An Equilibrium Theory for Multiperson Decision Making with Multiple Probabilistic Models

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Abstract—**This paper develops an equilibrium theory for two-person two-criteria stochastic decision problems with static information patterns, wherein the decision makers (DM's) have different probabilistic models of the underlying process, the objective functionals are quadratic, and the decision spaces are general inner-product spaces. Under two different modes of decision making (viz. symmetric and asymmetric), sufficient conditions are obtained for the existence and uniqueness of equilibrium solutions (stable in the former case), and in each case a uniformly convergent iterative scheme is developed whereby the equilibrium policies of the DM's can be obtained by evaluating a number of conditional expectations. When the probability measures are Gaussian, the equilibrium solution is linear under the symmetric mode of decision making, whereas it is generically nonlinear in the asymmetric case, with the linear structure prevailing only in some special cases which are delineated in the paper.**

I. Introduction

A TEAM is defined as a group of agents who work together in a coordinated effort, in a possibly hostile and uncertain environment, in order to achieve a common goal. In achieving this goal, the members of the team do not necessarily acquire the same information, and hence they have to operate in a decentralized mode of decision making. The scientific approach to formulation and analysis of team problems has involved 1) a quantification of the underlying common goal in the form of a (mathematical) objective function which is sought to be optimized jointly by the agents, and 2) a modeling of the uncertain environment and the possible measurements made by the agents on this environment in the form of a probability space together with an appropriate information structure [14], [7], [15], [16]. The underlying stipulation here has been the existence of a probability space that is *common* to all the agents, so that through their priors all members of the team "see the world" in exactly the same way.

One question that readily comes into mind at this point is the robustness of such a mathematical model, and the "optimum" solutions it produces, to slight variations in the underlying assumptions. In particular, what if the agents perceive the outside world in slightly different ways? Would the solution obtained under the assumption of common prior probability measures change drastically if there are discrepancies in the agents' perceptions of the probabilistic description of the outside world? In order to be able to answer these queries satisfactorily and effectively, we need a theory of equilibrium for decision problems in which the decision makers (DM's) have different probabilistic models of the system; such a general theory will clearly subsume the currently available results on teams which use a common probability space.

Consider a static team decision problem, formulated in the standard manner as in [7], with the only difference being in the underlying probability space. In particular, assume that the DM's assign different subjective probabilities to the uncertain events, in which case there will not exist a common probability space, thereby leading to a different expected (average) cost function for each DM. Hence, once we relax the assumption of existence of a common probability space, the team problem is no longer a stochastic optimization problem with a *single objective functional,* and we inevitably have to treat it as a nonzero-sum stochastic game [5], [8], [12]. Furthermore, even though the original team decision problem with a common probability space will admit the same team-optimal solution(s) regardless of the mode of decision making (that is, regardless of whether the roles of the DM's are symmetric or whether there is a hierarchy and dominance in decision making), this feature ceases to hold true when there exists a discrepancy between the perceived probability measures. When there are only two members, for example, two possibilities emerge in the presence of discrepancies: the totally symmetric roles, corresponding to the Nash equilibrium solution, and the hierarchical mode, corresponding to the Stackelberg equilibrium solution.

Motivated by these considerations, we treat in this paper a more general (than team) class of two-person stochastic decision problems which can be viewed as static stochastic nonzero-sum games with the DM's having different subjective probability measures. Adopting both the symmetric and asymmetric modes of decision making, we develop in each case a general theory of equilibrium when the objective functionals are quadratic and the decision spaces are appropriate Hilbert spaces. Such a formulation includes both finite-dimensional (discrete) and continuous-time decision problems, and involves arbitrary probability measures which are, though, restricted *a posteriori* by the conditions of existence and uniqueness developed in this paper. The special case of Gaussian distributions is studied in considerable depth, and some explicit solutions are obtained with appealing features.

The organization of the paper is as follows. Section II provides a precise problem formulation, and introduces the two solution concepts adopted in the paper. Section III develops general conditions for existence and uniqueness of a stable equilibrium solution under the symmetric mode of decision making, and elucidates the extent of the restrictions imposed on the problem by these conditions. Section IV presents a counterpart of the results of Section III under the asymmetric mode of decision making, with the mathematical machinery used being inherently different from that of Section III. Section V deals with the special class of Gaussian distributions, under both symmetric and asymmetric modes of decision making. In the former case it is shown that the unique stable equilibrium solution is affine in the measurements and can be obtained explicitly. In the latter case, however, the solution is generically nonlinear, and contains summation of terms which involve products of linear functions of measurements with

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exponential terms (whose exponents are quadratic in the measurements). The section also contains some discussion on finite-dimensional and continuous-time problems, treated as special cases. Section VI is devoted to discussions on possible extensions of these results in different directions, provides some interpretation of the general approach and results, and includes some concluding remarks. The paper ends with five Appendices which include results used in the main body of the paper.

II. Mathematical Formulation and Some Basic Results

A. Probability Spaces

Let $\Omega = \mathbb{R}^n \times \mathbb{R}^{m1} \times \mathbb{R}^{m2} \triangleq X \times Y_1 \times Y_2$, $\mathcal{B}$ denote the Borel field of subsets of Ω, and $\mathcal{B}^k$ denote the Borel field of subsets of $\mathbb{R}^k$, $k = n, m_1, m_2$. Let $\mathcal{P}$ denote the set of all probability measures on $(\Omega, \mathcal{B})$ with finite second moments, and for each $P \in \mathcal{P}$ denote the corresponding marginal measures on $\mathcal{B}^n$, $\mathcal{B}^{m1}$, and $\mathcal{B}^{m2}$ by P_x, P_{y_1}, and P_{y_2}, respectively. Furthermore, let the collection of all such probability measures be denoted by $\mathcal{P}_x$, $\mathcal{P}_{y_1}$, and $\mathcal{P}_{y_2}$, respectively. Then, for each $P \in \mathcal{P}$, the vector $z = (x', y_1', y_2')'$, taking values in Ω, becomes a well-defined random vector on $(\Omega, \mathcal{B}, P)$, and likewise x is a random vector on $(\mathbb{R}^n, \mathcal{B}^n, P_x)$ and y_i is a random vector on $(\mathbb{R}^{m_i}, \mathcal{B}^{m_i}, P_{y_i})$.

Here, x denotes the unknown state of nature, and y_i denotes an observation of DMi (ith decision maker) which is correlated with x. We now choose two elements out of $\mathcal{P}$, P^1 and P^2, which denote the subjective probabilities assigned to z by DM1 and DM2, respectively. For technical reasons, we place some further restrictions on the choices of P^1 and P^2 through the marginals $P^i_{y_i}$; in particular, we assume the following.

Condition 1: $P^1_{y_2}$ and $P^2_{y_1}$ are absolutely continuous [1] with respect to $P^2_{y_2}$ and $P^1_{y_1}$, respectively; that is, using the standard notation in probability theory

$$P^1_{y_2} \langle\langle P^2_{y_2},\ P^2_{y_1} \langle\langle P^1_{y_1}. \tag{1}$$

Condition 2: The Radon–Nikodym (R–N) derivative [1]

$$g^i(\xi) = dP^j_{y_i}/dP^i_{y_i}, \qquad j \neq i \tag{2}$$

is uniformly bounded a.e. $P^i_{y_i}$, $i = 1, 2$.

The necessity of these two conditions in the formulation of our problem will be made clear in the sequel. We should note, however, that for the special case when P^1 is equivalent to P^2, both of these conditions are satisfied (in the latter case the bound is equal to 1) and we have the standard decision theoretic framework [2] with a single probability space.

B. Decision and Policy Spaces

The decision variable of DMi will be denoted by u_i which belongs to a real separable Hilbert space U_i with inner product $(\cdot, \cdot)_i$. *Permissible policies* (decision rules) for DMi are measurable mappings

$$\gamma_i : \mathbb{R}^{m_i} \to U_i, \quad \int \|\gamma_i(\xi)\|_i^2 P^i_{y_i}\,(d\xi) < \infty \tag{3}$$

where $\|\cdot\|_i$ is the natural norm derived from $(\cdot, \cdot)_i$. Let Γ_i denote the space of all such policies, which is further equipped with the inner product

$$\langle \gamma, \beta \rangle_i = \int_{Y_i} (\gamma(\xi), \beta(\xi))_i P^i_{y_i}\,(d\xi). \tag{4}$$

Then, we have the following two results: the first of which is standard [3] and the second one involves a change of measures using the R-N derivative.

Lemma 1: Γ_i is a Hilbert space. □

Lemma 2: If Conditions 1 and 2 are satisfied, every element of Γ_i has bounded second-order moments also under $P^j_{y_i}$, $j \neq i$.

C. Cost Functionals

Let $D^i_{jj}: U_j \to U_j (i \neq j,\ i, j = 1, 2)$ be strongly positive[1] bounded linear operators, and $F^i_j: X \to U_j$ be bounded linear operators for all $i, j = 1, 2$. Furthermore, let $E^i[\mu^i(z)|y_i]$ denote the mathematical expectation of a z-measurable random variable $\mu^i(z)$ taking values in U_i conditioned on the random variable y_i, and under the probability measure P^i, i.e.,

$$E^i[\mu(z)|y_i] = \int_\Omega \mu(z) P^i_{|y_i}\,(dz|y_i) \tag{5}$$

where the second term of the integrand is the conditional probability measure derived from P^i. Then, for each pair $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$, we have a quadratic expected cost functional for each DM, defined from DMi by

$$\begin{aligned} J_i(\gamma_1, \gamma_2) = \frac{1}{2}\langle \gamma_i, \gamma_i \rangle_i + \frac{1}{2} & \cdot \int_{Y_j} (\gamma_j(\xi), D^i_{jj}\gamma_j(\xi))_j P^i_{y_j}\,(d\xi) \\ & - \int_{X \times Y_j} (\gamma_j(\xi), F^i_j x)_j P^i\,(dx, Y_i, d\xi) \\ & - \langle \gamma_i, E^i[D^i_{ij}\gamma_j(y_j)|y_i] \rangle_i - \langle \gamma_i, E^i[F^i_i x|y_i] \rangle_i \end{aligned} \tag{6}$$

every term of which can be shown to be finite, in view of Lemmas 1 and 2. Note that in the absence of Conditions 1 and 2, J_i is not necessarily finite, and hence the problem is not well defined.

It is worth mentioning here that J_i describes a most general type of quadratic cost functional which is strictly convex in u_i, and that the formulation here covers also the cases of *team* problems ($D^i_{jj} = I$, $D^1_{12} = D^{2*}_{21}$,[2] $F^1_i = F^2_i$, $i, j = 1, 2$, $i \neq j$) and *zero-sum games* ($D^i_{jj} = -I$, $D^1_{12} = -D^{2*}_{21}$, $F^1_i = -F^2_i$, $i, j = 1, 2, i \neq j$). But even in these "single loss-functional" problems, the DM's will have inherently different expected cost functions whenever P^1 and P^2 are different, since then a common probability space does not exist. This forces us to formulate the problem as a multicriteria optimization problem and introduce equilibrium solution concepts that would be appropriate in this framework.

D. Equilibrium Solution Under the Symmetric Mode of Decision Making

Since the expected cost functions (6), together with the policy spaces, provide a normal (strategic) form description, regardless of the presence of multiple probability measures, the standard definition of noncooperative (Nash) equilibrium [5] remains intact, which is the most reasonable solution concept here under the symmetric mode of decision making.

Definition 1: A pair of policies $(\gamma_1^\circ, \gamma_2^\circ) \in \Gamma_1 \times \Gamma_2$ constitutes a Nash equilibrium solution if

$$\begin{aligned} & J_1(\gamma_1^\circ, \gamma_2^\circ) \leq J_1(\gamma_1, \gamma_2^\circ), \\ & J_2(\gamma_1^\circ, \gamma_2^\circ) \leq J_2(\gamma_1^\circ, \gamma_2), \quad \forall \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2. \end{aligned} \tag{7}$$

□

Definition 2: A Nash equilibrium solution $(\gamma_1^\circ, \gamma_2^\circ)$ is *stable* if for all $(\gamma_1^{(0)}, \gamma_2^{(0)}) \in \Gamma_1 \times \Gamma_2$

[1] That is, there exists $\alpha > 0$ such that $(u, D^i_{jj}u)_j \geq \alpha(u, u)_j$ for all $u \in U_j$.

[2] A superscript * designates the adjoint of a given linear operator defined on a Hilbert space, and I designates the identity operator.

$$\lim_{k\to\infty} \gamma_i^{(k)} = \gamma_i^o \text{ in } \Gamma_i, \qquad i=1,2 \tag{8}$$

where

$$\gamma_1^{(k)} = \arg\min_{\Gamma_1} J_1(\gamma_1, \gamma_2^{(k-1)}) \tag{9a}$$

$$\gamma_2^{(k)} = \arg\min_{\Gamma_2} J_2(\gamma_1^{(k-1)}, \gamma_2), \qquad k=1,2,\cdots. \tag{9b}$$

□

Remark 1: The notion of stable equilibrium makes particular sense (and is of paramount importance) in decision problems wherein the DM's have different priors on the uncertain quantities because it is determined as the outcome of a natural iterative process. In this process, each DM responds optimally (using his priors) to the most recent decision (policy) of the other DM, with the priors on which this decision is based being irrelevant. In other words, even though the computation of the Nash equilibrium solution will depend on the different prior probability measures perceived by two DM's, in the iterative procedure that leads to this equilibrium each DM has to know *only* his own prior and the other one's announced policy at the previous step. For an earlier utilization of this concept in a deterministic setting we refer the reader to [28]. □

E. Equilibrium Solution Under an Asymmetric Mode of Decision Making

In the case of the asymmetric mode there is a hierarchy in decision making, which permits one DM (say DM1—*leader*) to announce and enforce his policy on the other DM (*follower*). The relevant solution concept here is the leader–follower (Stackelberg) solution which is introduced below.

