tight*, in the sense that it coincides with Radner's theorem [2] for the special case of a quadratic team problem.

In general it is not possible to obtain the unique solution of (11) [with $M=2$] in closed-form. However, the contraction principle that was employed in the proof of Theorem 1 does suggest that the unique solution can be obtained as the limit of a sequence of Picard iterates [19]. This idea is now formalized in the following Corollary to Theorem 1, which directly follows from the Theorem and properties of Picard iterates in a Banach space:

Corollary 1.2: Let the condition of Theorem 1 be satisfied, without any loss of generality, for a member Λ^* of $\mathcal{C}_1$, which is related to $D_{11}^{-1}D_{12}D_{22}^{-1}D_{21}$ via the similarity transformation matrix $\hat{\Pi}$. Then, the unique equilibrium strategy of $DM1$ is given by

$$\gamma_1^*(z_1)=\lim_n \xi_1^n(z_1), \quad (17)$$

where lim denotes the limiting operation under the $\mathcal{L}_{r_1}^{\eta_1}$-norm (10c) and ξ_1^n denotes a convergent sequence in $\mathcal{L}_{r_1}^{\eta_1}$ defined by

$$\xi_1^n(z_1)=k_1(z_1)+\hat{\Pi}^{-1}\sum_{m=1}^{n}(\Lambda^*E_{2|1}E_{1|2})^m(\hat{\Pi}k_1(z_1)) \quad (18)$$

where $k_1(\cdot)$ is defined by (12b). The equilibrium strategy of $DM2$ can then be obtained by substitution of (17) into (11) for $j=2$ and with $M=2$.

Remark 2: As we have mentioned before, the convergent sequence $\{\xi_1^n\}$ might not lead to a closed-form expression when x, z_1, and z_2 have arbitrary probability distributions. When this is the case it is quite possible that a few terms of the sequence will give quite a close approximation to the actual solution. Investigation of: 1) the conditions under which this will generally be true and 2) how many terms one should take to attain a fairly good approximation is a promising avenue for future research.

One of the few cases when (17) admits a closed-form solution is when x, z_1, and z_2 are taken to be jointly Gaussian. Then it is clear from (18) that $\xi_1^n(z_1)$ will be affine and, hence, $\lim_n \xi_1^n(z_1)=Az_1+a$ for some $r_1\times m_1$ matrix A and an r_1 vector a. These parameters can actually be explicitly determined. To that end, let

$$z_i=H_ix+w_i, \quad x\sim N(\bar{x},Q), \quad w_i\sim N(0,R_i), \quad i=1,2^2 \quad (19)$$

with $R_i>0$ and x, w_1, w_2 independent. Then, we have

Theorem 2: Assume that the condition of Theorem 1 is satisfied, and x, z_1, z_2 are related as given by (19). Then, the unique decision laws of $DM1$ and $DM2$ are affine in the observations and

$$\gamma_1^*(z_1)=A_1\bar{x}+B_1(\hat{x}_1-\bar{x}) \quad (20a)$$

[2] $N(\bar{x},Q)$ denotes the Gaussian distribution with mean $\bar{x}$, covariance $Q\geq 0$.

$$\gamma_2^*(z_2)=A_2\bar{x}+B_2(\hat{x}_2-\bar{x}) \tag{20b}$$

with

$$A_i=-\left[I-D_{ii}^{-1}D_{ij}D_{jj}^{-1}D_{ji}\right]^{-1}D_{ii}^{-1}\left[C_i-D_{ij}D_{jj}^{-1}C_j\right],\quad i\neq j,\ i,j=1,2, \tag{21a}$$

$$\hat{x}_i\triangleq E[x|z_i]=\bar{x}+QH_i^T(H_iQH_i^T+R_i)^{-1}(z_i-H_i\bar{x}),\quad i=1,2, \tag{21b}$$

and B_1 is the unique solution of the Lyapunov-type matrix equation

$$B_1+PB_1L=M \tag{22a}$$

where

$$P\triangleq -D_{11}^{-1}D_{12}D_{22}^{-1}D_{21} \tag{22b}$$

$$L\triangleq QH_1^T(H_1QH_1^T+R_1)^{-1}H_1QH_2^T(H_2QH_2^T+R_2)^{-1}H_2 \tag{22c}$$

$$M\triangleq -D_{11}^{-1}C_1+D_{11}^{-1}D_{12}D_{22}^{-1}C_2QH_2^T(H_2QH_2^T+R_2)^{-1}H_2 \tag{22d}$$

and

$$B_2=-D_{22}^{-1}D_{21}B_1QH_1^T(H_1QH_1^T+R_1)^{-1}H_1-D_{22}^{-1}C_2. \tag{22e}$$

