A Dynamic Games Approach to Controller Design: Disturbance Rejection in Discrete-Time

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Abstract—**We show that the discrete-time disturbance rejection problem, formulated in finite and infinite horizons, and under perfect state measurements, can be solved by making direct use of some results on linear-quadratic zero-sum dynamic games. For the finite-horizon problem an optimal (minimax) controller exists (in contrast with the continuous-time H^∞ control problem), and can be expressed in terms of a generalized (time-varying) discrete-time Riccati equation. The existence of an optimum also holds in the infinite-horizon case, under an appropriate observability condition, with the optimal control, given in terms of a generalized algebraic Riccati equation, also being stabilizing. In both cases, the corresponding worst-case disturbances turn out to be correlated random sequences with discrete distributions, which means that the problem (viewed as a dynamic game between the controller and the disturbance) does not admit a pure-strategy saddle point. The paper also presents results for the delayed state measurement and the nonzero initial state cases. Furthermore, it formulates a stochastic version of the problem, where the disturbance is a partially stochastic process with fixed higher order moments (other than the mean). In this case, the minimax controller depends on the energy bound of the disturbance, provided that it is below a certain threshold. Several numerical studies included in the paper illustrate the main results.**

I. Introduciton

ONE of the major developments of this decade in control theory has been the formulation ([1]) of the disturbance rejection, model matching, and tracking problems for linear plants as an H^∞-optimization problem, and the derivation of a complete solution to this frequency-domain optimization problem ([2]-[4]). Initially, it was thought that the H^∞-optimal controllers require high-dimensional representations, but later work has shown that the maximum *McMillan* degree of these controllers is in fact in the order of the *McMillan degree* of the overall system transfer matrix [5]; furthermore, it has been shown that in a time-domain (state-space) characterization of these controllers a generalized Riccati equation of the type that arises in linear-quadratic differential games or risk-sensitive linear-exponentiated quadratic stochastic control plays a key role ([6]-[11]). These correspondences have prompted further accelerated interest on the topic, with current work devoted to making the relationships between H^∞-optimal control, linear-quadratic differential games, and risk sensitive (entropy minimizing) control more precise, and searching for more direct time-domain derivations for (and thereby better insight into) the H^∞-optimal control problems ([12], [13]). Such a time-domain derivation was, in fact, first given in [14] using a game-theoretic approach (and using some earlier results from [15]), which provided the solution to the (infinite-horizon) continuous-time H^∞-optimal control problem with perfect state measurements—much before the problem was originally formulated in the frequency domain. It should be noted that, in contrast to the frequency-domain formulation, the time-domain approach also allows one to formulate finite-horizon versions of the original problem, and to introduce additional robustness measures, such as local or global admissibility of the minimax controllers [16].

This paper contributes to the current efforts for developing a complete time-domain based theory of H^∞-optimal control, by studying in detail the relationship between linear-quadratic dynamic games and the discrete-time disturbance rejection problem formulated in both finite and infinite horizons. The main message to be derived from this study is that some of the available results in linear-quadratic games (for example from [17]), properly interpreted and extended, readily solve the discrete-time disturbance rejection problem with perfect state measurements, leading to a linear controller that achieves the optimum (minimax) performance bound. The game-theoretic approach also allows us to consider other types of information structures (such as 1-step delayed state information), and problems with partially stochastic disturbances, both of which are discussed in the paper. Some concurrent work on the discrete-time H^∞-optimal control problem can be found in [18].

The organization of this paper is as follows. After this Introduction, Section II provides a precise problem formulation, which is followed in Section III by the formulation and complete solution of a related auxiliary dynamic game under different information patterns. Solutions to the finite-horizon disturbance attenuation problem with perfect and 1-step delay state measurements are obtained in Section IV, using the saddle-point solution of the auxiliary linear-quadratic game, and these results are extended to the infinite-horizon case in

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Section V. Section VI discusses three extensions to the basic problem: the case when the disturbance is a partially stochastic process (with unknown mean, but known covariance), plants where the control and disturbance have direct links to the output, and plants with *nonzero* initial states. The paper ends with the concluding remarks of Section VII.

II. The Discrete-Time Disturbance Attenuation Problem

We consider in this paper the (finite-horizon, time-varying) linear disturbance attenuation model

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \qquad x_1 = 0 \quad (2.1a)$$

$$z_k = H_k x_k + G_{k-1} w_{k-1} + F_{k-1} u_{k-1} \quad (2.1b)$$

where x_k is the state vector, z_k is the controlled output, u_k is the control vector, and w_k is the disturbance, with each vector belonging to an appropriate dimensional Euclidean space. The upper case letters denote matrices of compatible dimensions, and the index "k" denotes the discrete-time index, taking values in the set of integers: $\boldsymbol{K} := \{1, 2, \cdots, K\}$. Letting

$$x_{[1,k]} := (x_1, x_2, \cdots, x_k) \quad (2.2)$$

we let the permissible controls to be closed-loop controls $u_k = \mu_k(x_{[1,k]})$, where μ_k is a Borel measurable mapping for each $k \in \boldsymbol{K}$. The mapping μ_k, $k \in \boldsymbol{K}$, is commonly called the *control law* or *strategy*. For future reference, we introduce the function space $\boldsymbol{M}_k$, as the (control) space where each permissible μ_k lies. The disturbance w_k, $k \in \boldsymbol{K}$, is any l^2 sequence, and without any loss of generality we let $w_k \in W_k$, where W_k is an appropriate dimensional Euclidean space.

With this linear system, we associate a quadratic performance index

$$\tilde{J}(u_{[1,K]}, w_{[1,K]}) = \sum_{k=1}^{K} \{ |z_{k+1}|^2_{Q_{k+1}} + |u_k|^2 \} \equiv \|z\|^2_Q + \|u\|^2 \quad (2.3)$$

where $Q_{k+1} \geq 0$ for all $k \in \boldsymbol{K}$, $|\cdot|_S$ denotes an appropriate Euclidean (semi-) norm weighted by a nonnegative definite matrix S (the absence of such a weighting matrix implies that we have the standard Euclidean norm), and $\|\cdot\|$ denotes an appropriate (corresponding) l^2 norm. The disturbance attenuation problem, succinctly stated, is the design of a closed-loop controller $\mu^*_{[1,K]}$ which minimizes the maximum of $\tilde{J}$ over all energy bounded disturbances. In mathematical terms

$$\sup_{\|w\|^2 \leq b^2} \tilde{J}(\mu^*_{[1,K]}, w_{[1,K]}) = \min_{\{\mu_k \in \boldsymbol{M}_k,\, k \in \boldsymbol{K}\}} \sup_{\|w\|^2 \leq b^2} \tilde{J}(\mu_{[1,K]}, w_{[1,K]}) =: \bar{\tilde{J}} \quad (2.4)$$

where b^2 is some positive (energy) bound and the "bar" on $\tilde{J}$ indicates that the resulting value of $\tilde{J}$ is the *upper value* of the underlying dynamic game.[1] If the permissible controllers are linear, then one can show (rather trivially) that the precise value of b is immaterial, and the optimal controller is independent of b.[2] In this case, we can equivalently formulate the disturbance attenuation problem as

$$\sup_{\{w_k \in W_k,\, k \in \boldsymbol{K}\}} \tilde{T}(\mu^*_{[1,K]}, w_{[1,K]}) = \min_{\{\mu_k \in \boldsymbol{M}^l_k,\, k \in \boldsymbol{K}\}} \sup_{\{w_k \in W_k,\, k \in \boldsymbol{K}\}} \tilde{T}(\mu_{[1,K]}, w_{[1,K]}) =: (\gamma^*)^2 \quad (2.5)$$

where $\boldsymbol{M}^l_k$ is the class of linear mappings and $\tilde{T}(\mu_{[1,K]}, w_{[1,K]})$ is the (square of the) attenuation of the disturbance $w_{[1,K]}$ at the output (measured in terms of $\tilde{J}$) under the control law $\mu_{[1,K]}$

$$\tilde{T}(\mu_{[1,K]}, w_{[1,K]}) := \begin{cases} \tilde{J}(\mu_{[1,K]}, w_{[1,K]}) / \|w\|^2, & w_{[1,K]} \neq 0 \\ -\infty, & \text{else.} \end{cases} \quad (2.6)$$

The nonnegative quantity γ^* is the optimum (minimax) attenuation level, assuming of course that a $\mu^*_{[1,K]}$ exists, satisfying (2.5). If there is no such $\mu^*_{[1,K]}$, then we define γ^* again as in (2.5), but with "min" replaced by "infimum." A more general problem of interest is the characterization of all controllers $\mu^\gamma_{[1,K]}$, such that

$$\sup_{\{w_k \in W_k,\, k \in \boldsymbol{K}\}} \tilde{T}(\mu^\gamma_{[1,K]}, w_{[1,K]}) \leq (\gamma)^2 \quad (2.7)$$

for a given $\gamma > \gamma^*$.

It is important to note that in (2.4) and (2.5), and also in (2.7), one can allow the disturbance to depend on the state, without modifying the definitions. It is not generally possible, however, to interchange the "min" and "sup" operations in (2.4) and (2.5), since the underlying dynamic game does not generally admit a saddle point (in pure policies), as already demonstrated in [16]. If one seeks a saddle point, then it will be necessary to extend the disturbance class to include "all" random sequences, as was done in [16] for a specific design problem. We will have an occasion to comment more fully on this point in Section IV.