Definition 3: A pair of policies $(\gamma_1^s, \gamma_2^s) \in \Gamma_1 \times \Gamma_2$ constitutes a *leader-follower* (*Stackelberg*) equilibrium solution with unique follower responses, if there exists a unique mapping $T_2: \Gamma_1 \to \Gamma_2$ satisfying

$$J_2(\gamma_1, T_2[\gamma_1]) \le J_2(\gamma_1, \gamma_2), \qquad \forall (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \tag{10}$$

and furthermore

$$J_1(\gamma_1^s, T_2[\gamma_1^s]) \le J_1(\gamma_1, T_2[\gamma_1]), \qquad \forall \gamma_1 \in \Gamma_1 \tag{11}$$

with

$$\gamma_2^s = T_2[\gamma_1^s].$$

Remark 2: The uniqueness condition on T_2 is satisfied in our case because J_2 is strictly convex (and quadratic) in γ_2. □

Remark 3: The solution introduced above may not, at first glance, appear to be an equilibrium solution because of the strict ordering of the DM's. However, it can be shown, by following an argument first developed in [17], that the Stackelberg solution can be viewed as the so-called "strong equilibrium" of a decision problem with a modified (dynamic) information pattern (see Appendix E). □.

III. General Conditions for a Stable Equilibrium Solution Under the Symmetric Mode

We now obtain some general conditions for existence of stable equilibrium solutions under the symmetric mode of decision making, and also consider some special cases when the probability measures of both DM's are absolutely continuous with respect to the Lebesgue measure (i.e., when densities exist). First, we have the following.

Proposition 1: A pair of policies $(\gamma_1^o, \gamma_2^o) \in \Gamma_1 \times \Gamma_2$ constitutes a Nash *equilibrium solution* to the decision problem of Section II, if, and only if, it satisfies the pair of equations [under the notation of (5)]

$$\gamma_1^o(y_1) = D_{12}^1 E^1[\gamma_2^o(y_2)|y_1] + F_1^1 E^1[x|y_1] \tag{12a}$$

$$\gamma_2^o(y_2) = D_{21}^2 E^2[\gamma_1^o(y_1)|y_2] + F_2^2 E^2[x|y_2]. \tag{12b}$$

Proof: This result follows from a simple minimization of two quadratic forms $J_1(\gamma_1, \gamma_2^o)$ and $J_2(\gamma_1^o, \gamma_2)$ on the two Hilbert spaces Γ_1 and Γ_2, respectively, and by virtue of the fact that these two quadratic forms are positive definite in the relevant variables. □

By the same argument used in the proof of Proposition 1, relations (9a) and (9b) in Definition 2 can equivalently be written as

$$\gamma_1^{(k)}(y_1) = D_{12}^1 E^1[\gamma_2^{(k-1)}(y_2)|y_1] + F_1^1 E^1[x|y_1] \tag{13a}$$

$$\gamma_2^{(k)}(y_2) = D_{21}^2 E^2[\gamma_1^{(k-1)}(y_1)|y_2] + F_2^2 E^2[x|y_2], \qquad k=1,2,\cdots. \tag{13b}$$

Now, substituting (13b) into (13a), and also (13a) into (13b), by appropriately matching the superscripts, we arrive at the following two recursive relations:

$$\gamma_i^{(k)}(y_i) = D_{ij}^i D_{ji}^j E^i[E^j[\gamma_i^{(k-2)}(y_i)|y_j]|y_i] + F_i^i E^i[x|y_i] + D_{ij}^i F_j^j E^i[E^j[x|y_j]|y_i],$$
$$j, i = 1, 2;\ j \ne i;\ k=2,4,\cdots \text{ or } k=3,5,\cdots. \tag{14}$$

Note that if the recursive scheme (14) converges for even values of k, it also converges (to the same limit) for odd values of k [this follows from expressions (13a)–(13b)]. Hence, we confine attention only to even values of k and obtain the following results as a direct consequence of the foregoing analysis.

Proposition 2: A pair of policies $(\gamma_1^o, \gamma_2^o) \in \Gamma_1 \times \Gamma_2$ constitutes a *stable* Nash equilibrium solution if, and only if, for all $(\gamma_1^{(0)}, \gamma_2^{(0)}) \in \Gamma_1 \times \Gamma_2$

$$\gamma_i^o(y_i) = \lim_{\bar{k}\to\infty} \gamma_i^{(2\bar{k})}(y_i) \text{ in } \Gamma_i \tag{15}$$

where $\gamma_i^{(2\bar{k})}$, $\bar{k} = 1, 2, \cdots$, is given recursively by (14). Furthermore, such a stable equilibrium solution is necessarily unique. □

Let us now introduce linear operators $\mathcal{S}_i: \Gamma_i \to \Gamma_i$, $i = 1, 2$, by

$$\mathcal{S}_i(\gamma) = D_{ij}^i D_{ji}^j E^i[E^j[\gamma(y_i)|y_j]|y_i], \qquad j \ne i;\ i, j = 1, 2. \tag{16}$$

Note that $\mathcal{S}_i$ indeed maps Γ_i into Γ_i because the conditional expectation $E^j[D_{ji}^j \gamma(y_i)|y_j]$ maps Γ_i into $\Gamma_j (j \ne i)$ when the probability measures satisfy Conditions 1 and 2, and every element of Γ_i is square-integrable under both $P_{y_i}^i$ and $P_{y_i}^j$ (cf. Lemma 2).

Furthermore, let us introduce the notation $\langle\langle \mathcal{S} \rangle\rangle_i$ to denote the *norm* of a linear bounded operator $\mathcal{S}: \Gamma_i \to \Gamma_i$, which is defined by

$$\langle\langle \mathcal{S} \rangle\rangle_i = \sup_{\gamma \in \Gamma_i} [\langle \mathcal{S}\gamma, \mathcal{S}\gamma \rangle_i / \langle \gamma, \gamma \rangle_i]^{1/2} \tag{17a}$$

and $r_i(\mathcal{S})$ to denote the *spectral radius* of $\mathcal{S}$, which is defined by (see Appendix A)

$$r_i(\mathcal{S}) = \limsup_{k\to\infty} [\langle\langle \mathcal{S}^k \rangle\rangle_i]^{1/k} \tag{17b}$$

where $\mathcal{S}^k$ denotes the kth power of $\mathcal{S}$. Finally, let us introduce the linear operators

$$\bar{D}^i \triangleq D_{ij}^i D_{ji}^j \tag{18a}$$

and

$$\bar{P}_{i|i} \triangleq E^i[E^j[\cdot\,|y_j]y_i] \tag{18b}$$

both of which map Γ_i into itself (the former also maps U_i into itself). Then, the following proposition, whose proof depends on a contraction mapping argument (see Appendix B), provides a set of necessary and sufficient conditions for existence of the unique equilibrium solution alluded to in Proposition 2.

Theorem 1:

i) Under Conditions 1 and 2, the decision problem of Section II admits a *unique* stable Nash equilibrium solution given by (15) if, and only if, there exists, for at least one $i = 1, 2$, *a* ρ^i, $0 < \rho^i < 1$, such that

$$r_i(\mathcal{S}_i) = r_i(\bar{D}^i \bar{P}_{i|i}) \leq \rho^i. \tag{19}$$

ii) A set of sufficient conditions for (19) to hold true is the existence of a pair of positive scalars (ρ_1^i, ρ_2^i), such that

$$\rho_1^i \rho_2^i < 1, \; r_i(\bar{D}^i) \leq \rho_1^i, \; r_i(\bar{P}_{i|i}) \leq \rho_2^i. \tag{20a}$$

Furthermore, a set of sufficient conditions for the latter two is

$$\langle\langle \bar{D}^i \rangle\rangle_i \equiv \|\bar{D}^i\|_i \leq \rho_1^i, \; \langle\langle \bar{P}_{i|i} \rangle\rangle_i \leq \rho_2^i \tag{20b}$$

where $\|\cdot\|_i$ denotes the operator norm on U_i, as a counterpart of (17a).

Proof: See Appendix B. □

Part ii) of Theorem 1 provides a partial separation (in terms of sufficient conditions) of the deterministic and stochastic parts of the system. Now, if the decision problem is a *team* problem with a common loss functional (which requires $D_{22}^1 = I$, $D_{12}^1 = D_{21}^{2*}$, $F_1^1 = F_1^2$, and $F_2^1 = F_2^2$), and if team cost is strictly convex in the pair (u_1, u_2) (which is true if and only if $\|D_{12}^1 D_{12}^{1*}\|_1 = \bar{\rho} < 1$), it follows that the first inequality holds with $\rho_1^1 = \rho_1^2 < 1$. If, furthermore, the subjective probability measures assigned to the pair (y_1, y_2) by the two DM's are equivalent, $\bar{P}_{i|i}$ becomes the product of two projection operators, thus leading to satisfaction of the second inequality in (20b) with $\rho_2^1 = \rho_2^2 = 1$, and thereby to satisfaction of (20a). Hence, as a corollary to the second part of Proposition 3, we obtain the following result which is known in different contexts [7]–[9].[3]

Corollary 1: For the strictly convex quadratic team problem with equivalent subjective probability measures assigned by the two DM's to (y_1, y_2), there exists a unique stable equilibrium solution (the so-called *team-optimal* solution), irrespective of the underlying common probability measure. □

For team problems with $P^1 \not\equiv P^2$, a result along the lines of Corollary 1 does not in general hold because the operator $P_{i|i}$ is not necessarily the product of two projection operators. Then, the general condition is (19) [or the stronger one, (20a)] which places some restrictions on the parameters of the cost functional, as well as the probability measures P^1 and P^2. To delineate the extent of these restrictions, we now study the second inequality of (20b) somewhat further and obtain the following sufficient condition.

Corollary 2: For a given ρ_2^i, the second inequality of (20b) is satisfied if the expression

$$g^i(y_i) E^j[g^j(y_j) | y_i] = g^i(y^i) \int_{Y_j} g^j(\eta) P^j_{y_j|y_i}(d\eta | \xi = y_i) \tag{21a}$$

is uniformly bounded from above by $(\rho_2^i)^2$ a.e. $P^i_{y_i}$. Furthermore, if the probability measures P^1 and P^2 are absolutely continuous with respect to the Lebesgue measure, this condition can be expressed equivalently in terms of the probability densities $p^i(y_i, y_j)$ as follows:

$$\frac{p^j_{y_i}(y_i)}{p^i_{y_i}(y_i)} \int_{Y_j} [p^i_{y_j}(\eta) p^j_{y_j y_i}(\eta | y_i) / p^j_{y_j}(\eta)] \, d\eta \leq (\rho_2^i)^2. \tag{21b}$$

Proof: For (21a) see Appendix C; (21b) follows readily from (21a). □

[3] This result is slightly more general than the related ones that can be found in [7]–[9], since here P_x^1 is allowed to be different from P_x^2, although still a restriction is imposed on these (indirectly) via the equivalence between $P^1_{y_1 y_2}$ and $P^2_{y_1, y_2}$.

IV. General Sufficient Conditions for a Stackelberg Equilibrium Solution

We now turn our attention to the asymmetric mode of decision making, obtain some general sufficient conditions for existence of a Stackelberg equilibrium solution, and provide a complete characterization of the solution. Subsequently, we consider some special cases with some further structure imposed on the cost functionals and the probability measures.

First, we obtain an expression for DM2's unique reaction $T_2:\Gamma_1 \to \Gamma_2$, as defined by (10), using Proposition 1

$$T_2[\gamma_1](y_2) = \gamma_2^o(y_2) = D_{21}^2 E^2[\gamma_1(y_1) | y_2] + F_2^2 E^2[x | y_2]. \tag{22}$$

Hence, the derivation of the leader's Stackelberg policy $\gamma_1^s \in \Gamma_1$ involves [in view of (11)] the minimization of J_1 over Γ_1 after γ_2^o given by (22) is substituted in. This substitution yields

$$\begin{aligned}\tilde{J}(\gamma) \triangleq J_1(\gamma, \gamma_2^o) = {} & \frac{1}{2} \langle \gamma, \gamma \rangle_1 + \frac{1}{2} \int_{Y_2} (F_2^2 E^2[x | y_2] \\ & + D_{21}^2 E^2[\gamma(y_1) | y_2], \; D_{22}^1 D_{21}^2 E^2[\gamma(y_1 | y_2] \\ & + D_{22}^1 F_2^2 E^2[x | y_2])_2 P^1_{y_2}(d\xi) \\ & - \langle \gamma, E^1[F_1^1 x | y_1] \rangle_1 + \int_{X x Y_2} (D_{21}^2 E^2[\gamma(y_1) | y_2] \\ & + F_2^2 E^2[x | y_2], \; F_2^1 x)_2 \, P^1(dx, Y_1, d\xi) \\ & - \langle \gamma, E^1[D_{12}^1 D_{21}^2 E^2[\gamma(y_1) | y_2] | y_1] + E^1[D_{12}^1 F_2^2 E^2[x | y_2] | y_1] \rangle_1\end{aligned} \tag{23}$$

where we have deleted the subscript 1 in γ_1 in order to simplify the notation. Now, since Γ_1 is a linear space, and $\tilde{J}$ is the sum of terms homogeneous of degree zero, one, and two (maximum), any minimizing solution $\gamma \in \Gamma_1$ will have to satisfy

$$\Delta \tilde{J}(\gamma; h) = \tilde{J}(\gamma + h) - \tilde{J}(\gamma)) = \delta \tilde{J}(\gamma; h) + \delta^2 \tilde{J}(\gamma; h) \geq 0 \forall h \in \Gamma_1 \tag{24}$$

where $\delta^i \tilde{J}(\gamma; h)$ is the Gateaux variation of $\tilde{J}(\gamma)$ of degree i.[4] Extensive manipulations, details of which are given in Appendix D lead to the following expressions for $\delta \tilde{J}$ and $\delta^2 \tilde{J}$:

$$\begin{aligned}\delta \tilde{J}(\gamma; h) = {} & \langle h, \gamma \rangle_1 - \int_{Y_1} (h(y_1), (\mathcal{Z}\gamma)(y_1))_1 P^1_{y_1}(dy_1) \\ & - \int_{Y_1} (h(y_1), \beta(y_1))_1 P^1_{y_1}(dy_1)\end{aligned} \tag{25}$$

$$\begin{aligned}\delta^2 \tilde{J}(\gamma; h) = {} & \frac{1}{2} \langle h, h \rangle_1 + \frac{1}{2} \int_{Y_1} (h(y_1), g^1(y_1) E^2[g^2(y_2) D_{21}^{2*} D_{22}^1 D_{21}^2 \\ & \cdot E^2[h(\xi) | y_2] | y_1])_1 P^1_{y_1}(dy_1) - \langle h, D_{12}^1 D_{21}^2 \bar{P}_{1|1} h \rangle_1\end{aligned} \tag{26}$$

where $\mathcal{Z}:\Gamma_1 \to \Gamma_1$ and $\beta \in \Gamma_1$ are defined by

$$\begin{aligned}(\mathcal{Z}\gamma)(y_1) = {} & (D_{12}^1 D_{21}^2 \mathcal{P}_{1|1} + D_{21}^{2*} D_{12}^{1*} \mathcal{P}^*_{1|1}) \gamma(y_1) \\ & - D_{21}^{2*} D_{22}^1 D_{21}^2 g^1(y_1) E^2[g^2(y_2) E^2[\gamma(y_1) | y_2] | y_1]\end{aligned} \tag{27a}$$

$$\begin{aligned}\beta(y_1) = {} & F_1^1 E[x | y_1] - D_{21}^{2*} D_{22}^1 F_2^2 g^1(y_1) E^2[g^2(y_2) E^2[x | y_2] | y_1] \\ & - D_{12}^1 F_2^2 E^1[E^2[x | y_2] | y_1] \\ & + D_{21}^{2*} F_2^1 g^1(y_1) E^2[g^2(y_2) E^1[x | y_2] | y_1].\end{aligned} \tag{27b}$$