Proof: That the unique solution should be affine is obvious from (18). That (22a) admits a unique solution under the condition of Theorem 1 follows from a proof similar to that of [15, theorem 3]. Direct substitution of (20) into (11) and verification of (21)–(22) completes the proof. □

We now return to the general M-person static decision problem introduced at the beginning of this section and take Lemma 2 as the starting point. For $M>2$, it is still possible to obtain a uniqueness result, but the conditions and the expressions involved are not as neat as in the two-person case. The main reason is that with $M>2$ it is in general not possible to eliminate all $M-1$ decision laws by recursive substitution and obtain an equation similar to (12a) involving the decision variable of only one DM. Hence, derivation of a set of sufficiency conditions for the general static decision problem will be as follows.

We first note that the set of relations (11) can equivalently be written as

$$\gamma_j(z_j)=-D_{jj}^{-1}C_jE[x|z_j]-P_jD_{jj}^{-1}\sum_{i\neq j}D_{ji}\gamma_i(z_i),\qquad j\in\hat{\theta}_M, \tag{23}$$

where $P_j\triangleq E[\cdot|z_j]$ is the projection operator mapping $\mathcal{L}_{r_j}^{\eta}$ into $\mathcal{L}_{r_j}^{\eta}$, and $\eta\triangleq\{z_1,\cdots,z_M\}$. These M relations can further equivalently be written as a single relation

$$\gamma(\eta)=k(\eta)-PK\gamma(\eta), \tag{24a}$$

where

$$\gamma(\eta)\triangleq\left[\gamma_1^T(z_1),\cdots,\gamma_M^T(z_M)\right]^T \tag{24b}$$

$$k(\eta)\triangleq-\left[\left(D_{11}^{-1}C_1E[x|z_1]\right)^T,\cdots,\left(D_{MM}^{-1}C_ME[x|z_M]\right)^T\right]^T, \tag{24c}$$

K is an $r\times r$ matrix ($r\triangleq\Sigma r_i$) comprised of blocks of $r_i\times r_j$ matrices with the ij'th block given by

$$[K]_{ij}=\begin{cases}0, & i=j\\ D_{ii}^{-1}D_{ij}, & i\neq j\end{cases}\qquad i,j\in\hat{\theta}_M, \tag{24d}$$

and P is an operator mapping $\mathcal{L}_r^{\eta}$ into itself, defined by

$$P=\operatorname{diag}(P_1,\cdots,P_M). \tag{24e}$$

Note that P is a projection operator since it is both idempotent and self-adjoint. We thus conclude that the operator norm of P is unity. Concerning existence of a unique solution to (24a), we can now prove the following result.

Theorem 3: Let $\mathcal{C}$ denote the class of $r\times r$ real matrices obtained from K through the similarity transformation $\Pi K\Pi^{-1}$, where Π is structured as

$$\Pi=\operatorname{diag}(\Pi_1,\cdots,\Pi_M) \tag{25a}$$

$$\Pi_i \text{ nonsingular and } r_i\times r_i. \tag{25b}$$

Then, if there exists at least one member Λ of $\mathcal{C}$ with the property

$$\Lambda^T\Lambda<I, \tag{26}$$

there exists a unique M-tuple $\gamma_j\in\mathcal{L}_{r_j}^{\eta}$, $j\in\theta_M$, that solves (11). Equivalently, under the above given condition, the static M-person decision problem, with $\{x,\ z_j,\ j\in\theta_M\}$ being second-order random variables with arbitrary probability distributions, admits a unique equilibrium solution.

Proof: Since P, as defined by (24e), is a projection operator, the theorem will be established if we can show that (24a) admits a unique solution $\gamma^*\in\mathcal{L}_r^{\eta}$ with the additional property $P\gamma^*(\cdot)\equiv\gamma^*(\cdot)$, that is, γ^* should be in the subspace of $\mathcal{L}_r^{\eta}$ induced by P. To this end, let us first assume that (26) is satisfied by an element of $\mathcal{C}$ and with the corresponding real similarity transformation matrix being Π. Premultiplying (24a) by Π, we obtain

$$\tilde{\gamma}(\eta)=\tilde{k}(\eta)-P\Lambda\tilde{\gamma}(\eta)\triangleq\boldsymbol{K}\tilde{\gamma}(\eta) \tag{27a}$$