In this paper, we will first solve completely the special case where the performance index is

$$J(u_{[1,K]}, w_{[1,K]}) = \sum_{k=1}^{K} \{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2 \} \quad (2.8)$$

[1] This problem can be viewed as a zero-sum dynamic game between two players—controller and energy-bounded disturbance—with the controller announcing its policy in advance [17].

[2] If we allow nonlinear controllers, however, there may exist a minimax controller that depends explicitly on the energy bound b^2. As demonstrated in [16], such a nonlinear minimax controller may lead to improved performance if the actual disturbance happens to be different from the computed worst-case disturbance, assuming of course that the value of b^2 is precisely known.

both for the finite- and infinite-horizon formulations. Note that in this case the controlled output is (a linear function of) the state variable. We will then discuss (in Section VI) the complete extension to the more general case where the performance index is instead given by (2.3).

We first consider, in the next section, an auxiliary dynamic game whose saddle-point solution plays an important role in the solution of the disturbance attenuation problem.

III. An Auxiliary LQ Dynamic Game and Its Solution

Consider the two-person linear-quadratic dynamic game described by the system dynamics (2.1a), with *nonzero* initial state, and with objective functional

$$L(u_{[1,K]}, w_{[1,K]}) = J(u_{[1,K]}, w_{[1,K]}) - (\gamma)^2 \sum_{k=1}^{K} |w_k|^2 + |x_1|^2_{Q_1} \qquad Q_1 \geq 0 \quad (3.1)$$

where $\gamma > 0$ is a fixed parameter. Player 1 (minimizer) controls $u_{[1,K]}$, as a nonanticipatory function of the state, and Player 2 (maximizer) controls $w_{[1,K]}$, either as state dependent or in an open-loop fashion. If w_k is allowed to depend on (the present and past values of) the state, we denote the corresponding mapping (policy for the disturbance) by ν_k, at stage $k \in \boldsymbol{K}$, and the corresponding policy space by N_k. We now seek a saddle-point solution $(\mu^*_{[1,K]}, \nu^*_{[1,K]})$, satisfying

$$L(\mu^*_{[1,K]}, \nu_{[1,K]}) \leq L(\mu^*_{[1,K]}, \nu^*_{[1,K]}) =: L^* \leq L(\mu_{[1,K]}, \nu^*_{[1,K]}) \quad (3.2)$$

for all $\mu_k \in \boldsymbol{M}_k$, $\nu_k \in N_k$, $k \in \boldsymbol{K}$. (Note that there are no norm constraints imposed on $w_{[1,K]}$.)

A. Some Properties and a Fundamental Theorem

At this point it is important to recall the following fact from [17]: in zero-sum dynamic games, when players have access to closed-loop state information (with memory), the inherent redundancy in information gives rise to a multiplicity of saddle-point equilibria—each one leading to the same value (in view of *ordered interchangeability* property of saddle-point equilibria; see [17, p. 24]), but not necessarily requiring the same existence conditions. Hence, in these situations one has to bring in a further refinement on the saddle-point solution, such as *strong time consistency*, or *noise insensitivity*. We call a (saddle-point) solution "strongly time consistent" if it provides a solution to any truncated version of the original game, regardless of the values of the new initial states. More precisely, we have the following definition.

Definition 3.1 – Strong-Time Consistency [19]: From the original game defined on the time interval $[1, K]$, construct a new game on a shorter time interval $[l, K]$, by setting $\mu_{[0,l-1]} = \beta^1_{[0,l-1]}$, $\nu_{[0,l-1]} = \beta^2_{[0,l-1]}$, where $\beta^i_{[0,l-1]}$, $i = 1, 2$, are fixed but arbitrarily chosen. Let $(\mu^*_{[1,K]} \in \boldsymbol{M}_{[1,K]}, \nu^*_{[1,K]} \in N_{[1,K]})$ be a saddle-point solution for the original game. Then, it is "strongly time consistent," if the pair $(\mu^*_{[l,K]}, \nu^*_{[l,K]})$ is a saddle-point solution of the new game (on the interval $[l, K]$), regardless of the choices for $\beta^i_{[0,l-1]}$, $i = 1, 2$, and for every l, $2 \leq l \leq K$. ◇

By "noise insensitivity" we mean that the saddle-point solution retains its equilibrium property even if the state equation (2.1a) has an additional additive term which is a zero-mean white noise sequence. It turns out that both refinement schemes lead to the same unique solution, which is the one obtained by solving in retrograde time a sequence of static games—in the spirit of dynamic programming. Such a special closed-loop saddle-point solution is known as a *feedback* saddle-point solution, which is given in Theorem 3.1 below. Toward this end, we first introduce a sequence of matrices, $\{M_k, k \in \boldsymbol{K}\}$, generated recursively by

$$M_k = Q_k + A_k^T M_{k+1} \Lambda_k^{-1} A_k; \qquad M_{K+1} = Q_{K+1} \quad (3.3)$$

where

$$\Lambda_k := I + (B_k B_k^T - \gamma^{-2} D_k D_k^T) M_{k+1}. \quad (3.4)$$

Then, we have the following theorem.

Theorem 3.1: For the two-person zero-sum dynamic game with closed-loop information structure, as formulated above

i) there exists a unique feedback saddle-point solution if and only if

$$\Xi_k := \gamma^2 I - D_k^T M_{k+1} D_k > 0, \qquad k \in \boldsymbol{K} \quad (3.5)$$

where the sequence of nonnegative definite matrices M_{k+1}, $k \in \boldsymbol{K}$, is generated by (3.3);

ii) under condition (3.5), the matrices Λ_k, $k \in \boldsymbol{K}$, are invertible, and the unique feedback saddle-point policies are

$$u_k^* = \mu_k^*(x_k) = -B_k^T M_{k+1} \Lambda_k^{-1} A_k x_k \quad (3.6)$$

$$w_k^* = \upsilon_k^*(x_k) = \gamma^{-2} D_k^T M_{k+1} \Lambda_k^{-1} A_k x_k, \qquad k \in \boldsymbol{K} \quad (3.7)$$

with the corresponding unique state trajectory generated by the difference equation

$$x_{k+1}^* = \Phi_k^* x_k^* \equiv \Lambda_k^{-1} A_k x_k^*, \qquad x_1^* = x_1 \quad (3.8)$$

and the saddle-point value is

$$L^* = x_1^T M_1 x_1; \quad (3.9)$$

iii) if the matrix Ξ_k in (3.5) has a negative eigenvalue for some $k \in \boldsymbol{K}$, then the game does not admit a saddle point under any information structure, and its upper value becomes unbounded.

Proof: We prove here only part iii), since it is so important for the development to follow. Proofs for other parts of the theorem can be found in [17, p. 257] where the basic tool is dynamic programming, which involves the solution of a sequence of quadratic games in retrograde time. For each k, existence of a unique saddle point to the corresponding static game is guaranteed by the positive definiteness of Ξ_k, and if there is some $\bar{k} \in \boldsymbol{K}$ such that $\Xi_{\bar{k}}$ has a negative eigenvalue, then the corresponding static game in the sequence does not admit a saddle point, and being a quadratic game this implies that the upper value of the game (which then is different from the lower value) is unbounded. If Ξ_k has a zero (but no negative) eigenvalue, then whether the

corresponding game admits a saddle point or not depends on the precise value of the initial state x_1, and, in particular, if $x_1 = 0$ one can allow Ξ_k to have a zero eigenvalue and still preserve the existence of a saddle point. Since the "zero-eigenvalue case" can be recovered as the limit of the "positive-eigenvalue case," we will henceforth not address the former.

Now, contrary to the statement of iii), suppose that there exists a policy for Player 1, say $\{\hat{\mu}_k \in \boldsymbol{M}_k,\ k \in \boldsymbol{K}\}$, under which the cost function L is bounded for all $w_{[1,K]}$, even though Ξ_k has a negative eigenvalue for some $k \in \boldsymbol{K}$; clearly, this policy cannot be the feedback policy (3.6). Let $\bar{k}$ be the largest integer in $\boldsymbol{K}$ for which Ξ_k has a negative eigenvalue, and $\Xi_{\bar{k}+1} > 0$. Furthermore, let a policy for Player 2 be chosen as

$$\hat{v}_k(x_{[1,k]}) = \begin{cases} 0 & \text{for } k < \bar{k} \\ \hat{w}_k & \text{for } k = \bar{k} \\ v_k^*(x_k) & \text{for } k > \bar{k} \end{cases}$$

where $\hat{w}_{\bar{k}}$ is (at this point) an arbitrary element of $W_{\bar{k}}$, and v_k^* is as defined by (3.7). Denote the state trajectory corresponding to the pair $(\hat{\mu}_{[1,K]}, \hat{v}_{[1,K]})$ by $\hat{x}_{[1,K]}$, and the corresponding *open-loop values* of the two policies by the two players by $\hat{u}_{[1,K]}$ and $\hat{w}_{[1,K]}$. Finally, we introduce the notation $L^{(k)}(u_{[k+1,K]}, w_{[k,K]}; \hat{x}_k)$ to denote the kernel of a linear-quadratic game, formulated in exactly the same way as the original game, with the only difference being that it is defined on the subinterval $[k, K]$ and has the initial state $\hat{x}_k$. Then, we have the following sequence of equalities and inequalities:

$$\begin{aligned}
&\sup_{w_{[1,K]}} L(\hat{\mu}_{[1,K]}, w_{[1,K]}) \\
&\quad \geq L(\hat{\mu}_{[1,K]}, \hat{v}_{[1,K]}) \\
&\quad = L^{(\bar{k}+1)}(\hat{\mu}_{[\bar{k}+1,K]}, v^*_{[\bar{k}+1,K]}; \hat{x}_{\bar{k}+1}) \\
&\qquad - \gamma^2 |\hat{w}_{\bar{k}}|^2 + \sum_{k=1}^{\bar{k}} \{|\hat{x}_k|^2_{Q_k} + |\hat{u}_k|^2\} \\
&\quad \geq L^{(\bar{k}+1)}(\mu^*_{[\bar{k}+1,K]}, v^*_{[\bar{k}+1,K]}; \hat{x}_{\bar{k}+1}) \\
&\qquad + |\hat{u}_{\bar{k}}|^2 - \gamma^2 |\hat{w}_{\bar{k}}|^2 + c_{\bar{k}} \\
&\quad = |\hat{x}_{[\bar{k}+1]}|^2_{M_{\bar{k}+1}} + |\hat{u}_{\bar{k}}|^2 - \gamma^2 |\hat{w}_{\bar{k}}|^2 + c_{\bar{k}}
\end{aligned}$$

where $c_{\bar{k}}$ is some nonnegative constant (determined only by the initial state x_1 and Player 1's control $\hat{u}_{[1,\bar{k}-1]}$). In the above, the inequality in the first line follows because $\hat{v}$ is not necessarily a maximizing policy for Player 2; the equality of the next line follows because v^* and $\hat{v}$ agree (by construction) on the subinterval $[\bar{k}+1, K]$; the inequality of the third line follows from the fact that the pair $(\mu^*_{[\bar{k}+1,K]}, v^*_{[\bar{k}+1,K]})$ provides a saddle-point solution to the truncated game with kernel $L^{(\bar{k}+1)}$ (this is true because the feedback saddle-point solution of the original game is *strongly time consistent*); and finally the equality of the last line follows from (3.9) with the subindex "$_1$" replaced by "$_{\bar{k}+1}$". Since, by hypothesis, $\Xi_{\bar{k}+1}$ has a negative eigenvalue, the function

$$\hat{x}^T_{\bar{k}+1} M_{\bar{k}+1} \hat{x}_{\bar{k}+1} - \gamma^2 \hat{w}^T_{\bar{k}} \hat{w}_{\bar{k}}$$

is not bounded above (in $\hat{w}_{\bar{k}}$), and since the choice of $\hat{w}_{\bar{k}}$ was arbitrary this shows that the kernel $L(\hat{\mu}_{[1,K]}, w_{[1,K]})$ can be made arbitrarily large—a contradiction to the initial hypothesis that the upper value of the game was bounded. ◇

Several important remarks are now in order, to place the various statements of the aforementioned theorem into proper perspective.

Remark 3.1: An important implication of the last statement of the theorem, just proved, is the property that under the closed-loop information pattern the feedback saddle-point solution requires the least stringent existence conditions; in other words, if the matrix in (3.5) has a negative eigenvalue for some $k \in \boldsymbol{K}$, then a saddle point will not exist (and the upper value will be unbounded) even if Player 1's policy is allowed to depend on past values of the state. It then follows from the *ordered interchangeability* property that saddle-point solutions under other information structures (provided that these saddle points exist) can be constructed from the feedback saddle-point solution (3.6)–(3.7) on the state trajectory (3.8). In other words, every other saddle-point solution of the original game is a *representation* of the feedback solution on the saddle-point trajectory. In mathematical terms, any closed-loop policy $u_k = \mu_k(x_{[1,k]})$, $k \in \boldsymbol{K}$, satisfying the following two side conditions, constitutes a saddle-point solution along with (3.7):

$$\begin{aligned}
&\text{i)}\quad \mu_k(x^*_{[1,k]}) = -B_k^T M_{k+1} \Lambda_k^{-1} A_k x_k^*, \qquad k \in \boldsymbol{K}. \\
&\text{ii)}\quad L(\mu_{[1,K]}, w_{[1,K]}) \text{ is strictly concave in } w_{[1,K]}.
\end{aligned} \tag{3.10}$$

Side condition ii) above leads, in general, (depending on μ_k, $k \in \boldsymbol{K}$) to a more (and never less) stringent condition than (3.5), an example in point being the open-loop solution $u_k^* = \mu_k^*(x_k^*)$, $k \in \boldsymbol{K}$, for which the "strict concavity" condition can be shown to be more restrictive than (3.5) [17, p. 258].

Remark 3.2: If the information available to Player 2 is not closed loop (or feedback), but rather open loop, the statement of the theorem remains equally valid, with only x_k in (3.7) replaced by x_k^* which is generated by (3.8). In fact, quite parallel to the statement of Remark 3.1, under a general closed-loop pattern for the maximizer, any closed-loop policy $w_k = v(x_{[1,k]})$, $k \in \boldsymbol{K}$, for the maximizer, that satisfies the following two side conditions, would be in saddle-point equilibrium with (3.6):

$$\text{i)}\quad \nu_k(x^*_{[1,k]}) = \gamma^{-2} D_k^T M_{k+1} \Lambda_k^{-1} A_k x_k^*, \qquad k \in \boldsymbol{K} \tag{3.11}$$

i.e., ν should be a *representation* of ν^* on the equilibrium trajectory (3.8);

ii) $L(u_{[1,K]}, \nu_{[1,K]})$ is strictly convex in $u_{[1,K]}$.

Remark 3.3: If Player 1 has the expanded information

structure where he is allowed to observe (in addition to the state) the current value of the control of Player 2 (i.e., the disturbance), then the condition of Theorem 3.1 can be relaxed, since (intuitively) this gives informational advantage to the minimizer. The problem then is a special type of a feedback Stackelberg game [17, p. 314], where the derivation of the saddle-point solution can be done recursively, with the difference from the one of Theorem 3.1 being that now the *max-min value* of each static game is considered in the recursion (instead of the *minimax* or *saddle-point value*). The resulting solution is both *strongly time consistent* and *noise insensitive*, and is given for Player 1 by

$$\tilde{u}_k = \tilde{u}_k(x_k, w_k) = -\left(I + B_k^T M_{k+1} B_k\right)^{-1} \cdot B_k^T M_{k+1}(D_k w_k + A_k x_k) \qquad k \in \boldsymbol{K} \quad (3.12a)$$

under the existence condition

$$\gamma^2 I - D_k^T\left(I + M_{k+1} B_k B_k^T\right)^{-1} M_{k+1} D_k > 0, \qquad k \in \boldsymbol{K} \quad (3.12b)$$

which is equivalent to the condition that the matrix Λ_k defined by (3.4) has only positive eigenvalues, for all $k \in \boldsymbol{K}$. The saddle-point controller for Player 2 is still given by (3.7), and the pair of policies (3.12a)–(3.7) constitutes the *unique* strongly time consistent and noise-insensitive saddle-point solution, provided that condition (3.12b) is satisfied. Conversely, if the matrix in (3.12b) has a negative eigenvalue for at least one $k \in \boldsymbol{K}$, the upper value of the game under this asymmetric information (which is also the lower value) becomes unbounded. It is worth noting that since condition (3.5) implies nonsingularity of Λ_k, and not vice versa, the increase in information[3] to Player 1 has led to a less stringent existence condition (as expected).

Remark 3.4: As we have stated prior to the statement of Theorem 3.1, the feedback saddle-point solution (3.6)–(3.7) is also *noise insensitive*, and it is the *unique* such solution. Toward exploring this point in somewhat more detail, consider a system described by a stochastic state equation which is (2.1a) with an additional additive independent white driving noise $\{\theta_k,\ k \in \boldsymbol{K}\}$ with zero-mean, positive definite covariance, and with a probability measure assigning positive probability to every open subset of the Euclidean space where θ_k belongs.[4] For this stochastic dynamic game, and under closed-loop information structure for both players, the solution provided in Theorem 3.1 constitutes the unique saddle-point solution. The existence (and uniqueness) condition is still (3.5), but the expected saddle-point value (the counterpart of (3.9)) now has an additional positive additive term linear in $\operatorname{cov}(\theta_k)$, $k \in \boldsymbol{K}$. If, however, Player 2 has only open-loop information, (3.6) is no longer a minimax controller for Player 1 (in a sense, he can do better). The unique minimax controller associated with the *pure-strategy* saddle point is a particular representation of (3.6)

$$\tilde{\mu}_k(x_k, x_1) = \mu_k^*(x_k^*) + P_k\left[x_k - x_k^*\right] \quad (3.13)$$

where x_k^* is generated by (3.8), and P_k, $k \in \boldsymbol{K}$, is a unique matrix sequence defined by

$$P_k = -B_k^T Z_{k+1}\left[I + B_k B_k^T Z_{k+1}\right]^{-1} A_k \quad (3.14a)$$

$$Z_k = Q_k + A_k^T Z_{k+1}\left[I + B_k B_k^T Z_{k+1}\right]^{-1} A_k; \qquad Z_{K+1} = Q_{K+1}. \quad (3.14b)$$