$\mathcal{P}_{1|1}:\mathcal{U}_1 \to \mathcal{U}_1$ is a linear operator given by

$$\mathcal{P}_{1|1} \gamma(y_1) = E^1[E^2[\gamma(y^1) | y_2] | y_1], \tag{28}$$

$\mathcal{U}_1$ is the space of y_1-measurable random variables taking values in U_1, and $g^i(\xi)$ are the R-N derivatives (2). Note that $\mathcal{P}_{1|1}$ is

[4] Here $\delta^1 \tilde{J}$ is written simply as $\delta \tilde{J}$.

related to $\bar{P}_{1|1}$ defined by (18b) by

$$\mathcal{P}_{1|1}[\gamma(y_1)] = (\bar{P}_{1|1}\gamma)(y_1)$$

where the latter (which is a mapping from Γ_1 into Γ_1) has been used in (26) and will also be used in the sequel whenever needed.

Now, since (24) is also equivalent to

$$\begin{aligned} \delta\tilde{J}(\gamma, h) &= 0 \qquad \forall h \in \Gamma_1 \\ \delta^2\tilde{J}(\gamma, h) &\geq 0 \qquad \forall h \in \Gamma_1 \end{aligned} \tag{29}$$

a Stackelberg solution $\gamma \in \Gamma_1$ will exist for the leader if, and only if,

i) (26) is nonnegative definite,

and [from 25)]

ii) $\gamma(y_1) - (\mathcal{Z}\gamma)(y_1) - \beta(y_1) = 0$, a.e. $P^1_{y_1}$. (30)

Since the first of these conditions does not depend on γ, the optimal solution is solely determined by (30), which can be rewritten as

$$\begin{aligned} \gamma(y_1) = {} & D^1_{12}D^2_{21}E^1[E^2[\gamma(y_1)|y_2]|y_1] \\ & + D^{2*}_{21}D^{1*}_{12}g^1(y_1)E^2[g^2(y_2)E^1[\gamma(y_1)|y_2]|y_1] \\ & + D^{2*}_{21}F^1_2 g^1(y_1)E^2[g^2(y_2)E^1[x|y_2]|y_1] \\ & - D^{2*}_{21}D^1_{22}D^2_{21}g^1(y_1)E^2[g^2(y_2)E^2[\gamma(y_1)|y_2]|y_1] \\ & + F^1_1 E^1[x|y_1] - D^{2*}_{21}D^1_{22}F^2_2 g^1(y_1)E^2[g^2(y_2)E^2[x|y_2]y_1] \\ & + D^1_{12}F^2_2 E^1[E^2[x|y_2]|y_1] \end{aligned} \tag{31}$$

where we have utilized the fact that the adjoint of $\mathcal{P}_{1|1}$ is a linear operator $\mathcal{P}^*_{1|1}: \mathcal{U}_1 \to \mathcal{U}_1$, given by (see Appendix D)

$$\begin{aligned} \mathcal{P}^*_{1|1}\gamma(y_1) &= \int_{Y_1} \gamma(\eta) \int_{Y_2} \frac{P^1_{y_1y_2}(d\eta \times dy_2)P^2_{y_1y_2}(dy_1 \times dy_2)}{P^2_{y_2}(dy_2)P^1_{y_1}(dy_1)} \\ &= g^1(y_1)E^2[g^2(y_2)E^1[\gamma(y_1)|y_2)]|y_1]. \end{aligned} \tag{32}$$

Furthermore, condition i) can be rewritten as

$$\mathfrak{A} \triangleq I + \frac{1}{2} D^{2*}_{21}D^1_{22}D^2_{21}(\mathcal{K} + \mathcal{K}^*) - D^1_{12}D^2_{21}\bar{P}_{1|1} - D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1} \geq 0 \tag{33}$$

where $I : \Gamma_1 \to \Gamma_1$ is the identity operator, and $\mathcal{K}: \Gamma_1 \to \Gamma_1$ is defined by

$$(\mathcal{K}\gamma)(y_1) = g^1(y_1)E^2[g^2(y_2)E^2[\gamma(y_1)|y_2]|y_1]. \tag{34}$$

We now summarize these results in the following proposition.

Proposition 3: Under Conditions 1) and 2), the decision problem with multiple probability measures admits a Stackelberg equilibrium solution if, and only if, $\mathfrak{A}$ is nonnegative definite and (31) admits a solution in Γ_1. □

Equation (31) will, in general, not admit a closed-form solution, even if all random variables are jointly Gaussian distributed (see Section V-C); therefore, we will have to resort to numerical computations which will involve a recursion of some type. Hence, in analyzing the conditions of existence of a solution to (31) we may also require that such a numerical scheme be globally convergent (or stable). One appealing scheme whereby a unique solution to (31) [or equivalently, (30)] can be obtained is the recursion

$$\gamma^{(k)}(y_1) = (\mathcal{Z}\gamma^{(k-1)})(y_1) + \beta(y_1), \qquad k = 1, 2, \cdots \tag{35}$$

where $\gamma^{(0)}$ is chosen as an arbitrary element of Γ_1. If the limit $\lim_{k\to\infty} \gamma^{(k)} \triangleq \gamma^s$ exists in Γ_1, for all such initial choices, then γ^s will necessarily constitute a solution to (31). A sufficient condition for this readily follows from Lemma B.1, which we give below as Proposition 4.

Proposition 4: In addition to the conditions of Proposition 3, assume that there exists a scalar ρ, $0 < \rho < 1$, such that

$$r(\mathcal{Z}) \leq \rho \tag{36}$$

where $r(\mathcal{Z})$ is the spectral radius of $\mathcal{Z}$. Then, the decision problem admits a unique Stackelberg equilibrium solution $(\gamma^s, T_2[\gamma^s])$, where $\gamma^s \in \Gamma_1$ is the limit of the iterative scheme (35), and T_2 is the affine operator (22). □

We now further elaborate on (36), so as to bring it to a form which separates out the contributions from the deterministic and probabilistic components of the problem. (Here, we are seeking sufficient conditions which would constitute the counterpart of (20) in this context). Towards this end, let us first note that using (34) in (25a):

$$r(\mathcal{Z}) = r(D^1_{12}D^2_{21}\bar{P}_{1|1} + D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1} - D^{2*}_{21}D^1_{22}D^2_{21}\mathcal{K}) \tag{37}$$

and utilizing the inequality relationship between the spectral radius and norm of an operator (see Appendix A, Lemma A.1) this can be bounded from above by

$$\leq \langle\langle D^1_{12}D^2_{21}\bar{P}_{1|1} + D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1} - D^{2*}_{21}D^1_{22}D^2_{21}\mathcal{K}\rangle\rangle_1$$

where $\langle\langle\cdot\rangle\rangle_1$ is the operator norm as defined in (17a). Using the standard (triangle inequality) property of norms, this can further be bounded from above by

$$\leq \langle\langle D^1_{12}D^2_{21}\bar{P}_{1|1} + D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1}\rangle\rangle_1 + \langle\langle D^{2*}_{21}D^1_{22}D^2_{21}\mathcal{K}\rangle\rangle_1.$$

Now since both $D^{2*}_{21}D^1_{22}D^2_{21}$ and $\mathcal{K}$ map a Hilbert space (Γ_1) into itself, using the norm inequality for products of linear operators, we further have

$$\begin{aligned} &\leq \langle\langle D^1_{12}D^2_{21}\bar{P}_{1|1} + D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1}\rangle\rangle_1 + \langle\langle D^{2*}_{21}D^1_{22}D^2_{21}\rangle\rangle_1\langle\langle\mathcal{K}\rangle\rangle_1 \\ &= r(D^1_{12}D^2_{21}\bar{P}_{1|1} + D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1}) + r(D^{2*}_{21}D^1_{22}D^2_{21})[r(\mathcal{K}^*\mathcal{K})]^{1/2} \end{aligned}$$

where the equality follows because i) the spectral radius and norm of a self-adjoint linear operator are equal [13, p. 514], ii) norm of a "nonself-adjoint" linear operator $\mathcal{K}$ is equal to the square root of the spectral radius of the self-adjoint operator $\mathcal{K}^*\mathcal{K}$ (see Appendix A, Lemma A.1). Finally, using the result of Lemma A.2 (Appendix A), the latter is bounded from above by

$$\begin{aligned} r(\mathcal{Z}) \leq {} & 2[(D^1_{12}D^2_{21}D^{2*}_{21}D^{1*}_{12})]^{1/2}[r(\bar{P}^*_{1|1}\bar{P}_{1|1}]^{1/2} \\ & + r(D^{2*}_{21}D^1_{22}D^2_{21})[r(\mathcal{K}^*\mathcal{K})]^{1/2}. \end{aligned} \tag{38}$$

Now, let us assume the following.

Condition 3: There exist four positive scalars ρ_1, ρ_2, ρ_3, ρ_4, satisfying

$$2\rho_1\rho_2 + \rho_3\rho_4 < 1 \tag{39}$$

such that

$$r(D^1_{12}D^2_{21}D^{2*}_{21}D^{1*}_{12}) \leq (\rho_1)^2, \quad r(D^{2*}_{21}D^1_{22}D^2_{21}) \leq \rho_3 \tag{40a}$$

$$r(\bar{P}^*_{1|1}\bar{P}_{1|1}) \leq (\rho_2)^2, \quad r(\mathcal{K}^*\mathcal{K}) \leq (\rho_4)^2. \tag{40b}$$

□

Then, we have the following.

Theorem 2: Under Conditions 1 and 2 of Section II and Condition 3 given above, the decision problem admits a *unique Stackelberg equilibrium solution* $(\gamma^s, T_2[\gamma^s])$, where $\gamma^s \in \Gamma_1$ is the limit of the iterative scheme (35), and T_2 is given by (22).

Proof: The result follows from Proposition 4 and the discussion and derivation that leads to Condition 3, provided we show that the given three conditions subsume (33), i.e., nonnegativity of operator $\mathcal{A}$. We now verify that Condition 3 in fact implies that $\mathcal{A}$ is a strongly positive operator. First note that $\mathcal{A}$ is self-adjoint because $\mathcal{K}$ commutes with $D_{21}^{2*}D_{22}^{1}D_{21}^{2}$. Hence, using Lemma A.3 (Appendix A), we can write down the inequality

$$r(\mathcal{A}-\mathcal{I})\leq\frac{1}{2}\,r(D_{21}^{2*}D_{22}^{1}D_{21}^{2}(\mathcal{K}+\mathcal{K}^{*}))+r(D_{12}^{1}D_{21}^{2}\bar{P}_{1|1}$$
$$+D_{21}^{2*}D_{12}^{1*}\bar{P}_{1|1}^{*}).$$

Then, using the line of arguments that led to (38) from (37), and the spectral radius inequality for the product of two self-adjoint operators, we obtain the bound

$$\begin{aligned}r(\mathcal{A}-\mathcal{I})&\leq\frac{1}{2}\,r(D_{21}^{2*}D_{22}^{1}D_{21}^{2})r(\mathcal{K}+\mathcal{K}^{*})\\&\quad+r[(D_{12}^{1}D_{21}^{2}D_{21}^{2*}D_{12}^{1*})]^{1/2}[r(\bar{P}_{1|1}^{*}\bar{P}_{1|1})]^{1/2}\\&\leq\frac{1}{2}\,\rho_3 r(\mathcal{K}+\mathcal{K}^{*})+\rho_1\rho_2.\end{aligned}$$

But note that

$$\begin{aligned}r(\mathcal{K}+\mathcal{K}^{*})&=\sup_{\gamma\in\Gamma_1}\;[\langle\gamma,\;(\mathcal{K}+\mathcal{K}^{*})\gamma\rangle_1|\langle\gamma,\;\gamma\rangle_1]\\&=2\sup_{\gamma\in\Gamma_1}\;[\langle\gamma,\;\mathcal{K}\gamma\rangle_1|\langle\gamma,\;\gamma\rangle_1]\end{aligned}$$

and since, from the Cauchy–Schwartz inequality of inner products

$$|\langle\gamma,\;\mathcal{K}\gamma\rangle_1|^2\leq|\langle\gamma,\;\gamma\rangle_1||\langle\mathcal{K}\gamma,\;\mathcal{K}\gamma\rangle_1|$$

we have

$$\begin{aligned}r(\mathcal{K}+\mathcal{K}^{*})&\leq2\sup_{\gamma\in\Gamma_1}\;[\langle\mathcal{K}\gamma,\;\mathcal{K}\gamma\rangle_1|\langle\gamma,\;\gamma\rangle_1]^{1/2}\\&=2\sup_{\gamma\in\Gamma_1}\;[\langle\gamma,\;\mathcal{K}^{*}\mathcal{K}\rangle_1|\langle\gamma,\;\gamma\rangle_1]^{1/2}=2[r(\mathcal{K}^{*}\mathcal{K})]^{1/2}\leq2\rho_4.\end{aligned}$$

Thus, $r(\mathcal{A}-\mathcal{I})\leq\rho_3\rho_4+\rho_1\rho_2<1$, implying that the spectrum of the self-adjoint operator $\mathcal{A}-\mathcal{I}$ is uniformly in the unit sphere. Hence, $\mathcal{A}$ is strongly positive. □

For the special class of strictly convex team problems (cf. Section III) with multiple probability measures, several simplifications can be made. In this case (31) simplifies to

$$\begin{aligned}\gamma(y_1)&=D_{12}^{1}D_{12}^{1*}\{E^1[E^2[\gamma(y_1)|y_2]\\&\quad+g^1(y_1)E^2[g^2(y_2)\{E^1[\gamma(y_1)|y_2]\\&\quad-E^2[\gamma(y_1)|y_2]\}|y_1]\}+F[E^1[x|y_1]\\&\quad+D_{12}^{1}F_2^{1}g^1(y_1)E^2[g^2(y_2)\{E^1[x|y_2]\\&\quad-E^2[x|y_2]\}|y_1]+D_{12}^{1}F_2^{1}E^1[E^2[x|y_2]|y_1]\end{aligned}\tag{41}$$

and in Condition 3 inequalities (40a) are replaced by the single inequality

$$[r(D_{12}^{1}D_{12}^{1*}D_{12}^{1}D_{12}^{1*})]^{1/2}=\langle\langle D_{12}^{1}D_{12}^{1*}\rangle\rangle_1=\bar{\rho}\leq\rho_1=\rho_3$$

where ρ_1 can be taken to be less than one. Hence, (39) reads

$$(2\rho_2+\rho_4)<1/\bar{\rho}.\tag{42}$$

We now summarize these results as a corollary to Theorem 2.