where

$$\tilde{\gamma}(\cdot)\triangleq\Pi\gamma(\cdot) \tag{27b}$$

$$\tilde{k}(\eta)\triangleq\Pi k(\eta). \tag{27c}$$

In writing down (27a), we have made use of the important property of Π that it commutes with P. Furthermore, since the subspace of $\mathcal{L}_r^{\eta}$ induced by P is invariant under the nonsingular transformation Π, the original existence and uniqueness question is now equivalent to investigation of existence of a unique solution $\tilde{\gamma}^* \in \mathfrak{L}_r^\eta$ to (27a), with the property $P\tilde{\gamma}^* \equiv \tilde{\gamma}^*$. To establish the desired result, we first note that $\boldsymbol{K}$ (whose definition follows from (27a) maps the linear space $\mathfrak{L}_r^\eta$ into itself. Defining a metric $d(\gamma,\xi)$ on this space by the norm $\|\gamma-\xi\|_r^z$, let us now consider the following sequence of inequalities:

$$d(\boldsymbol{K}\gamma,\boldsymbol{K}\xi)=\|P\Lambda(\xi-\gamma)\|_r^z \leqslant \|P\|\,\|\Lambda(\xi-\gamma)\|_r^z < \|\xi-\gamma\|_r^z = d(\gamma,\xi).$$

Here, the first inequality is a standard one in the Banach space of operators (where $\|P\|$ denotes the standard operator norm of P) [21], and the second inequality follows from (26) and the property $\|P\|=1$. This, therefore, implies that $\boldsymbol{K}$ is a contraction mapping and thus has a unique fixed point. It should also be clear from (27a) that this unique fixed point is in the subspace of $\mathfrak{L}_r^\eta$ induced by P, since every Picard iteration applied to (27a) results in an element of that subspace. □

Unfortunately, we do not have a counterpart of Corollary 1.1 for the $M>2$ case since K can never be diagonalized or brought to a canonical form within the class of similarity transformations characterized by (25). However, we do have a counterpart of Corollary 1.2, in the general case, since under the condition of Theorem 3 the unique solution of (27a) can be obtained as the limit of a convergent sequence of Picard iterates. The following Corollary thus follows directly from the proof of Theorem 3.

Corollary 3.1: Let the condition of Theorem 3 be satisfied for a member Λ^* of $\mathcal{C}$, which is related to $\boldsymbol{K}$ via the similarity transformation matrix Π^*. Then the unique equilibrium solution of the decision problem is given by

$$\gamma^*(\eta)=\lim_n \xi^n(\eta), \tag{28a}$$

where the limit is taken under the $\mathfrak{L}_r^\eta$-norm, and ξ^n denotes a convergent sequence in $\mathfrak{L}_r^\eta$ defined by

$$\xi^n(\eta)=k(\eta)+\Pi^{*-1}\sum_{m=1}^{n}(-P\Lambda^*)^m(\Pi^* k(\eta)) \tag{28b}$$

where $k(\cdot)$ is defined by (24c).

Remark 3: As in the 2-person case, the limit of this convergent series cannot in general be expressed in closed form. It should however be clear that when the random variables $\{x,z_i,\ i\in\hat{\theta}_M\}$ are jointly Gaussian, the limit of (28b) is necessarily affine in $\{z_1,\cdots,z_M\}$ whenever the sufficiency condition of Theorem 3 holds. This result is given below in Theorem 4:

Theorem 4: Let the condition of Theorem 3 be satisfied, and let x and $\{z_i,\ i\in\hat{\theta}_M\}$ be related as given by (19) for all $i\in\hat{\theta}_M$. Then the unique decision law of DMi, $i\in\hat{\theta}_M$, is affine in this observation and

$$\gamma_i^*(z_i)=A_i\bar{x}+B_i(\hat{x}_i-\bar{x}), \tag{29a}$$

where $\hat{x}_i$ is given by (21b) for all $i\in\hat{\theta}_M$, A_i is the unique solution of

$$A_i+D_{ii}^{-1}\sum_{j\neq i}D_{ij}A_j=-D_{ii}^{-1}C_i, \qquad i\in\hat{\theta}_M, \tag{29b}$$

and B_i satisfies uniquely

$$B_i+D_{ii}^{-1}\sum_{j\neq i}D_{ij}B_jQH_j^T\left(H_jQH_j^T+R_j\right)^{-1}H_j = -D_{ii}^{-1}C_i, \quad i\in\hat{\theta}_M. \tag{29c}$$

Proof: As in the proof of Theorem 2, it follows from (28b) that every Picard iterate that starts with a linear function of η results in an affine function of $\hat{x}_i$ for $\gamma_i(z_i)$. Hence, substitution of the structural form (29a) into (11) and some manipulation results in the relations (29b) and (29c). Existence and uniqueness follow from Theorem 3. □

IV. The Multistage Problem

After thus concluding investigation of the equilibrium solution of the single-stage prolem, we now return to the general M-criteria dynamic optimization problem of Section II.