(Note that P_k is the gain matrix in (3.6) with $D_k \equiv 0$.) The condition that replaces (3.5) of Theorem 3.1 in this case is the strict concavity of $L(\tilde{u}_{[1,K]}, w_{[1,K]})$ in $w_{[1,K]}$, which is

$$\gamma^2 I - D_k^T S_{k+1} D_k > 0, \qquad k \in \boldsymbol{K} \quad (3.15)$$

where S_k is defined recursively by

$$S_k = Q_k + P_k^T P_k + \hat{A}_k^T S_{k+1} \hat{A}_k + \hat{A}_k^T S_{k+1} D_k \cdot \left[\gamma^2 I - D_k^T S_{k+1} D_k\right]^{-1} D_k^T S_{k+1} \hat{A}_k; \qquad S_{K+1} = Q_{K+1} \quad (3.16a)$$

and

$$\hat{A}_k := A_k + B_k P_k. \quad (3.16b)$$

If condition (3.15) is not satisfied, there may still exist a minimax controller, but this time associated with a *mixed-strategy* saddle point, with the maximizer's saddle-point policy being a random sequence. In fact, one can push the region of existence of a minimax controller to that defined by (3.5), by using linear control laws in the form

$$\mu_k(x_{[1,k]}) = \sum_{i=1}^{k} P_i^k x_i, \qquad k \in \boldsymbol{K}. \quad (3.17)$$

If $\{\theta_k\}$ is a Gaussian sequence, then the worst-case disturbance (maximizer's policy) is a correlated Gaussian sequence, conditioned on x_1, and the control law (3.17) forms an equilibrium with that Gaussian sequence. A complete verification of this result and derivation of precise expressions for $\{P_i^k\}$, are too involved to be included here;[5] these details are also not essential for the development to follow. ◇

B. Closed-Loop 1-Step Delay Information for Both Players

Another possible realistic information structure of Player 1 is the one where the control is allowed to depend on the state with a delay of one time unit. Endowing Player 2 also with a similar information structure, the permissible policies for the two players will now be

$$\mu_k(x_{[1,k-1]}), \qquad k \in \boldsymbol{K}; \ \nu_k(x_{[1,k-1]}), \qquad k \in \boldsymbol{K}.$$

In this case, for the original deterministic game the saddle-point solution is again not unique. However, requiring the solution to be additive noise insensitive leads to a unique

[3] In game-theoretic terminology, this new information structure does not permit simultaneous play, and in the context of the disturbance attenuation problem treated in this paper, it gives an unnatural (nonimplementable) advantage to the controller.

[4] The Gaussian distribution would satisfy this latter requirement.

[5] They can be found in [20]; see also Section II-C for an illustrative example.

solution by using Theorem 3.1, the *ordered interchangeability* property, and a dynamic programming type argument. The solution is essentially a particular representation of the feedback solution (3.6)–(3.7) on the state trajectory (3.8). Toward deriving this representation, we first note that if the control policies were restricted to depend only on the most recently available value of the state, the unique saddle-point solution would be given by (using the notation of Theorem 3.1)[6]

$$\begin{aligned} u_k &= \hat{\mu}_k(x_{k-1}) \\ &= -B_k^T M_{k+1}\Lambda_k^{-1}A_k\Lambda_{k-1}^{-1}A_{k-1}x_{k-1}, \qquad k>1 \\ &= -B_1^T M_2\Lambda_1^{-1}A_1 x_1, \qquad k=1 \end{aligned} \tag{3.18a}$$

$$\begin{aligned} w_k &= \hat{\nu}_k(x_{k-1}) \\ &= \gamma^{-2}D_k^T M_{k+1}\Lambda_k^{-1}A_k\Lambda_{k-1}^{-1}A_{k-1}x_{k-1}, \qquad k>1 \\ &= \gamma^{-2}D_1^T M_2\Lambda_1^{-1}A_1 x_1, \qquad k=1 \end{aligned} \tag{3.18b}$$

which is a valid solution provided that the cost function $L(\hat{\mu}_{[1,K]}, w_{[1,K]})$ is strictly concave in $w_{[1,K]}$. It is not difficult to see that this is not a noise-insensitive solution. Now, to construct one (in fact, the unique one) that is noise insensitive (i.e., solves the stochastic game of Remark 3.4) and requires the least stringent existence condition, we rewrite (3.6) as

$$u_k = -B_k^T M_{k+1}\Lambda_k^{-1}A_k \cdot \underbrace{\left(A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + D_{k-1}w_{k-1}\right)}_{\xi_k}$$

where ξ_k will have to be expressed in terms of not only x_{k-1}, but also $x_{[1,k-2]}$, through u_{k-1} and w_{k-1}. Likewise, (3.7) is rewritten as

$$w_k = \gamma^{-2}D_k^T M_{k+1}\Lambda_k^{-1}A_k\xi_k$$

in view of which, for all $k \in \boldsymbol{K}$

$$\begin{aligned} \xi_{k+1} &= A_k x_k - \left(B_k B_k^T - \gamma^{-2}D_k D_k^T\right) \cdot M_{k+1}\Lambda_k^{-1}A_k\xi_k, \qquad \xi_1 = x_1 \\ \xi_{k+1} &= A_k x_k + \left(\Lambda_k^{-1} - I\right)A_k\xi_k, \qquad \xi_1 = x_1. \end{aligned} \tag{3.19}$$

This is an n-dimensional compensator, the output of which at stage k replaces x_k in (3.6)–(3.7). The condition for the resulting set of policies to be in saddle-point equilibrium is strict concavity of $L(u_{[1,K]}, w_{[1,K]})$ in $w_{[1,K]}$ when $u_{[1,K]}$ is given by

$$u_k^* = \mu_k^*(\xi_k) = -B_k^T M_{k+1}\Lambda_k^{-1}A_k\xi_k, \qquad k \in \boldsymbol{K}. \tag{3.20}$$

Let $\zeta_k := (x_k^T, \xi_k^T)^T$, which is generated by

$$\zeta_{k+1} = \tilde{A}_k\zeta_k + \tilde{D}_k w_k, \qquad \zeta_1 = \left(x_1^T, x_1^T\right)^T \tag{3.21a}$$

where

$$\tilde{A}_k := \begin{pmatrix} I & -B_k B_k^T M_{k+1}\Lambda_k^{-1} \\ I & \Lambda_k^{-1} - I \end{pmatrix} A_k; \quad \tilde{D}_k := \begin{pmatrix} D_k \\ 0 \end{pmatrix}. \tag{3.21b}$$

In terms of ζ_k, $k \in \boldsymbol{K}$, $L(u^*, w)$ can be written as

$$L(u^*, w) = \sum_{k=1}^{K}\left\{|\zeta_{k+1}|^2_{\tilde{Q}_{k+1}} - \gamma^2|w_k|^2\right\} \tag{3.21c}$$

where

$$\begin{aligned} \tilde{Q}_{K+1} &= \operatorname{diag}(Q_{K+1}, 0); \\ \tilde{Q}_k &= \operatorname{diag}\left(Q_k, A_k^T(\Lambda_k^{-1})^T M_{k+1}B_k B_k^T M_{k+1}\Lambda_k^{-1}A_k\right), \\ &\qquad k \in \boldsymbol{K}. \end{aligned} \tag{3.21d}$$

The condition for strict concavity of (3.21c), in $w_{[1,K]}$, is

$$\gamma^2 I - \tilde{D}_k^T\tilde{S}_{k+1}\tilde{D}_k > 0, \qquad k \in \boldsymbol{K} \tag{3.22a}$$

where $\tilde{S}_{k+1}$, $k \in \boldsymbol{K}$, is generated by

$$\begin{aligned} \tilde{S}_k &= \tilde{Q}_k + \tilde{A}_k^T\tilde{S}_{k+1}\tilde{A}_k \\ &\quad + \tilde{A}_k^T\tilde{S}_{k+1}\tilde{D}_k\left[\gamma^2 I - \tilde{D}_k^T\tilde{S}_{k+1}\tilde{D}_k\right]^{-1}\tilde{D}_k^T\tilde{S}_{k+1}\tilde{A}_k; \\ \tilde{S}_{K=1} &= \tilde{Q}_{K+1}. \end{aligned} \tag{3.22b}$$

We are now in a position to state the following theorem.

Theorem 3.2: For the linear-quadratic zero-sum dynamic game with closed-loop 1-step delayed state information for both players the following hold.

i) Under condition (3.22a), there exists a unique noise-insensitive saddle-point solution, given by (3.6)–(3.7) with x_k replaced by ξ_k which is generated by (3.19). The saddle-point value is again given by (3.9).

ii) If the matrix in (3.22a) has a negative eigenvalue for at least one $k \in \boldsymbol{K}$, then the game does not admit a saddle point, and its upper value is unbounded.

Proof: The saddle-point property follows because the given pair is a particular representation of the feedback saddle-point solution (3.6)–(3.7) on the saddle-point trajectory, and the upper value is bounded under (3.22a), as discussed in the construction above. The noise insensitivity and uniqueness in this class follow from an alternative recursive derivation of the solution using a dynamic programming type argument. The proof of part ii), on the other hand, is a proof by contradiction, paralleling that of Theorem 3.1 iii). ◇

Remark 3.5: Note that, as compared to the closed-loop information case, here we need in addition (to (3.3)) the solution of a ($2n$-dimensional)[7] Riccati equation (3.22b) to check the condition of existence. It follows from Theorem 3.1 iii) that condition (3.22a) implies condition (3.5), so that under the former the matrix Λ_k is invertible for all $k \in \boldsymbol{K}$.

We also note that if Player 2 has access to full state information (and Player 1 still having 1-step delayed information) condition (3.22a) is still operative, and the saddle-point

[6]This result follows readily from Remark 3.1.