Corollary 3: Under Conditions 1 and 2 of Section II and (42) given above, the strictly convex quadratic team problem, with multiple probability measures and asymmetric mode of decision making, admits a *unique Stackelberg* equilibrium solution $(\gamma^s, T_2[\gamma^s])$, where $\gamma^s\in\Gamma_1$ is the limit of the iterative scheme (35) with

$$\begin{aligned}(\mathcal{Z}\gamma)(y_1)&=D_{12}^{1}D_{12}^{1*}[(\mathcal{P}_{1|1}+\mathcal{P}_{1|1}^{*})\gamma(y_1)\\&\quad-g^1(y_1)E^2[g^2(y_2)E^2[\gamma(y_1)|y_2]|y_1]]\end{aligned}$$

and T_2 is given by (22). □

Remark 3: When the original problem is a Stackelberg game, but the probability measures are identical, a study of the original condition (36) reveals the inequality

$$r(\mathcal{Z})\leq r(D_{12}^{1}D_{21}^{2}+D_{12}^{2*}D_{12}^{1*}-D_{21}^{2*}D_{22}^{1}D_{21}^{2})\leq\rho<1.$$

This is the existence condition associated with the standard stochastic Stackelberg game, which corroborates the earlier result obtained in [25]. □

We now conclude this section by presenting the counterpart of Corollary 2 in the present context, which provides a set of (simpler) sufficient conditions for (40b) to be satisfied.

Corollary 4: For a given pair (ρ_2,ρ_4), the first and second inequalities of (40b) are satisfied if, respectively,

$$g^1(y_1)\int_{Y_2}g^2(\eta)P_{y_2|y_1}^2\;(d\eta|\xi=y_1)\tag{43a}$$

and

$$g^1(y_1)\int_{Y_2}|g^2(\eta)|^2P_{y_2|y_1}^2\;(d\eta|\xi=y_1)\int_{Y_1}g^1(b)P_{y_1|y_2}^2\;(db|y_2=\eta)\tag{43b}$$

are uniformly bounded from above by $(\rho_2)^2$ and $(\rho_4)^2$.

Furthermore, if probability densities exist (with respect to the Lebesgue measure), these conditions can be expressed in terms of the corresponding probability density functions $p_{y_1y_2}^i(\cdot)$ as follows:

$$\frac{p_{y_1}^2(y_1)}{p_{y_1}^1(y_1)}\int_{Y_2}\frac{p_{y_2}^1(\eta)}{p_{y_2}^2(\eta)}\,p_{y_2|y_1}^2(\eta|y_1)d\eta\leq(\rho_2)^2\tag{44a}$$

$$\begin{aligned}&\frac{p_{y_1}^2(y_1)}{p_{y_1}^1(y_1)}\int_{Y_2}\left|\frac{p_{y_2}^1(\eta)}{p_{y_2}^2(\eta)}\right|^2p_{y_2|y_1}^2(\eta|y_1)\;d\eta\\&\quad\cdot\int_{Y_1}\frac{p_{y_1}^2(b)}{p_{y_1}^1(b)}\,p_{y_1|y_2}^2(b|\eta)db\leq(\rho_4)^2.\end{aligned}\tag{44b}$$

Proof: For (43a), (43b) see Appendix C; (44a), (44b), however, follow readily from (43a), (43b). □

V. Jointly Gaussian Distributions

In decision and control theory, one appealing class of probability distributions is the Gaussian distribution because it leads to tractable problems admitting, in most cases, closed-form solutions. Indeed when the probability measures of the two DM's are identical and Gaussian, equilibrium solutions have been shown to be affine functions of the observations for 1) quadratic stochastic team problems defined on Euclidean spaces [7], 2) quadratic stochastic Nash games on Euclidean spaces [8], 3) quadratic continuous-time stochastic team problems [9], 4) quadratic stochastic Stackelberg games on Euclidean spaces [25], and 5) quadratic continuous-time stochastic Stackelberg games [26]. In this section, we investigate possible extensions of this appealing structural feature to the case when discrepancies exist between the subjective Gaussian distributions, as reflected in the covariances of the random vectors (y_1, y_2). We could also have included discrepancies in the perceptions of the mean values, but such a more general treatment does not contribute substantially to the qualitative nature of the results obtained in the sequel, and besides it makes the expressions notationally cumbersome. Interested readers may find relevant expressions for the nonzero mean case in [27].

We first introduce notation and terminology, and delineate Conditions 1 and 2 of Section II (Section V-A). Then, we study the case of symmetric mode of decision making in Section V-B and show that the unique equilibrium solution of Theorem 1 is linear. Finally, in Section V-C we treat the case of asymmetric mode of decision making, and show that (in contradistinction with the result of Section V-B) the unique Stackelberg solution of Theorem 2 is generically nonlinear.

A. Notation and Terminology

Let (x, y_1, y_2) be zero-mean Gaussian random vectors under both P^1 and P^2, with

$$\text{covariance } (y_1, y_2) = \Sigma_y^i = \begin{pmatrix} \Sigma_{y_1}^i & \Sigma_{y_1y_2}^i \\ \Sigma_{y_2y_1}^i & \Sigma_{y_2}^i \end{pmatrix} > 0 \text{ under } P^i, \tag{45a}$$

$$\text{covariance } (x, y_1, y_2) = \text{cov } (x, y) = \Sigma^i = \begin{pmatrix} \Sigma_x^i & \Sigma_{xy}^i \\ \Sigma_{yx}^i & \Sigma_y^i \end{pmatrix} > 0 \text{ under } P^i. \tag{45b}$$

These probability distributions clearly satisfy the absolute continuity condition (Condition 1) of Theorems 1 and 2. Furthermore, since

$$g^i(\xi) = (\det \Sigma_{y_i}^i / \det \Sigma_{y_i}^j) \exp \left\{ -\frac{1}{2} \xi' W_i \xi \right\} \tag{46a}$$

$$W_i \triangleq \Sigma_{y_i}^{j\,-1} - \Sigma_{y_i}^{i\,-1}, \qquad j \neq i \tag{46b}$$

the uniform boundedness condition (Condition 2) of Theorems 1 and 2 is satisfied whenever

$$W_i \geq 0, \qquad i = 1, 2. \tag{47}$$

After making these observations, let us introduce the additional notation

$$N_i \triangleq M_{ii}^i - M_{ij}^i B_j^{-1} M_{ji}^i - \Sigma_{y_i}^{i\,-1}, \qquad j \neq i \tag{48a}$$

$$B_j = M_{jj}^i + W_j \tag{48b}$$

$$\begin{pmatrix} M_{11}^i & M_{12}^i \\ M_{21}^i & M_{22}^i \end{pmatrix} \triangleq \begin{pmatrix} \Sigma_{y_1}^i & \Sigma_{y_1y_2}^i \\ \Sigma_{y_2y_1}^i & \Sigma_{y_2}^i \end{pmatrix}^{-1} \tag{48c}$$

$$q^i \triangleq [\det \Sigma_{y_i}^i \cdot \det \Sigma_{y_j}^j / \det \Sigma_{y_j}^i \cdot \det B_j \cdot \det \Sigma_y^i]^{1/2} \tag{48d}$$

in terms of which we evaluate (21a) (using standard properties of Gaussian distributions) to be

$$g^i(y_i) E^i[g^j(y_j) | (y_i] = q^i \exp \left\{ -\frac{1}{2} y_i' D_i y_i \right\}. \tag{49}$$

We are now in a position to specialize the results of Theorems 1 and 2 to Gaussian distributions and obtain some explicit results.

B. Symmetric Mode of Decision Making

In order to apply Theorem 1 to the Gaussian decision problem formulated above, we first explore the satisfaction of various conditions given there. We have already shown above that Condition 1 is always satisfied and Condition 2 is satisfied whenever $W_i \geq 0$. For the remaining condition we study inequalities (20b). The second of these is satisfied, for a given ρ_2^i, if [using (21a)] expression (49) is uniformly bounded in y_i, and this bound is no greater than ρ_2^i. For uniform boundedness of (49) it is necessary and sufficient that

$$N_i \geq 0 \tag{50a}$$

under which the latter condition becomes

$$q^i \leq (\rho_2^i)^2.$$

Hence, going back to (20a), the condition

$$\|D_{ij}^i D_{ji}^j\|_i^2 < 1/q^i, \qquad \text{for at least one } i = 1, 2 \tag{50b}$$

becomes sufficient for (19). We are now in a position to state and prove the following theorem.

Theorem 3: Let (47) hold for $i = 1, 2$, and (50a), (50b) hold for at least one i. Then, the quadratic Gaussian decision problem formulated in this section admits a *unique stable Nash equilibrium solution* (γ_1^o, γ_2^o), where $u_i^o = \gamma_i^o(y_i)$ are linear in y_i, and are given by

$$\gamma_i^o(y_i) = L_i y_i \qquad i = 1, 2. \tag{51}$$

Here, $L_i: \mathbb{R}^{m_i} \to U^i$ are bounded linear operators, constituting the unique solution to the linear operator equations

$$\begin{aligned} &L_i y_i - D_{ij}^i D_{ji}^j L_i \Sigma_{y_i y_j}^j \Sigma_{y_j}^{i\,-1} \Sigma_{y_j y_i}^i \Sigma_{y_i}^{i\,-1} y_i \\ &\quad - D_{ij}^i F_j^i \Sigma_{xy_j}^j \Sigma_{y_j}^{j\,-1} \Sigma_{y_j y_i}^i \Sigma_{y_i}^{i\,-1} y_i \\ &\quad - F_i^i \Sigma_{xy_i}^i \Sigma_{y_i}^{i\,-1} y_i = 0, \ \forall y_i \in \mathbb{R}^{m_i}, \qquad i = 1, 2. \end{aligned} \tag{52}$$

Proof: The existence and uniqueness of the solution follows from Theorem 1, Corollary 2, and the discussion that precedes the statement of the theorem. The linearity of this unique solution, on the other hand, follows by noting that if the pair $(\gamma_1^{(o)}, \gamma_2^{(o)})$ is taken to be linear in (y_1, y_2) in (14), all the terms of the sequence are linear, and hence the limit (which exists as already proven) is linear. Hence, choosing γ_i as in (51), where $L_i: \mathbb{R}^{m_i} \to U_i$ are bounded linear operators, substituting this into (14) and requiring it to hold for all $y_i \in \mathbb{R}^{m_i}$ (since all probability measures are Gaussian), leads to the unique relations (52). □

Remark 4: Theorem 3 above extends the result of [8, Theorem 2] on quadratic Gaussian games to the case when a common probability space does not exist and the decision spaces are not necessarily finite dimensional, and shows that the appealing linear structure prevails when there exists a discrepancy in the perceptions of the two DM's of the underlying probability measures. The existence and uniqueness conditions here are, however, more restrictive than those of [8], and also involve the probabilistic structure [see (50b)]. Expression (21a) in the most general case (and (49) for the special Gaussian case) is not uniformly (in y^i) bounded by 1, unless $g^i(y_i) = g^j(y_j) = 1$ a.e. $P_{y_i}^i$ and $P_{y_j}^j$ (which corresponds to the case of equivalent probability measures), since R-N derivatives (if different from 1) will be both smaller and larger than unity on sets of nonzero measure. This then implies, in view of (47), and from (49), that $q^i \geq 1$, $i = 1, 2$, with the inequality being strict if $P_{y_i}^i$ is not equivalent to $P_{y_i}^j$ for at least one $i = 1, 2, j \neq i$. In such a case, even in *team* problems, a stable equilibrium solution may not exist, particularly if $1/q_i < \|D_{ij}^i D_{ji}^j\|_i^2 < 1$ for at least one $i = 1, 2; j \neq i$. This indicates, in general, the presence of a strong coupling between probabilistic and deterministic elements of the problem in terms of existence conditions. However, if the discrepancy between perceptions of the DM's on the probability measures (measured in terms of R-N derivatives) is sufficiently small, one would expect q_i to be sufficiently close to unity, which ensures satisfaction of condition (50c) for a fairly general class of quadratic strictly convex Gaussian team problems (since, $\|D_{ij}^i D_{ji}^j\|_i = \|D_{ji}^j D_{ij}^i\|_j = \bar{\rho} < 1$, for such team problems). For further discussion on this point we refer the reader to [10]. □

In the statement of Theorem 1, the condition (47) places some severe restrictions on the second moments of the underlying distributions (in case a discrepancy exists), which may however be relaxed if we are willing to consider equilibrium policies in a more restricted space. More specifically, satisfaction of (47) ensures that regardless of what initial set of policies the DM's start the infinite recursion (15) with, every element of this series is well defined, and under (50a), (50b) it will converge to a unique limit which is linear; in other words, even if the DM's start with nonlinear policies, the end result will be a linear equilibrium solution. The condition (47) is restrictive because we require (without imposing any constraints on the policy spaces) the series generated by (15) to be well defined even with nonlinear starting conditions. However, if we restrict the team agents to linear policies from the outset, under Gaussian distributions (and following the argument used in the proof of Theorem 1) elements of the series (15) will be well defined [without requiring (47)] and will converge to the equilibrium solution provided that (50a), (50b) hold for at least one $i = 1, 2$. This line of reasoning then leads to the following result which we give without a proof.