Let us mention at the outset that for this general problem it is possible to verify existence of unique affine equilibrium strategies and to obtain the corresponding expressions in closed form for each DM, under certain nonvoid sufficiency conditions, by employing the results of Theorem 3 in an appropriate way at every stage of the decision process. However, to write down the expressions for the equilibrium strategies and the conditions explicitly necessitates (inevitably) introduction of an excessive amount of notation, and limitations on space force us not to provide here the full account. Instead, the approach we will adopt in this section will be to give a proof of uniqueness and linearity without explicitly obtaining the equilibrium solution and the existence conditions, but by explaining in detail how they can be obtained. We will also delineate the pitfalls of what might at first seem to be a straightforward inductive procedure, especially with regard to the given information structure. Exact expressions for the solution and sufficiency conditions for the $M=2$ case can be found in [18].

The main theorem of this section is the following:

Theorem 5: Under the one-step-delay observation sharing pattern and under certain nonvoid sufficiency conditions, the equilibrium solution of the multicriteria dynamic LQG optimization (decision) problem is unique, and the corresponding strategy of every DM is affine in the dynamic information available. In other words, under certain conditions, the problem of Section II can only admit unique equilibrium strategies of the form

$${}^*\gamma_j^n(\eta_j^n)=\Lambda_j(j,n)z_j(n)+\sum_{i=1}^{M}\sum_{k=0}^{n-1}\Lambda_j(i,k)z_i(k)+\lambda_j(n), \tag{30}$$

where $\{\Lambda_j(i,k)\}$ and $\{\lambda_j(k)\}$ are appropriate dimensional matrices and vectors, respectively, for each $n \in \theta_N, j \in \hat{\theta}_M$.

Proof: Since every equilibrium solution is, by definition, also a recursive equilibrium solution, the desired result will have been established if we can show that under certain conditions inequalities (8) can only admit unique M-tuple of strategy sequences of the form (30). To achieve this, we start with inequalities (8) at stage $n = N-1$ and proceed inductively in a descending order by solving a static problem (similar to that of Section III) at every stage and by carefully taking into account the interrelations between the static problems at two consecutive stages. Details of this inductive argument are as follows:

Step 1: Without any loss of generality, we can start with inequalities (8) at stage $n = N-1$ and fix the strategies of the DM's at the previous stages as arbitrary elements of appropriate Banach spaces. The resulting set of M-tuple of inequalities defines a static M-person decision problem similar to the one of Section III with x replaced by $x(N-1)$ which is not necessarily Gaussian since the arbitrarily fixed previous decision laws are not necessarily affine. Furthermore, the quadratic objective functionals $\{J_j^{N-1}\}$ of this decision problem also include a $w(N-1)$ term, but since the additive state noise at stage $N-1$ is independent of all other random vectors and has zero mean, it separates out and does not affect the solution. Then, if the sequence of arbitrarily fixed policies at previous stages is denoted by γ^{N-2} (as in Section II), a direct application of Theorem 3 implies existence of a unique M-tuple $\{{}^*\gamma_j^{N-1} \in \mathfrak{L}_j^{\eta_j}(\gamma^{N-2}), j \in \theta_M\}$ under an appropriate sufficiency condition which is the counterpart of (26). This unique solution can further be realized as the limit of an appropriate convergent sequence of the type given in Corollary 3.1. We now note that the conditional probability distribution of $x(N-1)$ given η_j^{N-1} and γ^{N-2} is Gaussian for each $j \in \hat{\theta}_M$. This implies that $E[x(N-1)|\eta_j^{N-1}, \gamma^{N-2}]$ is necessarily affine, and thus the limit of the said convergent sequence is also affine, since the state and observation equations are linear. Consequently there exist matrices $\{A_j^{N-1}, B_j^{N-1}(i,n), C_j^{N-1}(i,n), i,j \in \hat{\theta}_M, n \in \theta_{N-1}\}$ and vectors c_j^{N-1} such that

$$
\begin{aligned}
{}^*\gamma_j^{N-1}(\eta_j^{N-1}) &= A_j^{N-1} z_j(N-1) \\
&\quad + \sum_{i=1}^{M} \sum_{n=0}^{N-2} \left[B_j^{N-1}(i,n) z_i(n) \right. \\
&\quad \left. + C_j^{N-1}(i,n) \gamma_i^n(\eta_i^n) \right] + c_j^{N-1}, \qquad j \in \hat{\theta}_M. \qquad (31a)
\end{aligned}
$$