[7]Henceforth n denotes the dimension of the state vector x.

solution given in the Theorem remains a saddle point (though no longer noise insensitive). In this case also the given saddle-point solution requires the least stringent condition among the infinitely many saddle-point solutions to the problem.

C. An Illustrative Example

To illustrate the results of Theorems 3.1 and 3.2 and some features outlined in Remarks 3.1–3.5, we now consider a scalar version of the dynamic game problem, where the state dynamics are

$$x_{k+1} = x_k + u_k + w_k, \qquad k + 1, 2, \cdots, \tag{3.23a}$$

and the objective function is

$$L = \sum_{k=1}^{K} \left[x_{k+1}^2 + u_k^2 + r w_k^2 \right] \tag{3.23b}$$

where $r = \gamma^2 > 0$ is taken as a variable.

i) With $K = 3$, condition (3.5) reads

$$r > 1, \quad r > 1 + \frac{1}{\sqrt{2}} \approx 1.7071, \quad r > 1.9275^{8}$$

which implies that the problem admits a unique saddle point in feedback (FB) policies if and only if $r > 1.9275$. One can show that as $K \to \infty$, the various constraints on r become nested, converging to $r > 2$.

Again, for $K = 3$, the minimax (saddle-point) controllers are

$$u_3 = \mu_3^*(x_3) = -\frac{r}{2r-1} x_3;$$

$$u_2 = \mu_2^*(x_2) = -\frac{r(3r-1)}{5r^2 - 5r + 1} x_2;$$

$$u_1 = \mu_1^*(x_1) = \frac{r(8r^2 - 6r + 1)}{13r^3 - 19r^2 + 8r - 1} x_1 \tag{3.24}$$

and the FP saddle-point policy for the maximzier is computed using the formula

$$w_k = \nu_k^*(x_k) = -\frac{1}{r} \mu_k^*(x_k). \tag{3.25}$$

The corresponding state trajectory is

$$x_2^* = \frac{5r^3 - 5r^2 + r}{13r^3 - 19r^2 + 8r - 1} x_1$$

$$x_3^* = \frac{r(2r-1)}{5r^2 - 5r - 1} x_2^*; \qquad x_4^* = \frac{r}{2r-1} x_3^*. \tag{3.26}$$

For comparison purposes, let us now determine the condition (on r) for the game to have a bounded upper value. In view of Remark 3.3, and particularly condition (3.12b), we have

$$r > \frac{1}{2}, \quad r > \frac{5 + \sqrt{5}}{10} \approx 0.7236, \quad r > 0.8347^{9}$$

[8]This bound is the largest root of the polynomial $5s^3 - 13s^2 + 7s - 1$.

[9]This bound is the largest root of the polynomial $13s^3 - 19s^2 + 8s - 1$.

and hence the lower (max–min) value of the game with $K = 3$ is unbounded if $r < 0.8347$, and it is bounded if $r > 0.8347$. This condition is clearly less stringent than that of boundedness of the upper (minimax) value.

ii) Let $K = 2$. Then the set of all saddle-point minimax controllers in the linear class is given by

$$u_1 = \mu_1(x_1) = -\frac{r(3r-1)}{5r^2 - 5r + 1} x_1 \tag{3.27a}$$

$$u_2 = \mu_2(x_2) = -\frac{r}{2r-1} x_2 + P\left(x_2 - \frac{r(2r-1)}{5r^2 - 5r + 1} x_1 \right) \tag{3.27b}$$

where P is a scalar satisfying the concavity condition (see Remark 3.1):

$$P^2 < \frac{(r-1)(2r^2 - 4r + 1)}{(2r-1)^2}. \tag{3.28}$$

As r approaches the boundary of the region associated with FB solution (i.e., $r \to 1 + 1/\sqrt{2}$), the R.H.S. of (3.28) goes to zero, implying that the FB solution of Theorem 3.1 indeed provides the least stringent condition on the parameter r.

iii) Now consider the stochastic version, with the state dynamics replaced by

$$x_{k+1} = x_k + u_k + w_k + \theta_k \qquad k = 1, 2, \cdots, \tag{3.29}$$

where $\{\theta_k\}$ is some zero-mean independent random sequence of the type described in Remark 3.4. For $K = 3$, the unique closed-loop saddle-point solution is again given by (3.24)–(3.25), under the condition $r > 1.9275$. If $K = 2$, then the condition becomes $r > 1 + 1/\sqrt{2} \approx 1.7071$. For this case, if the maximizer instead has only open-loop information (i.e., only the value of x_1), the relevant condition (for pure-strategy saddle point) (3.15) dictates $r > \frac{5}{4}(1 + 1/\sqrt{5}) \approx 1.809017$, under which the unique minimax controller is (using (3.13))

$$u_1 = \tilde{\mu}_1(x_1) = -\frac{r(3r-1)}{5r^2 - 5r + 1} x_1;$$

$$u_2 = \tilde{\mu}_2(x_2, x_1) = -\frac{1}{2} x_2 - \frac{r}{2(5r^2 - 5r + 1)} x_1. \tag{3.30}$$

Note that this is identical to (3.27a)–(3.27b), with P chosen as $P = 1/(4r - 2)$. Now, the gap between 1.809017 and 1.7071 can be filled, though by choosing another control law (to replace (3.30)) which does not correspond to (3.27a)–(3.27b). For $r \in (1 + 1/\sqrt{2}, \frac{5}{4}(1 + 1/\sqrt{5})]$, and with θ_1 and θ_2 taken as independent zero-mean Gaussian random variables with variances σ_1 and σ_2, respectively, it can be shown [20] that the saddle-point choice by Player 2 is the (conditionally) Gaussian random sequence[10]

$$w_2 = \frac{1}{r-1} \left[\left((1 + P)(1 + N) + S \right) x_1 + (1 + P) w_1 \right]$$

[10]For a related derivation and verification see [21].

$$w_1 \sim N(m, \lambda); \qquad \lambda := \frac{(r-1)(2P+1)\sigma_1}{1+P+(r-1)(2P+1)}$$

where m, N, S are some scalars (whose expressions are not given here), and

$$P = \frac{-r + \sqrt{r-1}\sqrt{2r^2 - 4r + 1}}{2r-1}.$$

The unique minimax controller for Player 1 that is in saddle-point equilibrium with this Gaussian sequence is

$$u_1 = \tilde{\mu}_1(x_1) = Nx_1; \qquad u_2 = \tilde{\mu}_2(x_2, x_1) = Px_2 + Sx_1.$$

It is important (and interesting) to note that $\lambda > 0$ for all $r \in (1 + 1/\sqrt{2}, \frac{5}{4}(1 + 1/\sqrt{5}))$, and is zero at the upper boundary, thus maintaining continuity with the pure-strategy saddle-point solution across the boundary. One can actually show that P, S, and N are all continuous across the boundary when compared to the corresponding gain coefficients in (3.30). We do not provide details of these features here, since the underlying issues are considerably technical and they are not essential for the main results of this paper.

iv) To illustrate the result of Theorem 3.2, we take $K = 4$, and compute condition (3.22a) to obtain

$$r > 4.87659$$

while condition (3.5) in this case dictates

$$r > 1.98218.$$

Note the degradation in the concavity condition due to loss in the information available to the players (particularly, Player 1). For comparison purposes, let us also compute the bound on r under the simpler (no-memory) controller (3.19a)

$$r > 6.31768$$

which is (as expected) the most stringent of the three.

IV. Solution to the Disturbance Attenuation Problem

A. General Closed-Loop Information

We now return to the disturbance attenuation problem formulated in Section II with the performance index (2.8), and see how the results of the previous section can directly be applied to obtain its solution. Toward this end, let us first note that in view of (3.9), and under condition (3.5)

$$L(\mu^*_{[1,K]}, w_{[1,K]}) \le L(\mu^*_{[1,K]}, \nu^*_{[1,K]}) = L^* = x_1^T M_1 x_1 \tag{4.1a}$$

with the inequality holding for all l^2 sequences $w_{[1,K]} := (w_1, \cdots, w_K)$. Using the relationship (3.1) between L and J, the inequality (4.1a) can equivalently be written as

$$J(\mu^*_{[1,K]}, w_{[1,K]}) \le x_1^T(M_1 - Q_1)x_1 + \gamma^2 \sum_{k=1}^{K} |w_k|^2 \tag{4.1b}$$

and letting $|x_1| \to 0$ we arrive at

$$J(\mu^*_{[1,K]}, w_{[1,K]}) \le \gamma^2 \sum_{k=1}^{K} |w_k|^2 \tag{4.2a}$$

$\Leftrightarrow$

$$T(\mu^*_{[1,K]}, w_{[1,K]}) \le \gamma^2 \tag{4.2b}$$

for all l^2 sequences $w_{[1,K]}$.

Hence, given any γ satisfying condition (3.5) of Theorem 3.1, the control law $\mu^*_{[1,K]}$ given by (3.6), which depends on this value of γ, delivers the desired bound (4.2b). Introduce the set

$$\Gamma := \{\gamma > 0: \quad \text{condition (3.5) holds}\}$$

which is a nonempty open set because the condition holds for γ sufficiently large, and if a particular γ is a permissible choice then $\gamma - \epsilon$ also is, for $\epsilon > 0$ sufficiently small. Let $\hat{\gamma}$ be the infimum of Γ, that is

$$\hat{\gamma} := \inf\{\gamma : \gamma \in \Gamma\}.$$

Clearly, by Theorem 3.1 iii), there does not exist any closed-loop controller which will deliver a bound in (4.2b) that is smaller than $\hat{\gamma}^2$, and hence

$$\gamma^* = \hat{\gamma}$$

where the former was defined by (2.5), with $\tilde{J}$ in (2.6) replaced by (2.8). Now let the control gain in (3.6) be denoted by $G_k(\gamma)$, to explicitly show the dependence of the minimax control law on the parameter γ. We first have the following useful Lemma.