Proposition 5: Let Γ_i^a be the class of all linear policies in the form (51), with $L_i : \mathbb{R}^{m_i} \to U^i$ a bounded linear operator, $i = 1,2$. On $\Gamma_1^a \times T_2^a$, the statement of Theorem 1 is valid even if (47) does not hold true. □

We now interpret these results in the context of two examples one of which is a scalar team problem and the other one is a continuous-time team problem, both with multiple subjective Gaussian probabilities.

Example 1: Consider a family of scalar Gaussian *team* problems, with $D_{22}^1 = D_{11}^2 = 1$, $D_{12}^1 = D_{21}^2 = d$, $|d| < 1$, $F_1^1 = f_1$, $F_2^2 = f_2$, $n = m_1 = m_2 = 1$, and

$$\Sigma_y^1 = \begin{pmatrix} a & e \\ e & \eta b \end{pmatrix};\ \Sigma_y^2 = \begin{pmatrix} \mu a & c \\ c & b \end{pmatrix}, \qquad \eta ab > e^2, \quad \mu ab > c^2. \quad (53)$$

To investigate the applicability of Theorem 1 to this class of problems, let us first observe that condition (47) is satisfied if, and only if, both

$$0 < \mu \le 1, \quad 0 < \eta \le 1. \quad (54)$$

For condition (50a), we evaluate N_i and require it to be nonnegative for either $i = 1$, or $i = 2$:

$$N_1 = (\mu ab - c^2)(1-\mu)[\mu a^2 - c^2 - (1-\mu)ab]/\{a[\mu^2 a^2 - (1-\mu)(\mu ab - c^2)]\} \ge 0 \quad (55a)$$

or

$$N_2 = (\eta ab - e^2)(1-\eta)[\eta b^2 - e^2 - (1-\eta)ab]/\{b[\eta^2 b^2 - (1-\eta)(\eta ab - e^2)]\} \ge 0. \quad (55b)$$

Finally, condition (50b) dictates *either*

$$\mu a^2 |d|^2 < \eta[\mu^2 a^2 - (1-\mu)(\mu ab - c^2)] \quad (56a)$$

or

$$\eta b^2 |d|^2 < \mu[\eta^2 b^2 - (1-\eta)(\eta ab - e^2)] \quad (56b)$$

provided that the terms on the right-hand side are positive (if not, then the inequalities will accordingly change direction).

The set of values for a, b, c, e, μ, η, that satisfy (54)–(56) is clearly not empty. To gain some further insight into these conditions, let us consider the class of team decision problems in which the discrepancies between the DM's perceptions of the variances of different Gaussian random variables is relatively small; that is, there exist sufficiently small $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\mu = 1 - \epsilon_1$, $\eta = 1 - \epsilon_2$, and furthermore $e \sim c$, and $|c|$ is considerably smaller than both a and b. Note that when $\epsilon_1 = \epsilon_2 = 0$, conditions (54), (56) are all satisfied (note that $|d| < 1$ because of strict convexity of the objective functional) regardless of the relative magnitudes of e and c. Hence, when the discrepancy is only in the perceptions of the correlation between y_1 and y_2, the scalar quadratic Gaussian team problem always admits a stable equilibrium solution. Now, for nonzero, but positive, and sufficiently small ϵ_1, the dominating term in (55a) is

$$N_1 \sim \epsilon_1(\mu ab - c^2)(\mu a^2 - c^2)/\mu^2 a^3$$

which is positive, in view of (53) and the initial hypothesis that $|a/c| \gg 1$. Likewise, D_2 is positive whenever $0 < \epsilon_2 \ll 1$ and $|b/e| \gg 1$. Furthermore, given $a\bar{d}$, $0 < \bar{d} < 1$, we can always find ϵ_1 and ϵ_2, both in (0, 1), so that both (56a) and (56b) are satisfied whenever $|d| < \bar{d}$. Hence, the conclusion is that when the deviations of the perceptions of the DM's from the common Gaussian probability measures are incremental [and satisfying (54)], the linear equilibrium solution of the Gaussian scalar team problem retains its stability property (but, of course, at a different (possibly close, in norm) equilibrium point). □

Example 2: As a second illustration of Theorem 1, for infinite-dimensional decision spaces, we consider here a class of stochastic Gaussian team problems defined in continuous time. More specifically, let $U_1 = U_2 = \mathcal{L}_2(0, T)$, the Hilbert space of all scalar-valued Lebesgue-integrable functions on the bounded interval $[0, T]$, endowed with the standard inner product $\int_0^T u(t)v(t)dt$, for $u, v \in \mathcal{L}_2$. Furthermore, let Y_1 and $Y_2 = \mathbb{R}$, and the Gaussian statistics have zero mean, and variances be as given in (53). Let $D_{11}^2 = D_{22}^1 = I$, the identity operator on $\mathcal{L}_2$, and $D_{12}^1 = D_{21}^{2*}$ be the Fredholm operator

$$D_{12}^1 u = \int_0^T K(t, s)u(s)ds \quad (57)$$

where $K(t, s)$ is a continuous kernel on $0 \le t, s \le T$, and finally let $F_i^i = f_i(t)$, $i = 1, 2$, which are continuous functions on $[0, T]$.

Now, conditions (47a) and (50a) depend only on the probabilistic structure, and are therefore again given by (54) and (55), respectively. For (50b), however, we have to obtain the counterpart of (56), by simply replacing $|d|^2$ with the norm of the operators $D_{12}^1 D_{12}^{1*}$ and $D_{12}^{1*} D_{12}^1$, respectively. Since $D_{12}^{1*} u = \int_0^T K(s, t)u(s)ds$, the self-adjoint operator $D_{12}^1 D_{12}^{1*}$ is given by

$$D_{12}^1 D_{12}^{1*} u = \int_0^T \int_0^T K(t, \tau)K(s, \tau)u(s)ds\, d\tau = \int_0^T \mathcal{K}(t, s)u(s)\, ds,$$

where

$$\mathcal{K}(t, s) \triangleq \int_0^T K(t, \tau)K(s, \tau)\, d\tau. \quad (58a)$$

Let

$$\lambda = \left\{ \int_0^T \int_0^T |\mathcal{K}(t, s)|^2 dt\, ds \right\}^{1/2}. \quad (58b)$$

Then, $\|D_{12}^1 D_{12}^{1*} u\|_1^2 \triangleq \int_0^T |\int_0^T \mathcal{K}(t, s)u(s)ds|^2 dt \le \int_0^T [\int_0^T |\mathcal{K}(t, s)|^2 ds][\int_0^T |u(s)|^2 ds]dt = \lambda^2 \|u\|^2$ where the second step follows from the Cauchy–Schwarz inequality. Hence,

$$\|D_{12}^1 D_{12}^{1*}\|_1 \le \lambda$$

and because of symmetry $D_{12}^{1*} D_{12}^1$ is also bounded in norm by the same quantity. This then leads to the following counterpart of (56). A sufficient condition for satisfaction of (50b) is *either*

$$\mu a^2 \lambda < \eta[\mu^2 a^2 - (1-\mu)(\mu ab - c^2)] \quad (59a)$$

or

$$\eta b^2 \lambda < \mu[\eta^2 b^2 - (1-\eta)(\eta ab - e^2)] \quad (59b)$$

provided that the terms on the right-hand side are positive, where λ is defined by (58a), (58b).

Hence, under (54) *and* either (55a) and (59a) or (55b) and (59b), the continuous-time static decision problem formulated above admits a unique stable equilibrium solution, and this solution is given by (from Theorem 3)

$$\gamma_i^o(t, y_i) = k_i(t) y_i, \qquad i = 1, 2 \tag{60}$$

where $k_i(t)$ are continuous functions on $[0, T]$, satisfying

$$k_1(t) - \left(\frac{ec}{ab}\right)\int_0^T K(t, s) k_1(s)\, ds - (\sigma_{xy_2}^2 e/ab)\int_0^T K(t, s) f_1(s)\, ds - (\sigma_{xy_1}^1/a) f_1(t) = 0 \tag{61a}$$

$$k_2(t) - \left(\frac{ec}{ab}\right)\int_0^T K(s, t) k_2(s)\, ds - (\sigma_{xy_1}^1 c/ab)\int_0^T K(s, t) f_2(s)\, ds - (\sigma_{xy_2}^2/b) f_2(t) = 0. \tag{61b}$$

Note that $k_i(t)$ above stands for operator L_i in (52), and we have already shown that a unique solution to both (61a) and (61b) exists in $\mathcal{L}_2\, [0, T]$, under (54) and either (55a) and (59a) or (55b) and (59b), and this solution is also continuous.

Finally, if our interest lies only in the existence of a unique linear equilibrium solution (not necessarily stable), the required condition is the unique solvability of the integral equations (61a), (61b), for which a sufficient condition is [6]

$$(ec/ab)\lambda < 1$$

where λ is defined by (58b). □

C. Asymmetric Mode of Decision Making

To obtain the counterpart of the results of Section V-B under the asymmetric mode of decision making, we first investigate the possibility for the unique solution of Theorem 2 to be linear. Towards this end we first observe that the decision problem will admit a unique linear solution if, and only if, (31) is satisfied by the decision rule

$$\gamma(y_1) = A y_1 \tag{62}$$

for some linear bounded operator $A: \mathbb{R}^{m_1} \to U_1$. Hence, using (31), A should be the solution of (by pulling A out of the conditional expectations)

$$\begin{aligned} A y_1 = {} & D_{12}^1 D_{21}^2 A E^1[E^2[y_1|y_2]|y_1] \\ & + D_{21}^{2*} D_{12}^{1*} A g^1(y_1) E^2[g^2(y_2) E^1[y_1|y_2]|y_1] \\ & - D_{21}^{2*} D_{22}^1 D_{21}^2 A g^1(y_1) E^2[g^2(y_2) E^2[y_1|y_2]|y_1] \\ & + F_1^1 E^1[x|y_1] + D_{21}^{2*} F_2^1 g^1(y_1) E^2[g^2(y_2) E^1[x|y_2]|y_1] \\ & - D_{21}^{2*} D_{22}^1 F_2^2 g^1(y_1) E^2[g^2(y_2) E^2[x|y_2]|y_1] \\ & - D_{12}^1 F_2^2 E^1[E^2[x|y_2]|y_1], \qquad \forall y_1 \in \mathbb{R}^{m_1}. \end{aligned} \tag{63a}$$

Since the random variables are jointly Gaussian under both measures,

$$E^i[y_k|y_l] = S_{kl}^i y_l, \qquad k \neq l: i, k, l = 1, 2 \tag{63b}$$

$$E^i[x|y_l] = S_{0l}^i y_l, \qquad i, l = 1, 2 \tag{63c}$$

for some matrices S_{kl}^i and S_{0l}^i. In view of this, (63a) can be rewritten as

$$\begin{aligned} A y_1 = {} & (D_{12}^1 D_{21}^2 A S_{12}^2 S_{21}^1 + F_1^1 S_{01}^1 + D_{12}^1 F_2^2 S_{02}^2 S_{21}^1) y_1 \\ & + [D_{21}^{2*} D_{12}^{1*} A S_{12}^1 - D_{21}^{2*} D_{22}^1 D_{21}^2 A S_{12}^2 \\ & + D_{21}^{2*} F_2^1 S_{02}^1 - D_{21}^{2*} D_{22}^1 F_2^2 S_{02}^2] g^1(y_1) E^2[g^2(y_2) y_2 | y_1]. \end{aligned} \tag{64}$$

This then leads to the following proposition.

Proposition 6: Let (47) and Condition 3 be satisfied, and either $P_{y_1}^1 \neq P_{y_1}^2$ or $P_{y_2}^1 \neq P_{y_2}^2$. Then, the quadratic Gaussian decision problem with asymmetric mode of decision making admits a linear (Stackelberg) equilibrium solution if, and only if,

i) there exists a bounded linear operator $A: \mathbb{R}^{m_1} \to U_1$ satisfying

$$A = D_{12}^1 D_{21}^2 A S_{12}^2 S_{21}^1 + F_1^1 S_{01}^1 + D_{12}^1 F_2^2 S_{02}^2 S_{21}^1 \tag{65a}$$

ii) this solution also satisfies

$$\begin{aligned} & D_{21}^{2*} D_{12}^{1*} A S_{12}^1 - D_{21}^{2*} D_{22}^1 D_{21}^2 A S_{12}^2 + D_{21}^{2*} F_2^1 S_{02}^1 \\ & - D_{21}^{2*} D_{22}^1 F_2^2 S_{02}^2 = 0. \end{aligned} \tag{65b}$$

Proof: Since the "if" part is obvious in view of Theorem 2, we verify only the "only if" part of the proposition. (In what follows we adopt the notation $S \geq 0$ to imply that the nonnegative definite matrix S has at least one positive eigenvalue.) The proof proceeds by showing for three exclusive (and exhaustive) cases that $f(y_1) = g^1(y_1) E^2[g^2(y_2) y_2 | y_1]$ is a nonlinear function of y_1.

a) $P_{y_2}^2 = P_{y_2}^1$, and $P_{y_1}^1 \neq P_{y_1}^2$.

Here, $g^2(y_2) = 1$, and $g^1(y_1) = c_1 \exp\{-1/2 y_1' W_1 y_1\}$, where $W_1 \geq 0$, and $c_1 > 0$ is a constant. Hence, $f(y_1) = g^1(y_1) S_{21}^2 y_1$ which is nonlinear since $W_1 \geq 0$.

b) $P_{y_2}^2 \neq P_{y_2}^1$, $P_{y_1}^1 = P_{y_1}^2$.