The crucial observation that has to be made here is that ${}^*\gamma_j^{N-1}(\cdot)$ is necessarily a linear function of $z_j(N-1)$ with a unique coefficient matrix A_j^{N-1}. However, we cannot yet say that ${}^*\gamma_j^{N-1}(\cdot)$ is linear in the past observation, since its dependence on them is also partly through γ^{N-2} whose structure is not yet known. Now, if the previous decision laws (i.e., γ^{N-2}) were known to DMj (i.e., if his information pattern was the one-step-delay sharing pattern), then (31a) would constitute a well-defined optimal solution at stage $N-1$, though not unique since any other function $l_j^{N-1}(\cdot)$, with appropriate dimension and measurability properties and which is identically zero at equilibrium, could be added to the right hand side (RHS) of (31a) without altering its equilibrium value. Under the one-step-delay observation sharing pattern, however, (31a) does not yet constitute a well-defined strategy since it is not an element of $\mathfrak{L}_j^{\eta_j}(\gamma^{N-2})$. What we know though is that regardless of what γ^{N-2} is, ${}^*\gamma_j^{N-1}(\cdot)$ will be a linear function of $z_j(N-1)$ with the unique coefficient matrix A_j^{N-1} computable independent of γ^{N-2}. This implies that in seeking equilibria, γ_j^{N-1} can be taken, without any loss of generality, in the structural form

$$
\gamma_j^{N-1}(\eta_j^{N-1}) = A_j^{N-1} z_j(N-1) + k_j^{N-1}(Z_{N-2}), \qquad j \in \hat{\theta}_M, \qquad (31b)
$$

where k_j^{N-1} is an arbitrary element of $\mathfrak{L}_j^{Z_{N-2}}(\gamma^{N-3})$. We note that the dependence of $\gamma_j^{N-1}(\cdot)$ on $\{\gamma_i^{N-2}, i \in \hat{\theta}_M\}$ is only through the first term, as it should be, since $k_j^{N-1}(\cdot)$ involves measurements of stages $N-2, N-3, \cdots, 1, 0$.

Step 2: We now consider inequalities (8) for $n = N-2$, after substitution of (31b) into the appropriate expressions and by assuming γ^{N-3} to be fixed *a priori* within the permissible class. Since $z_j(N-1)$ is a linear function of $\{\gamma_i^{N-2}(\cdot), i \in \hat{\theta}_M\}$, the result is again a set of M-tuple inequalities which define a static quadratic decision problem of the type considered at Step 1, with $N-1$ replaced by $N-2$ and with each cost function containing an additional quadratic term in $\{k_j^{N-1}, j \in \hat{\theta}_M\}$. Consequently, at equilibrium, we have, as counterpart of relations (11), the equation

$$
\begin{aligned}
\gamma_j^{N-2}(\eta_j^{N-2}) &= \tilde{D}_{jj}^{N-2} E\left[x(N-2) | \eta_j^{N-2} \right] \\
&\quad + \sum_{i=1}^{M} \tilde{D}_{jj}^{N-2}(i) E\left[k_i^{N-1}(\cdot) | \eta_j^{N-2} \right] \\
&\quad + \sum_{i \neq j} \tilde{D}_{ji}^{N-2} E\left[\gamma_i^{N-2}(\eta_i^{N-2}) | \eta_j^{N-2} \right], \qquad (32a)
\end{aligned}
$$

where $\tilde{D}_{ji}^{N-2}$, $\tilde{D}_{jj}^{N-2}(i)$, $i,j \in \hat{\theta}_M$, are appropriate dimensional computable matrices. In solving this set of equations, we also have to make use of the optimal linear dependence of $k_j^{N-1}(\cdot)$ on γ_i^{N-2}, which can be obtained by comparison of (31a) and (31b). This property used in (32a) and rearranging terms yields a relation of the form

$$
\begin{aligned}
\gamma_j^{N-2}(\eta_j^{N-2}) &= \hat{D}_{jj}^{N-2} E\left[x(N-2) | \eta_j^{N-2} \right] \\
&\quad + \sum_{i \neq j} \hat{D}_{ji}^{N-2} E\left[\gamma_i^{N-2}(\cdot) | \eta_j^{N-2} \right] \\
&\quad + \sum_{i=1}^{M} \sum_{n=0}^{N-3} \left[\hat{B}_j^{N-2}(i,n) z_i(n) \right. \\
&\quad \left. + \hat{C}_j^{N-2}(i,n) \gamma_i^n(\eta_i^n) \right] + \hat{c}_j^{N-2}, \qquad j \in \hat{\theta}_M, \qquad (32b)
\end{aligned}
$$

where the weighting matrices and the vector $\hat{c}_j^{N-2}$ are appropriately determined. The last two terms above can be considered as constant under the information pattern η_j^{N-2} for DMj, and this set of equations to be solved for $\{\gamma_j^{N-2}, j\in\hat{\theta}_M\}$ is similar to (11). Theorem 3 again applies under appropriate nonvoid sufficiency conditions as well as Corollary 3.1. Since γ^{N-3} is fixed *a priori* and $E[x(N-2)|\eta_j^{N-2},\gamma^{N-3}]$ is affine, it follows, as in Step 1, that (32b) admits a unique solution of the form