Lemma 4.1: Let $\{\gamma^{(n)}, n = 1, 2, \cdots,\}$ be any decreasing sequence of positive numbers, with limit point $\hat{\gamma}$. Then, for each $k \in \boldsymbol{K}$, the sequence of matrix functions $\{G_k(\gamma^{(n)}), n = 1, 2, \cdots,\}$ is right continuous at $\hat{\gamma}$.

Proof: All we need to show is that the control gain $G_k(\gamma)$ is bounded as $\gamma \downarrow \hat{\gamma}$. For this, we need only consider the case where $\Lambda_k(\hat{\gamma})$ is singular for at least one $k \in \boldsymbol{K}$, say $\bar{k}$, which will occur only if

$$\hat{\gamma}^2 I - D_k^T M_{k+1} D_k \ge 0, \qquad k = \bar{k} \tag{4.3}$$

with at least one zero eigenvalue. With $N(\cdot)$ denoting the "null space" operator, it follows from (3.4) and (3.5) that

$$N(\Lambda_{\bar{k}}) = N(B_{\bar{k}}^T M_{\bar{k}+1}) \cap N(I - \hat{\gamma}^{-2} D_{\bar{k}} D_{\bar{k}}^T M_{\bar{k}+1})$$

which actually holds not only at $\gamma = \hat{\gamma}$, but also for $\gamma > \hat{\gamma}$. This then implies that the matrix

$$B_{\bar{k}}^T M_{\bar{k}+1} \Lambda_{\bar{k}}^{-1}$$

(with the inverse on $\Lambda_{\bar{k}}$ interpreted in a limiting sense) is bounded for all $\gamma > \hat{\gamma}$, and also in the limit as $\gamma \downarrow \hat{\gamma}$. This proves the right continuity of $\{G_{\bar{k}}(\gamma)\}$ at $\gamma = \hat{\gamma}$. ◇

Now, we are in a position to present the following result.

Theorem 4.1: Let $\hat{\gamma} \ge 0$ be the scalar defined above (as the infimum of Γ), and $\mu^*_{\hat{\gamma}} := \mu^*_{[1,K]}$ be the feedback control law (3.6) with $\gamma = \hat{\gamma}$. Then, for the finite-horizon disturbance attenuation problem of Section II, with performance

index (2.8)

i) $$\inf_{\{\mu_k \in M_k^l,\ k \in K\}} \sup_{\{w_k \in W_k,\ k \in K\}} T(\mu_{[1,K]}, w_{[1,K]}) = \sup_{\{w_k \in W_k,\ k \in K\}} T(\mu_{\hat\gamma}^*, w_{[1,K]}) = \hat\gamma^2 \quad (4.4)$$

that is, the feedback controller $\mu_{\hat\gamma}^*$ solves the disturbance attenuation problem, and the optimum attenuation level γ^* exists and is equal to $\hat\gamma$.

ii) $$\inf_{\{\mu_k \in M_k,\ k \in K\}} \sup_{\|w\| \le b} J(\mu_{[1,K]}, w_{[1,K]}) = \sup_{\|w\| \le b} J(\mu_{\hat\gamma}^*, w_{[1,K]}) = \hat\gamma^2 b^2 \quad (4.5)$$

for every $b > 0$.

iii) Given any $\tilde\gamma > \hat\gamma$, the set of all linear control laws that satisfy the bound (2.7) with $\gamma = \hat\gamma$ is a subset of $\{M_k^l, k \in K\}$, consisting of those elements that satisfy the two side conditions i) and ii) of Remark 3.1, for $\gamma \in [\tilde\gamma, \hat\gamma]$.

Proof: The proof follows basically from Theorem 3.1, Remarks 3.1–3.2, Lemma 4.1, and the discussion in the opening paragraph of Section IV-A. We provide here a formal proof for the *necessity* part of the result, which is that the value in (4.4) [which is the upper value of the disturbance attenuation game] cannot be smaller than $\hat\gamma$. Suppose that, to the contrary, the value is $\tilde\gamma < \hat\gamma$. Then, there exists a controller $\{\tilde\mu_k \in M_k,\ k \in K\}$ such that for some $\epsilon > 0$ (sufficiently small, so that $\tilde\gamma^2 + \epsilon < \hat\gamma^2$)

$$\sup_{\{w_k \in W_k,\ k \in K\}} T(\tilde\mu_{[1,K]}, w_{[1,K]}) = \tilde\gamma^2 + \epsilon =: \tilde{\tilde\gamma}^2 < \hat\gamma^2$$

which is equivalent to the inequality

$$J(\tilde\mu_{[1,K]}, w_{[1,K]}) - \tilde{\tilde\gamma}^2 \sum_{k=1}^{K} |w_k|^2 \le 0 \quad \text{for all } w_{[1,K]} \in W_{[1,K]}.$$

This, however, is impossible by Theorem 3.1 iii) (see also Remark 3.1) because for $\gamma < \hat\gamma$ the upper value of the auxiliary (*soft-constraint*) game of Section III is unbounded. The fact that the bound is achieved under $\mu_{\hat\gamma}^*$ follows from the continuous dependence of the eigenvalues of a (symmetric) matrix on its entries.

Note that since the control policy space $\{M_k,\ k \in K\}$ also includes nonlinear controllers, use of a nonlinear controller cannot improve upon the attenuation bound $\hat\gamma$. ◇

Remark 4.1: Two aspects of the solution presented above are worth emphasizing. The first is that while in the continuous-time disturbance attenuation H_∞-optimal control) problem with perfect state measurements there is generally no optimum[11] (see, for example, [8], [6], [22], [23]), for the discrete-time version the existence of an optimal controller is always guaranteed (which we will shortly see to be the case also for the infinite-horizon version). The second point is the cautionary remark that the theorem does not attribute any (pure-strategy) saddle point to the disturbance attenuation problem; in other words, there is no claim that "inf" and "sup" operations in (2.4) can be interchanged. Even though the related (soft-constraint) dynamic game problem of Section III admitted a pure-strategy saddle point, the disturbance attenuation problem in fact does not admit one. What one can show, however, is that the minimax controller $\mu_{\hat\gamma}^*$ is in equilibrium with a "mixed" l^2 policy for the disturbance, which features a discrete distribution. A precise characterization of this worst case distribution is given later in the section, in Proposition 4.1. To motivate this result, we will first obtain (in the sequel) the worst case distribution for the two and three stage examples treated earlier in Section III-C. ◇

[11]The control gain generally becomes unbounded as $\gamma \downarrow \gamma^*$.

B. Illustrative Example (Continued)

Let us return to the example of Section III-C, treated now as a disturbance attenuation problem, with the objective function (3.23b) replaced by

$$J = \sum_{k=1}^{K} \left[x_{k+1}^2 + u_k^2 \right] \quad (4.6)$$

and with the initial state taken to be $x_1 = 0$.

i) With $K = 3$, the attenuation constant $\hat\gamma$ used in Theorem 4.1 is

$$\hat\gamma = \sqrt{1.9275} = 1.3883 \quad (4.7)$$

leading to the unique feedback minimax control law (from (3.25))

$$\mu_3^*(x_3) = -0.675131 x_3, \qquad \mu_2^*(x_2) = -0.927505 x_2, \qquad \mu_1^* = 0. \quad (4.8)$$

The corresponding mixed (worst-case) disturbance is obtained from (3.25) as follows: we first compute the limit of the open-loop saddle-point disturbance (Obtained by taking $x_1 \neq 0$) as $\gamma \downarrow \hat\gamma$:

$$w_3 = 0.100616 x_1, \quad w_2 = 0.249647 x_1, \quad w_1 = 0.518807 x_1.$$

This, of course, in itself is meaningless since for the problem at hand $x_1 = 0$. However, the ratios of w_3 and w_2 to w_1 are meaningful quantities, independent of x_1, and if we further normalize the total energy (i.e., b in (2.4)) to unity (without any loss of generality), we arrive at the following distribution of unit energy across stages:

$$w_3^+ = 0.172149, \quad w_2^+ = 0.427133, \quad w_1^+ = 0.887651. \quad (4.9)$$

Then, the worst-case disturbance (under unit energy constraint) that is in the saddle-point equilibrium with the minimax controller (4.8) is the mixed policy

$$w_k^* = \xi w_k^+, \qquad k = 1, 2, 3 \quad (4.10)$$

where ξ is a discrete random variable taking the values $+1$ and -1 with equal probability $1/2$, and w_k^+ is given by (4.9). This is unique up to a normalization constant, which is determined by the energy bound b.