Here, $g^1(y_1) = 1$, and $g^2(y_2) = c_2 \exp\{-1/2 y_2' W_2 y_2\}$, where $W_2 \geq 0$, and $c_2 > 0$ is a constant. In this case, f can be evaluated to be

$$f(y_1) = c(V + W_2)^{-1} V S_{21}^2 y_1 \exp\left\{-\frac{1}{2} y_1' B y_1\right\}$$

where

$$V \triangleq E^2\{(y_2 - S_{21}^2 y_1)(y_2 - S_{21}^2 y_1)'\}$$

$$B = S_{21}^{2\prime} V S_{21}^{2\prime} - V'(V + W_2)^{-1} V S_{21}^2 \geq 0,$$

and c is a constant. Since $W_2 \geq 0$, B has at least one positive eigenvalue, and hence $f(y_1)$ is again nonlinear in y_1.

c) $P_{y_2}^2 \neq P_{y_2}^1$ and $P_{y_1}^1 \neq P_{y_1}^2$.

In this case, following the same lines as above we find

$$f(y_1) = \bar{c}(V + W_2)^{-1} V S_{21}^2 y_1 \exp\left\{-\frac{1}{2} y_1' (B + W_1) y_1\right\}$$

which is nonlinear since both $B \geq 0$, $W_1 \geq 0$.

Hence, in view of the preceding analysis, a necessary condition for existence of a solution to (64) is that the last term should vanish [i.e., (65b)] for an A that solves (65a). □

Remark 5: A sufficient condition for (65a) to admit a unique solution in the Banach space of linear bounded operators mapping $\mathbb{R}^{m_1}$ into U_1 is

$$r(D_{12}^1 D_{21}^2 D_{21}^{2*} D_{12}^{1*})\ \mathrm{tr}\ \{S_{12}^2 S_{21}^1 S_{21}^{1*} S_{12}^{2*}\} < 1$$

which is clearly satisfied under condition 3. □

The conditions of Proposition 6 are clearly nonvoid because, given the unique solution of (65a), it may be possible to find F_2^1, F_2^2, S_{02}^1, and S_{02}^2 so that (65b) is satisfied. However, it should also be clear that satisfaction of (65b) places some severe restrictions on the parameters of the problem, which in general will not be met. Hence, it is fair to say that if either $P_{y_1}^1 \neq P_{y_1}^2$ or $P_{y_2}^1 \neq P_{y_2}^2$, generically the problem does not admit a linear equilibrium solution, even if it is a team problem; that is the following.

Corollary 5: If either $P^1_{y_1} \neq P^2_{y_1}$ or $P^1_{y_2} \neq P^2_{y_2}$ (or both), the quadratic Gaussian decision problem does *not* admit (generically) a linear Stackelberg equilibrium solution. The unique solution, which exists under (47) and Condition 3, is *nonlinear*. □

The conditions of the preceding corollary involve only the marginal distributions of y_1 and y_2; in the compliment of these conditions we can derive the following linear solution.

Proposition 7: For the quadratic Gaussian decision problem, let both $P^1_{y_1} = P^2_{y_1}$ and $P^1_{y_1} = P^2_{y_2}$ (but *not necessarily* $P^1_{y_1y_2x} = P^2_{y_1y_2x}$, and even $P^1_{y_1y_2} = P^2_{y_1y_2}$). Then, if

$$2[r(D^1_{12}D^2_{21}D^{2*}_{21}D^{1*}_{12}D^{1*}_{12})]^{1/2} + [r(D^{2*}_{21}D^1_{22}D^2_{21})]^{1/2} < 1 \quad (66)$$

the problem admits a unique Stackelberg equilibrium solution for DM1 (the leader) which is linear in y_1:

$$\gamma^s_1(y_1) = Ay_1 \quad (67a)$$

where $A: \mathbb{R}^{m_2} \to U_1$ is the unique bounded linear operator solving

$$\begin{aligned} Ay_1 = (&D^1_{12}D^2_{21}AS^2_{12}S^1_{21} + D^{2*}_{21}D^{1*}_{12}AS^1_{12}S^2_{21} - D^{2*}_{21}D^1_{22}D^2_{21}AS^2_{12}S^2_{21} \\ &+ F^1_1S^1_{01} + D^1_{12}F^2_2S^2_{02}S^1_{21} + D^{2*}_{21}F^1_2S^1_{02}S^2_{21} \\ &- D^{2*}_{21}D^1_{22}F^2_2S^2_{02}S^2_{21})y_1, \qquad \forall y_1 \in \mathbb{R}^{m_1} \end{aligned} \quad (67b)$$

and S^i_{kl} are defined by (31b), (31c), and S^i_{0i} is defined by $E^i[x|y_i] = S^i_{0i}y_i$.

Proof: When $P^1_{y_1} = P^2_{y_2}$ and $P^1_{y_2} = P^2_{y_2}$, $g^1(y_1) = g^2(y_2) = 1$, and hence Conditions 1 and 2 of Theorem 2 are always satisfied, and in Condition 3, $\rho_2 = \rho_4 = 1$. Then, (66) is the counterpart of (39), and hence existence and uniqueness follow from Theorem 2. Linearity, on the other hand, follows by noting that if we start iteration (35) with $\gamma^{(0)}_1 = 0$, since $g^1(y_1) = g^2(y_2) = 1$ every term will be linear in y_1 [see also (64)], and hence the limit (which exists by Theorem 2) will be linear. Then, substituting $\gamma^1(y_1) = Ay_1$ in (31), we obtain (67b), by simply letting $g^1(y_1) = g^2(y_2) = 1$ in (64). □

When there is a discrepancy between the DM's perceptions of the variances of either y_1 or y_2, Proposition 7 will not hold, and the problem will admit (generically) a nonlinear equilibrium solution, as proven earlier in Proposition 6 and Corollary 5. In this case, an explicit closed-form solution cannot be obtained; however, an approximate solution can be derived by using the iteration (35) which, for the Gaussian problem becomes

$$\begin{aligned} \gamma^{(k+1)}(y_1) = &D^1_{12}D^2_{21}E^1[E^2[\gamma^{(k)}(y_1)|y_2]|y_1] + D^{2*}_{21}D^{1*}_{12}g^1(y_1) \\ &\cdot E^2[g^2(y_2)E^1[\gamma^{(k)}(y_1)|y_2]|y_1] - D^{2*}_{21}D^1_{22}D^2_{21}g^1(y_1) \\ &\cdot E^2[g^2(y_2)E^2[\gamma^{(k)}(y_1)|y_2]|y_1] \\ &+ (F^1_1S^1_{01} + D^1_{12}F^2_2S^2_{02}S^1_{21})y_1 \\ &+ (D^{2*}_{21}F^1_2S^1_{02} - D^{2*}_{21}D^1_{22}F^2_2S^2_{02})g^1(y_1)E^2[g^2(y_2)y_2|y_1]. \end{aligned} \quad (68)$$

If we start this iteration with $\gamma^{(0)}(y_1) = 0$, or any linear function of y_1, at every iteration we obtain linear combinations of terms of the type ($A^{(k)}y_1$ and $B^{(k)}y_1 \exp\{-1/2y_1'V^{(k)}y_1\}$, where $A^{(k)}$ and $B^{(k)}$ are linear operators, and $V^{(k)} \geq 0$ is an $m_1 \times m_1$ matrix. Since this is a successive approximation technique under Condition 3, even stopping the iteration after a finite number of terms will provide a solution sufficiently close to the unique optimum. Hence, generically, a suboptimal policy for DM1, which is sufficiently close to the unique solution of (31), will be of the form

$$\gamma_1(y_1) = A^{(N)}y_1 + \sum_{l \leq N} B^{(l)}y_1 \quad \exp\left\{-\frac{1}{2}y_1'V^{(l)}y_1\right\}$$

where N is a sufficiently large integer [related to the number of iterations taken in (68)], and $A^{(N)}$, $B^{(l)}$, $V^{(l)}$ are generated via the iteration (68). Note that as $N \to \infty$ this solution will uniformly converge to the unique optimum.

Yet another suboptimal solution can be obtained by restricting DM1's policies, at the outset, to linear functions of y_1, i.e., to the form (62) where A is a variable linear operator. DM2's response to any such policy will also be linear (in y_2), thus making T_2 in (10) a linear operator. Then, the problem faced by DM1 is minimization of (11), with $\gamma(y_1) = Ay_1$, over all linear bounded operators A. The solution of this minimization problem will provide DM1 with a linear policy that is (in general) inferior to the limiting solution of (68), unless, of course, $g^1(y_1) = g^2(y_2) = 1$ in which case the two solutions will be the same [satisfying (67b)]. We do not pursue here the details of the derivation of the best linear solution for the general case (as outlined above).

Furthermore, it is possible to work out the various conditions for the special cases of the scalar and continuous-time team problems (formulated as in Examples 1 and 2) and write down the equilibrium solution explicitly whenever it is linear. Such an analysis would routinely follow the lines of the discussion of Examples 1 and 2, and hence will not be included here mainly because of space limitations.

VI. Discussion of Possible Extensions, and Concluding Remarks

In the preceding sections, we have developed an equilibrium theory for two-person quadratic decision problems with static information patterns, wherein the decision makers (DM's) do not necessarily have the same perception of the underlying probability space; that is, our formulation allows for discrepancies in the way different DM's perceive the probability space. As indicated earlier, when such discrepancies exist, even team problems have to be analyzed in the framework of nonzero-sum stochastic games, and in such a framework the Nash solution concept is the most suitable equilibrium concept if the DM's occupy symmetric (nonhierarchical) positions in the decision process, and the Stackelberg solution concept becomes more meaningful if there is a hierarchy in decision making.

Section III of the paper has provided a set of sufficient conditions for existence and uniqueness of Nash equilibrium in the case of symmetric mode of decision making, with the additional feature that it be stable. This is an appealing feature of the solution because, in order to arrive at equilibrium (as a consequence of an infinite number of response iterations), each DM does not have to know the subjective probability measures perceived by the other DM, but has to know only the policy adopted by the other DM at the most recent step of the iteration.

In Section IV we have presented a counterpart of the results of Section III under the asymmetric mode of decision making. The conditions derived ensure that the equilibrium policy of the leader can be obtained as the limit of an infinite sequence which involves conditional expectations under two different probability measures. This sequence [(35), (27)] is structurally different from its counterpart in Section III (see 14), even for team problems, and it contains R-N derivatives of the two probability measures as multiplying factors [which are absent in (14)].

In Section V we have shown that when the underlying probability distributions belong to a Gaussian class, the Nash equilibrium solution will be linear (affine, if mean values are nonzero) in the available static measurements, with the gain operator satisfying a Lyapunov-type operator equation (cf. Theorem 3). This solution and the associated existence conditions have been studied further in the context of two examples which involve scalar and continuous-time stochastic team problems with multiple probability models. In developing a counterpart of Theorem 3 for asymmetric mode of decision making, we have arrived at a seemingly surprising (unexpected) result—the unique Stackelberg equilibrium solution being generically *nonlinear* in the measurements (even under Gaussian multiple probability measures). This constitutes the first *unique* nonlinear solution reported in the literature for a quadratic Gaussian static game or team problem.[5] It should be noted that we have not given a closed-form expression for this nonlinear solution, but have instead provided a recursive scheme which generates admissible policies that come arbitrarily close to the optimum solution.

Several extensions of the results presented in this paper seem to be possible. First, we should note that the general Hilbert-space framework adopted in this paper and the general solutions presented for the Gaussian problems in Section V (Theorems 3 and 4) apply to other models also, such as the ones similar to the continous-time team problem treated in [9] and the Stackelberg problem of [26], but with the DM's having different probability models. It is expected that some explicit results (closed-form solutions) can also be obtained in these cases, but this point has not been pursued in this paper and is left for future research.

Another possible extension of the results of this paper would be to the class of problems in which the random state of nature (i.e., x) as well as the measurements (y_i) are stochastic processes. The general theories of Section III and IV could easily be extended so as to encompass this class of problems also, provided that the problem is set up under the right mathematical assumptions. In particular, if the random variables are taken to be Hilbert space valued weak random variables, with the inner product satisfying some continuity and boundedness conditions [11], Theorems 1–4 directly apply to this more general class of decision problems, when interpreted in the right framework. Furthermore, extension to dynamic (multistage) problems is also possible, by adopting the framework of (say) [8] for the linear-quadratic-Gaussian problem. Then, the unique Nash equilibrium solution under the one-step-delay observation sharing pattern can be obtained by basically following the approach of [8] and utilizing in the recursive derivation Theorem 3 of this paper instead of [8, Theorem 2]. Details of this derivation are, however, rather involved, and will be reported elsewhere.

Regarding the Nash equilibrium solution, yet another possible extension would be to multiple decision-maker problems with more than two (say, N) DM's. Even though the definition of Nash equilibrium (cf. Definition 1) admits a natural (unique) extension to such problems, that of stable equilibrium (cf. Definition 2) does not extend in a unique way. One viable alternative is to assume that each DM reacts optimally to the set of most recent policies of all the other DM's, which leads to a set of N relations similar to (9). In this case, (12) will be replaced by N equations with the right-hand side expressions involving N-1 policies of different DM's. However, the line of reasoning that took us from (13) to (14) does not have a counterpart if $N > 2$, and in general it is not possible to obtain N recursion relations each of which involves *only one* DM's policies at consecutive stages. Then, the counterpart of (13) will have to be treated as a "multivalued" operator equation, in which context an existence and uniqueness result will have to be established. This seems to be a challenging problem whose solution requires somewhat different mathematical techiques than the ones employed in this paper.

One source of motivation for the research reported in this paper has been (as discussed in Section I) the desire to investigate the sensitivity and robustness of team-optimal solutions (in stochastic teams) to independent variations in the perceptions of the DM's of the underlying probability space (and, in particular, the probability measure). The analysis of this paper indeed provides a framework for such a study when the roles of the DM's are either symmetric or asymmetric, since an equilibrium theory has been established in both cases within an "ϵ-neighborhood" of the team-optimal solution. Some further work is needed in order to determine the "satisfiability" of the several existence conditions obtained in the paper when the region of interest is an ϵ-neighborhood of a common probability space, and to further extend the analysis to an investigation of sensitivity and robustness properties of team solutions (obtained under the stipulation of existence of a common underlying probability space) in this ϵ-neighborhood.