$$
\begin{aligned}
{}^*\gamma_j^{N-2}(\eta_j^{N-2}) &= A_j^{N-2}z_j(N-2)\\
&\quad+\sum_{i=1}^{M}\sum_{n=0}^{N-3}\left[B_j^{N-2}(i,n)z_i(n)\right.\\
&\quad\left.+C_j^{N-2}(i,n)\gamma_i^n(\eta_i^n)\right]+c_j^{N-2}, \qquad j\in\hat{\theta}_M, \qquad (32c)
\end{aligned}
$$

for some matrices A_j^{N-2}, $B_j^{N-2}(i,n)$, $C_j^{N-2}(i,n)$ and vectors c_j^{N-2}. It should again be noted that ${}^*\gamma_j^{N-2}$ is necessarily a linear function of $z_j(N-2)$ with a unique coefficient matrix A_j^{N-2}. Furthermore, (32c) substituted into (31a) indicates that ${}^*\gamma_j^{N-1}$ is also a linear function of $\{z_i(N-2),\ i\in\hat{\theta}_M\}$. This then implies that, without any loss of generality, γ_j^{N-1} and γ_j^{N-2} can be taken in the structural forms

$$
\begin{aligned}
\gamma_j^{N-1}(\eta_j^{N-1}) &= A_j^{N-1}z_j(N-1)+\sum_{i=1}^{M}\left[B_j^{N-1}(i,N-2)\right.\\
&\quad\left.+C_j^{N-1}(i,N-2)A_i^{N-2}\right]z_i(N-2)\\
&\quad+k_{j,N-1}^{N-2}(Z_{N-3}) \qquad (33a)
\end{aligned}
$$

$$
\gamma_j^{N-2}(\eta_j^{N-2}) = A_j^{N-2}z_j(N-2)+k_{j,N-2}^{N-2}(Z_{N-3}) \qquad (33b)
$$

where $k_{j,N-1}^{N-2}$ and $k_{j,N-2}^{N-2}$ are arbitrary elements of $\mathcal{L}_{\Gamma_j}^{Z_{N-3}}(\gamma^{N-4})$.

Step 3: The next step is to consider inequalities (8) for $n=N-3$ after substitution of (33) and by assuming γ^{N-4} to be fixed *a priori* within the permissible class. Then, the procedure applied on the resulting static decision problem is no different than the procedure of Step 2, with only $N-2$ replaced by $N-3$; and this results in unique structural forms for γ_j^{N-1}, γ_j^{N-2}, and γ_j^{N-3} linear in $\{z_j(N-1), z_i(n),\ i\in\hat{\theta}_M,\ n=N-2,N-3\}$, $\{z_j(N-2), z_i(n),\ i\in\hat{\theta}_M,\ n=N-3\}$ and $z_j(N-3)$, respectively, as counterparts of (33).

If these steps are inductively followed up to the initial state, then it should be clear that the equilibrium strategies will be unique and affine in the structural form (30). Appearance of the $\lambda_j(\cdot)$ terms in (30) is a consequence of the assumption that the mean of x_o is not zero. □

Remark 5: The question that remains to be answered is what happens if the information pattern is taken as the one-step-delay sharing pattern, which allows each DM to have access to also the past decisions of the other DM's. In this case, as we have mentioned at Step 1, (31a) will not be the only strategy representation of DMj at stage $N-1$, and at Step 2 we might not take $k_j^{N-1}(\cdot)$ only as a function of $\{z_i(n),\ i\in\hat{\theta}_M,\ n\in\theta_{N-1}\}$. It could also depend on γ^{N-1} explicitly and this results in a different static problem to solve for every permissible choice of $k_j^{N-1}(\cdot)$. It should be noted that this construction of nonunique "representations" of (30) is within the spirit of the concept of nonunique representations introduced in [22] within the context of deterministic nonzero-sum dynamic games, but not quite the same. Nevertheless, it can be shown that essentially nonunique equilibrium solutions emerge from this construction under the amended information structure. We now provide a specific example to illustrate and verify our conjecture.