ii) With $K = 2$, the value of γ^* is

$$\gamma^* = \sqrt{1 + (1/\sqrt{2})} \approx 1.30656 \tag{4.11}$$

and the corresponding (unique feedback) minimax controller is (from (3.27), with $P = 0$)

$$\mu_2^*(x_2) = -(1/\sqrt{2})x_2 \qquad \mu_1^* = 0. \tag{4.12}$$

The worst-case disturbance, computed according to the procedure outlined above, is

$$w_1^* = \frac{\xi}{2}\sqrt{2+\sqrt{2}}, \qquad w_2^* = \frac{\xi}{2}\sqrt{2-\sqrt{2}} \tag{4.13}$$

where ξ is as defined above. This solution also appears in [16] where a different (more direct) method was used.

iii) If we let $K \to \infty$ in the auxiliary game problem with perfect state information, we arrive at the stationary minimax controller

$$u_k = \mu^*(x_k) = \left(M(\gamma)\Big/\left[1 + \frac{\gamma^2 - 1}{\gamma^2}M(\gamma)\right]\right)x_k, \quad k = 1, 2, \cdots,$$

provided that $\gamma > \sqrt{2}$. For the disturbance attenuation problem with infinite horizon, this leads to the attenuation constant $\gamma^* = \sqrt{2}$ and to the steady-state minimax controller

$$u_k = \mu(x_k) = -x_k, \qquad k = 1, 2, \cdots. \tag{4.14}$$

For the auxiliary game, the pure-feedback saddle-point policy for Player 2, as $\gamma \downarrow \sqrt{2}$, is

$$w_k = \frac{1}{2}x_k, \qquad k = 1, 2, \cdots, \tag{4.15a}$$

which, together with (4.14), leads to the equilibrium state trajectory (for the game)

$$x_{k+1}^* = (1/2)x_k^*. \tag{4.15b}$$

We can now construct the mixed saddle-point policy for the maximizer in the disturbance attenuation problem, by using the earlier procedure. Using again a normalized (to unity) energy bound, we arrive at (from (4.15a) and (4.15b))

$$w_k^* = \xi/2^k, \qquad k = 1, 2, \cdots,$$

as the worst-case disturbance input.

C. A Least Favorable Distribution for the Disturbance

We are now in a position to present formally a worst-case (*least favorable*) probability distribution for the disturbance in the disturbance attenuation problem of Section II. Consider the performance index given by (2.6), with $\tilde{J}$ replaced by J given by (2.8)

$$T(\mu_{[1,K]}, w_{[1,K]}) := \begin{cases} J(\mu_{[1,K]}, w_{[1,K]})/\|w\|^2, & w_{[1,K]} \neq 0 \\ -\infty, & \text{else}. \end{cases} \tag{4.16}$$

Then, if $\{w_k^*, k \in \boldsymbol{K}\}$ is a least favorable random sequence, we should have

$$\min_{\mu_{[1,K]}} E\{T(\mu_{[1,K]}, w_{[1,K]}^*)\} = E\{T(\mu_{\hat{\gamma}}^*, w_{[1,K]}^*)\} = \hat{\gamma}^2 \tag{4.17}$$

where $E\{\cdot\}$ denotes the expectation operator under the probability measure of the least favorable random sequence, and $\mu_{\hat{\gamma}}^*$ is the optimum controller introduced in Theorem 4.1. In words, if the least favorable distribution were made known to the control designer (as *a priori* information), the optimal controller (that now minimizes the expected value of T) would still be the one given in Theorem 4.1. The following Proposition provides such a least favorable sequence.

Proposition 4.1: Let N be a nonnegative definite matrix defined by

$$N := M_2 + \hat{\gamma}^{-2}\sum_{k=2}^{K}\Phi_{k,2}^{*T}A_k^T\Lambda_k^{T^{-1}}M_{k+1}^T \cdot D_kD_k^TM_{k+1}\Lambda_k^{-1}A_k\Phi_{k,2}^* \tag{4.18a}$$

where $\Phi_{k,2}^*$ is defined by (using (3.8))

$$x_k^* = \Phi_{k+1}^*x_{k-1}^* =: \Phi_{k,2}^*x_2^* \tag{4.18b}$$

with γ taken to be equal the value $\hat{\gamma}$. Let D_1 have nonzero rank, and η be an eigenvector corresponding to a maximum eigenvalue of the matrix $D_1^TND_1$, where M_2 is generated by (3.3) with $\gamma = \hat{\gamma}$. Let ξ be a random vector taking the values η and $-\eta$ with equal probability $\frac{1}{2}$. Furthermore, let the random vector sequence $\{\xi_k, k \in \boldsymbol{K}\}$ be generated by

$$\xi_{k+1} = \Lambda_k^{-1}A_k\xi_k, \qquad \xi_2 = D_1\xi \tag{4.19}$$

where again we take $\gamma = \hat{\gamma}$. Then, a least favorable random sequence for the disturbance is given by

$$w_k^* = \hat{\gamma}^{-2}D_k^TM_{k+1}\Lambda_k^{-1}A_k\xi_k, \quad k = 2, \cdots, K; \qquad w_1^* = \xi \tag{4.20}$$

Proof: Even though $\{w_k^*, k \le K\}$ is a random sequence, it is highly correlated across stages, so that with $u_1 = 0$ we have $x_2 = \xi_2$, which makes the entire future values ξ_k known to the minimizer. In view of this, the optimality of μ_k^*, $k \ge 2$, follows from Theorem 3.1 with initial state x_2, and by taking the maximizer's policy to be open-loop. Let $n^* := \sum_{k=1}^{K}|w_k^*|^2$, which is independent of the sample path. Then

$$\begin{aligned} E\left\{T\left(\{\mu_k^*\}_{k=2}^K, u_1; \{w_k^*\}_{k=1}^K\right)\right\} &= \frac{1}{n^*}E\{x_2^TNx_2 + u_1^Tu_1\} \\ &= \frac{1}{n^*}E\{|B_1u_1 + D_1w_1^*|_N^2 + u_1^Tu_1\} \\ &= \frac{1}{n^*}E\{|D_1w_1^*|_N^2 + u_1^T(B_1^TNB_1 + I)u_1\} \end{aligned}$$

where the last step follows since w_1^* has a symmetric two-point distribution. Clearly, the unique minimizing solution is $u_1^* = 0$, which completes the proof as far as the optimality of

the controller goes. The fact that any one of the two sample paths of $\{w_k^*, k \le K\}$ maximizes $T(\mu_\gamma^*, w_{[1,K]})$ follows from the saddle-point property of the open-loop version of (3.7) (see Remark 3.2) together with the observation that under a fixed energy bound on the disturbance, w_1^* should maximize the quantity $|D_1 w_1|_N$. Clearly, if the matrix $D_1^T N D_1$ has maximum eigenvalues of multiplicity greater than one, the least favorable random sequence may not be unique (even under the hard energy bound, as in (2.4)). ◇

D. Optimum Controller Under the 1-Step Delay Information Pattern

The counterpart of Theorem 4.1 can be obtained under the 1-step delay information pattern for the controller, by using this time Theorem 3.2. Toward this end, let γ^* be the optimum attenuation level for the perfect-state information case (à la Theorem 3.1), and let Γ_{1D} be the set of all $\gamma \ge \gamma^*$ which further satisfy (3.22a). Let

$$\gamma^\circ := \inf\{\gamma : \gamma \in \Gamma_{1D}\}. \tag{4.21}$$

Then, following a reasoning similar to the one that led to Theorem 4.1, we arrive at the following result for the disturbance attenuation problem with 1-step delayed state measurements.

Theorem 4.2: Let $\gamma^\circ \ge 0$ be the scalar defined above, and assume that $\Lambda_k(\gamma^\circ)$ is nonsingular for all $k \in \boldsymbol{K}$.[12] Let $\mu_{\gamma^\circ}^\circ$ be the controller

$$\mu_k^\circ(\xi_k) = -B_k^T M_{k+1}(\gamma^\circ)\Lambda_k(\gamma^\circ)^{-1} A_k \xi_k, \qquad k \in \boldsymbol{K}, \tag{4.22a}$$

$$\xi_{k+1} = A_k x_k + \left(\Lambda_k(\gamma^\circ)^{-1} - I\right) A_k \xi_k, \qquad \xi_1 = 0. \tag{4.22b}$$

Then

$$\inf_{\{\mu_{[1,K]} \in \boldsymbol{M}_{CLD}\}} \sup_{\{w_k \in \boldsymbol{W}_k,\, k \in \boldsymbol{K}\}} T(\mu_{[1,K]}, w_{[1,K]}) = \sup_{\{w_k \in \boldsymbol{W}_k,\, k \in \boldsymbol{K}\}} T(\mu_{\gamma^\circ}^\circ, w_{[1,K]}) = (\gamma^\circ)^2 \tag{4.23}$$

where $\boldsymbol{M}_{CLD}$ is the class of all controllers adapted to the 1-step delay information pattern. ◇

Remark 4.2: If the controller were allowed to depend only on x_{k-1} at state k, and not also on $x_{[1,k-2]}$, then the only representation of the feedback controller (3.6) in this class would be

$$\begin{aligned} \mu_k(x_{k-1}) &= -B_k^T M_{k+1}\Lambda_k^{-1} A_k \Lambda_{k-1}^{-1} A_{k-1} x_{k-1}, & k \ge 2 \\ &= -B_1^T M_2 \Lambda_1^{-1} A_1 x_1, & k = 1 \end{aligned} \tag{4.24}$$

which however requires more stringent conditions on γ than the controller (4.22a). If the resulting performance level is acceptable, however, the advantage here is that the controller is static, while (4.22a) is dynamic (of the same order as that of the state). ◇

[12] This condition holds when $\gamma^\circ > \gamma^*$.