An aspect of the decision problem studied here, which is worth bringing forth, is that the subjective probability measures perceived by each DM is fixed in advance and the DM's do not attempt to change their subjective priors during the course of the decision process. Hence, in this sense, the problem treated here is categorically different from the class of problems treated in [18]–[21], where the objective was for the DM's to arrive at a common (consistent) set of probabilistic descriptions of the unknown variables. In the symmetric mode, there is, however, an implicit learning process built in the recursive process that leads to the stable equilibrium decision rules for each DM, since the DM's do not necessarily have access to each other's perception of the priors.

Yet another aspect of the problem treated in this paper is that the general formulation could be viewed as a multimodeling in multiple decision maker problems; however, as opposed to the singular perturbations approach of [22]–[24], here the multimodeling is in the probabilistic description of the decision problem, with each DM having a different probabilistic model of the "rest of the world."

Appendix A

In this Appendix we state a number of results concerning the spectral radii of linear bounded operators.

Let $A:\Gamma \to \Gamma$ and $B:\Gamma \to \Gamma$ be two linear bounded operators where Γ is a Hilbert space equipped with the inner product $\langle\cdot\rangle$. Then the spectral radius of A is defined by

$$r(A) = \limsup_{l\to\infty} \; [\langle\langle A^k\rangle\rangle]^{1/k} \tag{A-1}$$

where $\langle\langle A\rangle\rangle$ is the norm of A, given by

$$\langle\langle A\rangle\rangle = \sup_{\gamma\in\Gamma} \; [\langle A\gamma,\, A\gamma\rangle / \langle\gamma,\, \gamma\rangle]^{1/2}. \tag{A-2}$$

For self-adjoint operators there is an equivalence between the spectral radius and norm of an operator; specifically, if A is self-adjoint,

$$r(A) = \langle\langle A\rangle\rangle = \sup_{\gamma\in\Gamma} \; \{|\langle\gamma,\, A\gamma\rangle| / \langle\gamma,\, \gamma\rangle\} \tag{A-3}$$

(see [13, p. 514]). However, for operators which are not self-adjoint, such an equivalence does not exist, and one can only provide bounds on $r(A)$.

Lemma A.1: For any linear bounded operator A,

$$r(A) \le \langle\langle A\rangle\rangle = [r(A^*A)]^{1/2}.$$

Proof: Since A belongs to a Banach algebra, $\langle\langle A^k\rangle\rangle \le |\langle\langle A\rangle\rangle|^k$ and hence

$$r(A) \le \limsup_{k\to\infty} \; \{|\langle\langle A\rangle\rangle|^k\}^{1/k} = \langle\langle A\rangle\rangle.$$

Furthermore, $\langle\langle A\rangle\rangle = \sup_{\gamma\in\Gamma}[\langle\gamma,\, A^*A\gamma\rangle/\langle\gamma,\, \gamma\rangle]^{1/2}$ which is $[r(A^*A)]^{1/2}$ by (A-3) because A^*A is self adjoint. □

Lemma A.2: Let A and B be two linear bounded operators which commute. Then,

i) $r(AB + A^*B^*) \le 2[r(AA^*)r(B^*B)]^{1/2} = 2[r(A^*A)r(BB^*)]^{1/2}$

ii) $r(AB) \le r(A)r(B)$.

[5] Reference [12] also reports on existence of nonlinear (Nash) solutions for quadratic Gaussian nonzero-sum games, but there the nonlinear solution is one of many solutions one of which is linear, and is due to nonunique intersection of reaction functions (which disappears under appropriate conditions).

Proof:
i) Since $AB + A^*B^*$ is self-adjoint, using (A-3)

$$r(AB+A^*B^*)=\sup_{\gamma\in\Gamma}\{|\langle\gamma,\ (AB+A^*B^*)\gamma\rangle|/\langle\gamma,\ \gamma\rangle\}$$
$$=2\sup_{\gamma\in\Gamma}\{|\langle A\gamma,\ B^*\gamma\rangle|/\langle\gamma,\ \gamma\rangle\}$$

where the equality has followed since A and B commute. Using Cauchy–Schwarz inequality [3], this expression can be bounded from above by

$$\le 2\sup_{\gamma\in\Gamma}\left\{\frac{|\langle A\gamma,\ A\gamma\rangle|^{1/2}|\langle B^*\gamma,\ B^*\gamma\rangle|^{1/2}}{\langle\gamma,\ \gamma\rangle}\right\}$$

and performing individual supremization we further obtain the bound

$$\le 2\sup_{\gamma\in\Gamma}\left[\frac{|\langle A\gamma,\ A\gamma\rangle|}{\langle\gamma,\ \gamma\rangle}\right]^{1/2}\sup_{\gamma\in\Gamma}\left[\frac{|\langle B^*\gamma,\ B^*\gamma\rangle|}{\langle\gamma,\ \gamma\rangle}\right]^{1/2}$$
$$=2\langle\langle A\rangle\rangle\langle\langle B^*\rangle\rangle=2[r(A^*A)r(BB^*)]^{1/2}$$

where the last line has followed from Lemma A.1. Note that this expression can be written in different ways because $r(A^*A) = r(AA^*)$, $r(BB^*) = r(B^*B)$.

ii) First, note that

$$r(AB)=\limsup_{k\to\infty}\ [\langle\langle (AB)^k\rangle\rangle]^{1/k}=\limsup_{k\to\infty}\ [\langle\langle A^kB^k\rangle\rangle]^{1/k}\quad (*)$$

where the last equality has followed because A and B commute. Now, since A, B belong to a Banach algebra, $\langle\langle A^kB^k\rangle\rangle \le \langle\langle A^k\rangle\rangle\langle\langle B^k\rangle\rangle$ for every k

$$\Leftrightarrow [\langle\langle A^kB^k\rangle\rangle]^{1/k}\le[\langle\langle A^k\rangle\rangle\langle\langle B^k\rangle\rangle]^{1/k}$$
$$=[\langle\langle A^k\rangle\rangle]^{1/k}[\langle\langle B^k\rangle\rangle]^{1/k}\qquad \text{for every } k$$

and taking lim sup of both sides, and using (*)

$$r(AB)\le\limsup_{k\to\infty}\ \{[\langle\langle A^k\rangle\rangle]^{1/k}[\langle\langle B^k\rangle\rangle]^{1/k}\le r(A)r(B)$$

which proves the desired result. □

Lemma A.3: Let A and B be both self-adjoint. Then,

$$r(A+B)\le r(A)+r(B).$$

Proof: This follows from (A-3) and the triangle inequality applied to norm $\langle\langle\cdot\rangle\rangle$. □

Appendix B

Proof of Theorem 1

Let us first recall the following result from functional analysis (see, for example [13, Chapter XIII, Theorem 3]).

Lemma B.1: Let $\mathcal{S}$ be a linear bounded operator mapping a Hilbert space Γ into itself, and consider the equation

$$\gamma=\mathcal{S}\gamma+\mu \tag{B-1}$$

defined on Γ. Furthermore, consider the "successive approximation"

$$\gamma^{(k+1)}=\mu+\mathcal{S}\gamma^{(k)},\qquad k=0,\ 1,\ \cdots \tag{B-2}$$

to the solution of (B-1). Then, the sequence generated by (B-2) converges to a unique element of Γ, for any starting point $\gamma^{(0)} \in \Gamma$, which is further a solution of (B-1), if, and only if, the spectral radius of $\mathcal{S}$ is less than unity, i.e., there exists a ρ, $0 < \rho < 1$, such that

$$r(\mathcal{S})\le\rho. \tag{B-3}$$

□

Now, applying this lemma to our problem, we identify $\mathcal{S}$ with either $\mathcal{S}_1$ or $\mathcal{S}_2$ [given by (16)], Γ with Γ_1 or Γ_2, the successive approximation (B-2) with (14), and condition (B-3) above with (19) for either $i = 1$ or 2. Then, the statement of Theorem 1 i) readily follows from the preceding lemma, in view of Proposition 2.

Furthermore, since $\mathcal{S}_i$ can be written as the product of two commuting operators, using Lemma A.2 ii) we obtain

$$r_i(\mathcal{S}_i)=r_i(\bar{D}^i\bar{P}_{i|i})\le r_i(\bar{D}^i)r_i(\bar{P}_{i|i}).$$

Under (20a) this can be bounded from above by $\rho_1^i\rho_2^i = \rho^i < 1$, thereby ensuring (19). On the other hand, since the spectral radius of a bounded linear operator is bounded from above by its norm [13], and that $\|\bar{D}^i\|_i = \langle\langle\bar{D}^i\rangle\rangle_i$ because $\bar{D}^i$ also maps U_i into U_i (in addition to being a mapping from Γ_i into itself), (20b) follows. This completes the proof of Theorem 1. □

Appendix C

Proof of Corollary 2 (Section III)

Here we verify that the second inequality of (20b) is implied by the condition that (21a) is uniformly bounded by ρ_2^i. Towards this end, we first have, for each $\gamma \in \Gamma_i$, from the Cauchy–Buniakowski (Schwarz) inequality [3] applied to Γ_i[6]

$$\|\bar{P}_{i|i}\gamma\|_i^2=\left\|\int_{Y_j}P^i_{y_j|y_i}\ (d\eta|y_i)\int_{Y_i}\gamma(\xi)P^j_{y_i|y_j}(d\xi|\eta)\right\|_i^2$$
$$\le\left\|\int_{Y_i}\gamma(\xi)P^j_{y_i|y_j}\ (d\xi|\eta)\right\|_i^2$$
$$=\int_{Y_j}\left(\int_{Y_i}\gamma(\xi)P^j_{y_i}(d\xi|\eta),\ \int_{Y_i}\gamma(\xi)P^j_{y_i|y_j}(d\xi|\eta),\right)_i$$
$$\cdot\ P^j_{y_j}(d\eta)g^j(\eta)$$

where the last equality has involved a change of measures, using the R–N derivative $g^j(\eta)$. Now, again using the Cauchy–Schwarz inequality, this expression can be bounded from above by

$$\le\int_{Y_j}\int_{Y_i}\gamma(\xi),\ (\gamma(\xi))_iP^j_{y_i|y_j}\ (d\xi|\eta)g^j(\eta)P^j_{y_j}(d\eta)$$

$$=\int_{Y_i}(\gamma(\xi),\ \gamma(\xi))_iP^i_{y_i}\ (d\xi)g^i(\xi)\int_{Y_j}P^j_{y_j|y_i}\ (d\eta|\xi)g^j(\eta)$$

where the last equality has followed from Bayes' theorem. It now readily follows that under the condition of Corollary 2, the last expression is bounded from above by $\le \rho_2^i\|\gamma\|_i^2$, thus proving the desired result for $i = 1, 2$. □

Proof of Corollary 4 (Section IV)

The fact that uniform boundedness of (43a) (by $(\rho_2)^2$) implies the first inequality of (40b) follows readily from the proof given

[6] In the following we have abused the notation and have used $\|\cdot\|_i$ to stand also for the natural norm derived from $\langle\cdot,\ \cdot\rangle_i$; but this should not create any source of confusion.

above, since the spectral radius of $\bar{P}_1^*|_1\bar{P}_1|_1$ is equal to the square of the norm of $\bar{P}_1|_1$. Now, to verify that uniform boundedness of (43b) implies the second inequality of (40b) we follow basically the same line of reasoning, but the details of the proof are more involved. Towards this end we first note that for each $\gamma \in \Gamma_1$,

$$\|\mathcal{K}\gamma\|_1^2 \equiv \|g^1(\xi)\int_{Y_2} P^2_{y_2|y_1}(d\eta|y_1=\xi)g^2(\eta) \cdot \int_{\eta} P^2_{y_1|y_2}(db|y_2=\eta)\gamma(b)\|_1^2$$

$$= \|\sqrt{g^1(\xi)}\int_{Y_2} P^2_{y_2|y_1}(d\eta|y_1=\xi)g^2(\eta) \cdot \int_{Y_1} P^2_{y_1|y_2}(db|y_2=\eta)\gamma(b)\|_2^2$$

$$\le \|\sqrt{g^1(\xi)}g^2(\eta)\int_{Y_1} P^2_{y_1|y_2}(db|y_2=\eta)\gamma(b)\|_2^2$$

where the second equality follows from a change of measures, and the last bound follows form the Cauchy-Schwarz inequality. It should be pointed out that here we have abused the notation and have used $\|\cdot\|_2$ to mean

$$\|m(\xi,\eta)\|_2 = \left\{\int_{Y_1}\int_{Y_2}(m(\xi,\eta),\ m(\xi,\eta))_1 P^2_{y_1y_2}(d\xi\times d\eta)\right\}^{1/2}$$

where m is a (y_1, y_2)—measurable random variable taking values in U_1; hence, the subindex "2" indicates that the probability space is the one determined by the subjective probability measure of DM2.