Example 1: Consider (as a special case of the general problem) a scalar 2-stage 2-criteria optimization problem with $2DM$'s and described by the state equation

$$
\begin{aligned}
x(2) &= x(1)+u_1(1)\\
x(1) &= x_o+u_1(0)+u_2(0), \qquad x_o\sim N(0,1) \qquad (34a)
\end{aligned}
$$

and objective functionals

$$
\begin{aligned}
J_1 &= x^2(2)+u_1^2(1)+u_1^2(0)\\
J_2 &= x^2(2)+u_2^2(0). \qquad (34b)
\end{aligned}
$$

The observation equations are given by

$$
\begin{aligned}
z_3 &\triangleq z_1(1) = x(1)+v_1(1), \qquad v_1(1)\sim N(0,1)\\
z_1 &\triangleq z_1(0) = x_o+v_1(0) \qquad (34c)\\
z_2 &\triangleq z_2(0) = x_o+v_2(0), \qquad v_i(0)\sim N(0,1), \quad i=1,2.
\end{aligned}
$$

Now, under the original "one-step-delay observation sharing pattern", we follow the procedure outlined in this section to obtain the unique equilibrium solution

$$
\begin{aligned}
{}^*\gamma_1^o(z_1) &= -(39/269)z_1\\
{}^*\gamma_2^o(z_2) &= -(35/269)z_2 \qquad (35)\\
{}^*\gamma_1^1(z_1,z_2,z_3) &= -(19/269)z_1-(41/269)(1/2)z_2-(1/8)z_3.
\end{aligned}
$$
[3]

Under the "one-step-delay sharing pattern", however, again by following the outlined procedure and this time taking nonunique linear representation for $k_1^1(\cdot)$ at Step 2, we obtain the set of *essentially nonunique* equilibrium solutions[4]

$$
\begin{aligned}
{}^*\gamma_1^o(z_1) &= -\left[(39+8p)/(269+88p)\right]z_1\\
{}^*\gamma_2^o(z_2) &= -\left[(35+40p)/(269+88p)\right]z_2\\
{}^*\gamma_1^1(z_1,z_2,z_3,u_2(0)) &= -\tfrac{1}{8}z_3-\left[(19+8p)/(269+88p)\right]z_1\\
&\quad-\left[(41-8p)/2(269+88p)\right]z_2\\
&\quad+p\left[u_2(0)+\left[(35+40p)/(269+88p)\right]z_2\right], \qquad (36)
\end{aligned}
$$

[3]It can be directly verified that (35) satisfies inequalities (7).
[4]Details of this construction can be found in [18, sec. 5].

where $p > -8/23$ is the parameter characterizing the nonuniqueness. It can actually be tested by direct substitution into (7) that for every $p > -8/23$ (36) is indeed an equilibrium solution under the amended information pattern; this implies that if $DM1$ is given access to the past action of $DM2$, the equilibrium solution is nonunique, and it can easily be made nonlinear by an appropriate choice of representation. We also note that (36) corresponds to (35) for a specific value of p, which is $p = 0$.

Remark 6: When specialized to the case of dynamic M-member team problems, Theorem 5 implies that under certain convexity conditions the LQG team problems with one-step-delay observation sharing pattern admit unique team-optimal solutions affine in the available information. When the information structure is one-step-delay sharing pattern, we have seen above that for the multicriteria problem a possibility of nonunique solutions emerges. For the special case of team problems, however, the latter information structure cannot produce better solutions (with lower average cost) than the former information structure, as the following argument validates: Assume the contrary for the moment, that is, there exists an M-tuple of strategies depending explicitly on past actions and yielding a lower average value for the team cost function. However, at equilibrium every action can uniquely be expressed in terms of corresponding observation vectors, and this produces a strategy M-tuple measurable with respect to the field generated by the one-step-delay observation sharing pattern. This implies that in a team problem one can, without any loss of generality, deal with the original information structure. Hence, for the special case of a team problem, Theorem 5 provides a verification of the conjecture made in [7] concerning linearity of optimal team-solutions for this class of problems and under the one-step-delay sharing pattern.

Remark 7: The deterministic version of the problem of Section IV, but with additive noise term in the state dynamics and with closed-loop information for all players, has been considered in [22], and in this context uniqueness of Nash equilibria has been established. That particular equilibrium solution has the property that it is in Nash equilibrium at every stage of the game, in other words, at every stage one has to solve a static Nash game whose objective functionals are the "cost to go" expressions in the sense of dynamic programming. Furthermore, the unique Nash strategies, at every stage, are completely determined from the solutions of those static games. This is definitely a very desirable property, and besides, the resulting strategies are computationally very feasible. At this point one might raise the question whether it is possible to obtain a Nash equilibrium solution for the stochastic problem of Section IV, with such a property. First of all, it should be clear that this is not possible under the one-step-delay observation sharing pattern, since the equilibrium solution mentioned in Theorem 5 is unique and it does not possess such a property. Under the one-step-delay sharing pattern, however, we have seen that there exists nonunique equilibrium solutions, and one of these could very well possess that property.[5] Actually such a solution has recently been presented in [26] by applying dynamic programming arguments and by making repeated use of Theorem 3 of [15] at every stage. Though computationally feasible, such an equilibrium solution does not seem to have much of a significance, at the present, since it is one of an infinite number of equilibrium solutions, and the property that we have outlined above has not yet been made mathematically precise. It should be noted that one possible approach is to take the delayed-commitment point of view of [23]–[24] under the one-step-delay sharing pattern. But this requires further investigation.

V. Concluding Remarks

Our aim, in this paper, has been to develop certain general results concerning the (Nash) equilibrium solutions of LQG multicriteria optimization problems and under quasiclassical information patterns. One of the main results is that under the one-step-delay observation sharing pattern, the equilibrium solution is unique and affine. Hence, in solving a problem of this type, one may without any loss of generality assume an affine strategy for each DM, and then optimize only on the coefficient matrices. Development of efficient algorithms to achieve that objective is a problem that should be undertaken in the future. Under the one-step-delay sharing pattern, however, it has been shown that the (Nash) equilibrium solution concept alone does not make much sense, since such problems admit uncountably many solutions. In such a situation, one has to introduce some additional rational selection criterion, perhaps in the same spirit as the sensitivity and robustness criterion of [22, sec. IV] or an extension of the delayed-commitment approach of [23] and [24]. Some further work is definitely needed in that context.

For the static LQ multicriteria optimization problems, we have obtained a uniqueness result under rather relaxed sufficiency conditions; and the fixed-point approach adopted to prove the result has also suggested the possibility of determining the optimal solution as the limit of a sequence of Picard iterates. This way of representing the solution is especially useful in the case of non-Gaussian distributions, when no closed-form solution can be obtained. Such a series solution approach has also recently been suggested within the context of static team problems [25], where the approach taken involves inversion of certain operators and it results in somewhat more complex expressions even for the team problem. What should definitely be considered, in the future, within that context, is determination of the speed of convergence of the Picard iterates for different probability distributions, and determination of the number of terms that should be taken

[5]Within the context of Example 1, this will correspond to (36) for a particular value of p.

in each case so as to achieve a fairly good approximation to the optimal solution.

In some applications, the Stackelberg solution might be more appropriate than the Nash solution, and hence a thorough investigation of the properties of stochastic Stackelberg solutions of LQ multicriteria optimization problems would be very desirable. Some work in that direction has been done in [27], but there are still several structural problems that require further investigation. In particular, the structural sensitivity of the stochastic Stackelberg solution to changes in the information pattern has not yet been fully explored. For a start in that direction the reader is referred to [28].

One of the most interesting, challanging and important class of problems within the context of multicriteria stochastic optimization are those characterized by strictly nonclassical information patterns. Since the publication of Witsenhausen's counterexample [3], no significant research has been done in this area within the context of team problems. On the other hand, as we have mentioned before (in Section I), for zero-sum dynamic games of the LQG type, the strictly nonclassical information pattern does not really create any major difficulties, since the saddle-point solution is still attained by affine strategies [11]–[12]. This implies that for some class of LQG multicriteria optimization problems with strictly nonclassical information patterns, it is still possible to obtain unique linear equilibrium solutions. It is still not known precisely how large this class is, and it constitutes a challanging problem awaiting further consideration via an innovative approach. A thorough investigation of this class of problems will definitely also shed light on the solution of the LQG team problems with strictly nonclassical information patterns.

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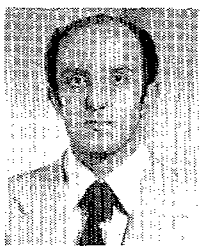

Tamer Başar (S'71–M'73) was born in Istanbul, Turkey, on January 19, 1946. He received the B.S.E.E. degree from Robert College, Boğaziqi University, Istanbul, Turkey, in 1969, and the M.S., M.Phil., and Ph.D. degrees from Yale University, New Haven, CT, in 1970, 1971, and 1972, respectively.

Between March 1972 and August 1973 he was a Research Fellow with the Decision and Control Group of the Division of Engineering and Applied Physics, Harvard University, Cambridge, MA, and since 1973 he has been with the Applied Mathematics Division, Marmara Scientific and Industrial Research Institute, Gebze-Kocaeli, Turkey, where he is presently a Senior Research Scientist. He is also affiliated as an associate professor with the Mathematics Department, Boğaziqi University. His current interests include Nash and Stackelberg solutions of deterministic and stochastic nonzero-sum differential games, information patterns and value of information in large scale decision problems, and application of control theoretic concepts in economic systems. He is an associate editor of the IFAC Journal *Automatica*, a reviewer for *Mathematical Reviews*, *Applied Mechanics Reviews* and *Zentralblatt für Mathematik*. He is the recipient of the 1976 TUBITAK (the Scientific and Technical Research Council of Turkey) Young Scientist Award in Applied Mathematics.

Dr. Başar is a member of Sigma Xi.