Now, the latter bound can further be bounded above by

$$\le \int_{Y_1}\int_{Y_2} g^1(\xi)|g^2(\eta)|^2 P^2_{y_1y_2}(d\xi\times d\eta) \cdot \int_{Y_1} P^2_{y_1|y_2}(db|y_2=\eta)(\gamma(b),\ \gamma(b))_1$$

since i)

$$\left(\int_{Y_1} P^2_{y_1|y_2}(db|y_2=\eta)\gamma(b),\ \int_{Y_1} P^2_{y_1|y_2}(db|y_2=\eta)\gamma(b)\right)_1 \le \int_{Y_1} P^2_{y_1|y_2}(db|y_2=\eta)(\gamma(b),\ \gamma(b))_1$$

by the Cauchy-Schwarz inequality (because $P^2_{y_1}|_{y_2}$ is also a probability measure), and ii) $g^1(\xi)|g^2(\eta)|^2 \ge 0$. Hence, by interchanging the variables ξ and b,

$$\|\mathcal{K}\gamma\|_1^2 \le \int_{Y_1}\int_{Y_2}\int_{Y_1}(\gamma(\xi),\ \gamma(\xi))_1 g^1(b)|g^2(\eta)|^2 P^2_{y_1y_2} \cdot (db\times d\eta)\frac{P^2_{y_2|y_1}(d\eta|y_1=\xi)P^2_{y_1}(d\xi)}{P^2_{y_2}(d\eta)}$$

$$= \int_{Y_1} P^1_{y_1}(d\xi)(\gamma(\xi),\ \gamma(\xi))_1\int_{Y_2}\int_{Y_1} g^1(\xi)g^1(b)|g_2(\eta)|^2 \cdot P^2_{y_1|y_2}(db|y_2=\eta)P^2_{y_2|y_1}(d\eta|y_1=\xi)$$

and under (43b) this can be bounded above by

$$\le \int_{Y_1} P^1_{y_1}(d\xi)(\gamma(\xi),\ \gamma(\xi))_1\rho_4^2 = \rho_4^2\|\gamma\|_1^2$$

which completes the proof. □

Appendix D

Derivation of First and Second Gateaux Variations (25), (26)

Starting with the expression for $\tilde{J}$ as given by (23), we first obtain

$$\Delta\tilde{J}(\gamma;\ h) \triangleq \tilde{J}(\gamma+h)-\tilde{J}(\gamma)$$
$$= \frac{1}{2}\langle h,\ \gamma\rangle_1 + \frac{1}{2}\langle\gamma,\ h\rangle_1 + \frac{1}{2}\langle h,\ h\rangle_1$$
$$+\frac{1}{2}\int_{Y_2}\{(F_2^2E^2[x|y_2] + D^2_{21}E^2[\gamma(y_1)|y_2],$$
$$\cdot D^1_{22}D^2_{21}E^2[h(y_1|y_2])_2$$
$$+(D^2_{21}E^2[h(y_1)|y_2],\ D^1_{22}F^2_2E^2[x|y_2]$$
$$+(D^1_{22}D^2_{21}E^2[\gamma(y_1)|y_2])_2$$
$$+(D^2_{21}E^2[h(y_1)|y_2],\ D^1_{22}D^2_{21}E^2[h(y_1)|y_2])_2\}P^1_{y_2}(d\xi)$$
$$-\langle h,\ E^1[F^1_1x|y_1]\rangle_1 - \int_{X\times Y_2}(D^2_{21}E^2[h(y_1)|y_2],\ F^1_2x)_2P^1$$
$$\cdot(dx,\ Y_1,\ d\xi)$$
$$-\langle h,\ E^1[D^1_{12}D^2_{21}E^2[\gamma(y_1)|y_2]y_1]$$
$$+E^1[D^1_{12}F^2_2E^2[x|y_2]|y_1]\rangle_1$$
$$-\langle\gamma,\ E^1[D^1_{12}D^2_{21}E^2[h(y_1)|y_2]|y_1]\rangle_1$$
$$-\langle h,\ D^1_{12}D^2_{21}E^1[E^2[h(y_1)|y_2]|y_1]\rangle_1$$
$$\equiv\delta\tilde{J}(\gamma;\ h)+\delta^2\tilde{J}(\gamma;\ h).$$

Now, since $\delta\tilde{J}(\gamma;h)$ is homogeneous of degree one, and $\delta^2\tilde{J}(\gamma;h)$ is homogeneous of degree two, $\Delta\tilde{J}(\gamma;\ h)$ admits a unique decomposition with the corresponding expressions being (after some simplification)

$$\delta\tilde{J}(\gamma;\ h) = \langle h,\ \gamma\rangle_1 + \int_{Y_2}(E^2[h(y_1)|y_2],\ D^{2*}_{21}D^1_{22}\{F^2_2E^2[x|y_2]$$
$$+D^2_{21}E^2[\gamma(y_1)|y_2]\})_1P^1_{y_2}(d\xi) - \langle h,\ E^1[F^1_1x|y_1]\rangle_1$$
$$-\int_{X\times Y_2}(E^2[h(y_1)|y_2],\ D^{2*}_{21}F^1_2x)_1P^1(dx,\ Y_1,\ d\xi)$$
$$-\langle h,\ D^1_{12}D^2_{21}\mathcal{P}_{1|1}\gamma(y_1)+D^1_{12}F^2_2E^1[E^2[x|y_2]|y_1]\rangle_1$$
$$-\langle\bar{P}_{1|1}h,\ D^{2*}_{21}D^{1*}_{12}\gamma\rangle_1 \qquad \text{(D-1)}$$

$$\delta^2\tilde{J}(\gamma;\ h) = \frac{1}{2}\langle h,\ h\rangle_1 + \frac{1}{2}\int_{Y_2}(E^2[h(y_1)|y_2],\ D^{2*}_{21}D^1_{22}D^2_{21} \cdot E^2[h(y_1)|y_2])_1P^1_{y_2}(d\xi) - \langle h,\ D^1_{12}D^2_{21}\bar{P}_{1|1}h\rangle_1 \qquad \text{(D-2)}$$

where we have used some properties of adjoint operators under inner products, and the notation introduced in (28); we have also made use of the fact that the bounded linear operator $D^1_{12}D^2_{21}:\mathcal{U}_1 \to \mathcal{U}_1$ commutes with the double conditional expectation operator $\bar{P}_1|_1$ (or $\mathcal{P}_1|_1$).

We now prove a lemma which will be used in simplifying these expressions further.

Lemma D.1: For $h(\cdot) \in \mathcal{U}_1$, $f(\cdot) \in \mathcal{U}_2$,

$$\int_{Y_2}(E^2[h(y_1)|y_2=\xi],\ f(\xi))_1P^1_{y_2}(d\xi)$$
$$\equiv \int_{Y_2}(h(\eta),\ g^1(\eta)E^2[g^2(y_2)f(y_2)|y_1=\eta]_1P^1_{y_1}(d\eta)$$
$$\triangleq \langle h,\ g^1(y_1)E^2[g^2(y_2)f(y_2)|y_1]\rangle_1$$

where $g^i(\cdot)$ are given by (2).

Proof: The proof follows from the following set of equalities where we are allowed to change orders of integration because $\mathcal{U}_1$ and $\mathcal{U}_2$ are Hilbert spaces of random variables well defined under both measures:

$$\int_{Y_2} E^2[h(y_1)|y_2=\xi],\ f(\xi))_1 P^1_{y_2}\ (d\xi)$$

$$=\int_{Y_2}\left(\left(\int_{Y_1} h(\eta)P^2_{y_1|y_2}\ (d\eta|\xi),\ f(\xi)\right)_1 P^1_{y_2}\ (d\xi)\right.$$

$$=\int_{Y_2}\int_{Y_1}\left(h(\eta),\ f(\xi)\right)_1 P^2_{y_1|y_2}\ (d\eta|\xi)P^1_{y_2}\ (d\xi)$$

$$=\int_{Y_2}\int_{Y_1}(h(\eta),\ f(\xi))_1 P^2_{y_2|y_1}\ (d\xi|y_1=\eta)g^1(\eta)g^2(\xi)P^1_{y_1}\ (d\eta)$$

where, in the next to the last line, we have used continuity property of inner product in pulling out $P^2_{y_1}|_{y_2}(dy_1|\xi)$. Now, pulling the integration over Y_2 into the inner product, we further obtain

$$=\int_{Y_1} P^1_{y_1}\ (d\eta)[h(\eta),\ \int_{Y_2} g^2(\xi)f(\xi)P^2_{y_2|y_1}\ (d\xi|y_1=\eta))_1 g^1(\eta)$$

$$=\int_{Y_1} P^1_{y_1}\ (d\eta)(h(\eta),\ g^1(\eta)E^2[g^2(y_2)f(y_2)|y_1=\eta])_1$$

which is the desired result. □

Now, using (D-3) in (D-2) we obtain

$$\delta^2\tilde{J}(\gamma;\ h)$$
$$=\frac{1}{2}\langle h,\ h\rangle_1+\frac{1}{2}$$
$$\cdot\int_{Y_1}(h(\eta),\ g^1(\eta)D^{2*}_{21}D^1_{22}D^2_{21}E^2[g^2(y_2)E^2[h(y_1)|y_2]|y_1=\eta])_1 P^1_{y_1}\ (d\eta)$$
$$-\frac{1}{2}\langle h,\ D^1_{12}D^2_{21}\bar{P}_{1|1}h\rangle_1-\frac{1}{2}\langle h,\ D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1}h\rangle_1$$

which verifies (26).

To verify (25), we apply the result of Lemma D.1 to (D-1) to obtain

$$\begin{aligned}\delta\tilde{J}(\gamma;\ h)=\langle h,\ \gamma\rangle_1+\langle h,\ & g^1(y_1)D^{2*}_{21}D^1_{22}\{F^2_2E^2[g^2(y_2)E^2[x|y_2]|y_1]\\ & +D^2_{21}E^2[g^2(y_2)E^2[\gamma(y_1)|y_2]\ |y_1]\}\rangle_1\\ & -\langle h,\ F^1_1E^1[x|y_1]\rangle_1\\ & -\langle h,\ g^1(y_1)D^{2*}_{21}F^1_2E^2[g^2(y_2)E^1[x|y_2]|y_1]\rangle_1\\ & -\langle h,\ (D^1_{12}D^2_{21}\bar{P}_{1|1}+D^{2*}_{21}D^{1*}_{12}\bar{P}^*_{1|1})\gamma\rangle_1\\ & -\langle h,\ D^1_{12}F^2_2E^1[E^2[x|y_2]|y_1]\rangle_1\\ \equiv\ & \langle h,\ \gamma\rangle_1-\langle h,\ \mathcal{Z}\gamma\rangle_1-\langle h,\ \beta\rangle_1\end{aligned}$$

where $\mathcal{Z}$ and β are defined by (27a) and (27b), respectively. This then completes the verification of (25) and (26).

Derivation of an Expression for $\mathcal{P}^*_{1|1}$, the Adjoint of $\mathcal{P}_{1|1}$

First note that

$$\int_{y_1}(\mathcal{P}^*_{1|1}\gamma(y_1),\ h(y_1))_1 P^1_{y_1}\ (dy_1)$$

$$=\int_{Y_1}(\gamma(y_1),\ \mathcal{P}_{1|1}h(y_1))_1 P^1_{y_1}\ (dy_1)\quad\equiv E^1[(\gamma(y_1),\ E^1[E^2[h(y_1)|y_2]|y_1])_1]$$

$$=E^1[(\gamma(y_1),\ E^2[h(y_1)|y_2])_1]$$

where we have used the smoothing property of conditional expectation under the probability measure $P^1_{y_1}$. Now, a further conditioning under $P^1_{y_1}|_{y_2}$ yields

$$=E^1[(E^1[\gamma(y_1)|y_2],\ E^2[h(y_1)|y_2])_1]$$

and using (D-3) (cf. Lemma D-1) this becomes equivalent to

$$=E^1[(g^1(y_1)E^2[g^2(y_2)E^1[\gamma(y_1)|y_2]|y_1],\ h(y_1))_1]$$

thus proving (32). The first expression in (32) follows by routine manipulations.

Appendix E

In this Appendix we show that the Stackelberg solution satisfying (10), (11) is indeed an equilibrium solution—the so-called *strong equilibrium* of a decision problem with a modified (dynamic) information pattern. Towards this end, let us replace the original decision problem with one in which the *decision* (action) *variables* are $\gamma_1\in\Gamma_1$ and $\gamma_2\in\Gamma_2$, for DM1 and DM2, respectively, and the information pattern is dynamic (for DM2), with DM2 having access to the decision γ_1 of DM1. Let $\tilde{\mathcal{U}}_1$ and $\tilde{\mathcal{U}}_2$ denote the strategy spaces of DM1 and DM2, respectively, under this new information pattern; furthermore, denote their generic elements by β_1 and β_2, respectively. Now, since DM1 has static information, all permissible policies β_1 will be constant mappings: $\rightarrow\Gamma_1$, and hence $\tilde{\mathcal{U}}_1=\Gamma_1$. For DM2, on the other hand, all permissible policies will be measurable mappings $\beta_2:\Gamma_1\rightarrow\Gamma_2$. Finally, let $\tilde{J}_i:\tilde{\mathcal{U}}_1\times\tilde{\mathcal{U}}_2\rightarrow\mathbb{R}$ be the cost function of DMi, satisfying the boundary condition

$$\tilde{J}_i(\beta_1,\ \beta_2)=J_i(\gamma_1,\ \gamma_2),\qquad\forall\beta_1\equiv\gamma_1\in\Gamma_1\equiv\mathcal{U}_1\tag{E-1}$$

where $\gamma_2\in\Gamma_2$ is uniquely defined for each $\gamma_1\in\Gamma_1$ by

$$\gamma_2=\beta_2(\gamma_1)\ \text{ in }\ \Gamma_2.\tag{E-2}$$

Now, let $(\gamma^s_1,\ \gamma^s_2)\in\Gamma_1\times\Gamma_2$ be a Stackelberg solution to the original decision problem with the unique mapping T_2 satisfying (10). Note that $T_2\in\mathcal{U}_2$, and hence relabeling T_2 as β^s_2, and γ^s_1 as β^s_1, in (10) and (11), we obtain in view of (E-1), (E-2)

$$\tilde{J}_1(\beta^s_1,\ \beta^s_2)\le\tilde{J}_1(\beta_1,\ \beta^s_2)\qquad\forall\beta_1\in\tilde{\mathcal{U}}_1$$
$$\tilde{J}_2(\beta_1,\ \beta^s_2)\le\tilde{J}_2(\beta_1,\ \beta_2)\qquad\forall(\beta_1,\ \beta_2)\in\tilde{\mathcal{U}}_1\times\tilde{\mathcal{U}}_2,$$

which clearly indicate that $(\beta^s_1,\ \beta^s_2)\in\tilde{\mathcal{U}}_1\times\tilde{\mathcal{U}}_2$ is a noncooperative Nash equilibrium. This is, in fact, a stronger equilibrium (called "strong equilibrium" [17]) because the second inequality is satisfied not only for $\beta_1=\beta^s_1$, but for all $\beta_1\in\mathcal{U}_2$.

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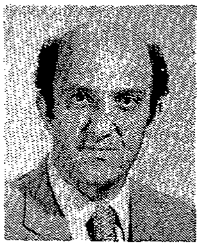

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