V. The Infinite-Horizon Case

The last result of the *illustrative* example of Section IV-B (i.e., the existence of a unique steady-state minimax controller, obtained as the limit of finite-horizon minimax controllers) prompts the question of whether a characterization of a stationary minimax controller can be obtained for the time-invariant version of the general disturbance attenuation problem of Section II. Said another way, the question is whether Theorem 4.1 admits a limiting case, as $K \to \infty$, when all the system matrices are time-invariant.[13] A further question is whether the resulting stationary controller is stabilizing. And a third question is whether the stabilizing stationary controller obtained as the limit of the finite-horizon optimal controller is indeed optimal for the infinite-horizon disturbance attenuation problem. The answers to all three questions are in the affirmative, under appropriate conditions on the system matrices, as to be established in the following.

For future reference, let us first write down the steady-state version of the feedback saddle-point solution of the auxiliary game problem covered by Theorem 3.1, along with the steady-state versions of (3.3)–(3.5). In all cases, the limiting values (as $K \to \infty$) are designated by an "overbar."

Saddle-point controllers:

$$\bar{\mu}_k = \bar{\mu}(x_k) = -B^T \overline{M}(\gamma)\overline{\Lambda}^{-1}(\gamma) A x_k, \qquad k \ge 1 \tag{5.1a}$$

$$\bar{w}_k = \bar{\nu}(x_k) = \gamma^{-2} D^T \overline{M}(\gamma)\overline{\Lambda}^{-1}(\gamma) A x_k, \qquad k \ge 1 \tag{5.1b}$$

$$\overline{\Lambda}(\gamma) := I + \left(BB^T - \gamma^{-2} DD^T\right)\overline{M}(\gamma) \tag{5.2a}$$

and $\overline{M}(\gamma)$ satisfies the generalized algebraic Riccati equation (GARE)

$$\overline{M}(\gamma) = Q + A^T \overline{M}(\gamma)\overline{\Lambda}^{-1}(\gamma) A. \tag{5.2b}$$

Existence condition (as the Counterpart of (3.5)):

$$\gamma^2 I - D^T \overline{M}(\gamma) D > 0. \tag{5.3}$$

Toward validating the above, we now first establish the monotonicity of the sequence generated by (3.3).

Lemma 5.1: Given an integer K, let γ be chosen such that condition (3.5) of Theorem 3.1 is satisfied for all $k \le K$. Let M_k, $k = K, K-1, \cdots, 1$, be the sequence of nonnegative definite matrices generated by iteration (3.3). Then

$$M_k \ge M_{k+1} \qquad \text{for all } k \le K$$

(i.e., $M_k - M_{k+1}$ is a nonnegative definite matrix).

Proof: The proof is similar to that of monotonicity of the solution of Riccati equation in linear-quadratic control, but now we use the saddle-point value function (of the auxiliary game) rather than the dynamic programming value function. Toward this end, we first note the following recursion associated with the "feedback" game of Section III (see

[13] Henceforth, in this section, we take all the system matrices, in (2.1a)–(2.1b) and (2.8), as constant matrices.

[17, p. 254])

$$V_k(x) = \min_u \max_w \{ V_{k+1}(Ax + Bu + Dw) + |x|_Q^2 + |u|^2 - \gamma^2 |w|^2 \}$$
$$V_{K+1}(x_{K+1}) = |x_{K+1}|_Q^2$$

where $V_k(x)$ is the saddle-point value of the dynamic game of Section III, with only $K - k + 1$ stages (i.e., starting at stage k with state x and running through K). We know from Theorem 3.1 that $V_k(x) = |x|_{M_k}^2$, and

$$|x|_{M_k}^2 = \min_u \max_w \{ |Ax + Bu + Dw|_{M_{k+1}}^2 + |x|_Q^2 + |u|^2 - \gamma^2 |w|^2 \} \quad (*)$$

for every $x \in \mathbb{R}^n$. Now, since

$$g(u, w) \geq f(u, w) \Rightarrow \inf_u \sup_w g(u, w) \geq \inf_u \sup_w f(u, w)$$

it follows from (*) that $M_{k+1} \geq M_{k+2}$ implies $M_k \geq M_{k+1}$. Hence, the proof of the Lemma will be completed (by induction) if we can show that $M_K \geq M_{K+1} = Q$. Since we know that under condition (3.5) with $k = K$

$$Q\Lambda_K^{-1} = Q[I + (BB^T - \gamma^{-2}DD^T)Q]^{-1} \geq 0$$

it follows that

$$M_K = Q + A^T Q \Lambda_K^{-1} A \geq Q = M_{K+1}$$

thus completing the proof. □

The next lemma provides a set of conditions under which the sequence $\{M_{k,K+1}\}_{k=K+1}^{1}$ is bounded, for every $K > 0$. Here we use a double index on M to indicate explicitly the dependence of the sequence on the terminal (starting) time point K. Of course, for the time-invariant problem, the elements of the sequence will depend only on the difference $K - k$ and not on k and K separately.

Lemma 5.2: Let $\gamma > 0$ be fixed, and $\overline{M}$ be a positive definite solution of the GARE (5.2b), which also satisfies the condition (5.3). Then, for every $K > 0$

$$\overline{M} - M_{k,K+1} \geq 0, \qquad \text{for all } k \leq K + 1 \quad (5.4)$$

where $M_{k,K}$ is generated by (3.3).

Proof: First, note that (5.3) implies nonsingularity of $\overline{\Lambda}$, and since by hypothesis $\overline{M} > 0$, it follows that $\overline{M}\overline{\Lambda}^{-1} > 0$. If this property is used in (5.2b), we immediately have $\overline{M} \geq Q$, which proves (5.4) for $k = K + 1$. We now show that the validity of (5.4) for $k + 1$ implies its validity for k. Toward this end let us first assume that $M_{k+1,K+1} > 0$, and note from (3.3) and (5.2b), that

$$\overline{M} - M_{k,K+1} = A^T(\overline{M}\overline{\Lambda}^{-1} - M_{k+1,K+1}\Lambda_{k,K+1}^{-1})A$$
$$= A^T\left(\left[\overline{M}^{-1} + BB^T - \frac{1}{\gamma^2}DD^T\right]^{-1} - \left[M_{k+1,K+1}^{-1} + BB^T - \frac{1}{\gamma^2}DD^T\right]^{-1}\right)A$$

which is nonnegative definite since $\overline{M} \geq M_{k+1,K+1}$ by the hypothesis of the inductive argument. By the continuous dependence of the eigenvalues of a matrix on its elements the result holds also for $M_{k+1,K+1} \geq 0$, since we can choose a matrix $N(\epsilon)$, and a sufficiently small positive parameter ϵ_0 such that $0 < N(\epsilon) \leq \overline{M}$, $0 < \epsilon < \epsilon_0$, and $N(0) = M_{k+1,K+1}$. This completes the proof of the lemma. ◇

The next lemma says that any nonnegative definite solution of (5.2b) has to be positive definite, if we take the pair (A, C) to be observable, where $C^T C = Q$ (this requirement henceforth referred to as "$(A, Q^{1/2})$ being observable"). The proof of this result is similar to that of the standard ARE which arises in linear regulatory theory [24], and is therefore omitted.[14]

Lemma 5.3: Let $(A, Q^{1/2})$ be observable. Then, if there exists a nonnegative definite solution to the GARE (5.2b), satisfying (5.3), it is positive definite. ◇

An important consequence of the above result, also in view of Lemmas 5.1 and 5.2, is that if there exist multiple positive-definite solutions to the GARE (5.2b), satisfying (5.3), there is a minimal such solution (minimal in the sense of matrix partial ordering), say $\overline{M}+$, and that

$$\lim_{K\to\infty} M_{k,K+1} = \overline{M}^+ > 0$$

whenever $(A, Q^{1/2})$ is observable. Furthermore, this minimal solution determines the value of the infinite-horizon game, in the sense that

$$\inf_{\mu_{(1,\infty)} \in \mathcal{M}_{(1,\infty)}} \sup_{w_{(1,\infty)}} L_\gamma^\infty(\mu_{[1,\infty]}, w_{[1,\infty]})$$
$$= x_1^T(\overline{M}^+(\gamma) - Q)x_1 = \sup_{\nu_{[1,\infty)} \in \mathcal{N}_{[1,\infty)}} \inf_{\mu_{[1,\infty)} \in \mathcal{M}_{[1,\infty)}} \cdot L_\gamma^\infty(\mu_{[1,\infty)}, \nu_{[1,\infty)}) \quad (5.5)$$

where L_γ^∞ is (3.1) with $K = \infty$, where we explicitly indicate the dependence on the parameter γ.

The following lemma now says that the existence of a positive definite solution to the GARE (5.2b), satisfying (5.3), is not only sufficient, but also necessary for the value of the game to be bounded.

Lemma 5.4: Let $(A, Q^{1/2})$ be observable. Then, if the GARE (5.2b) does not admit a positive definite solution satisfying (5.3), the upper value of the game $\{L_\gamma^\infty; \mathcal{M}_{[1,\infty)}, \mathcal{W}_{[1,\infty)}\}$ is unbounded.

Proof: Since the limit point of the monotonic sequence of nonnegative definite matrices $\{M_{k,K+1}\}$ has to constitute a solution to (5.2b), nonexistence of a positive definite solution to (5.2b) satisfying (5.3), (which also implies nonexistence of a nonnegative definite solution, in view of Lemma 5.3) implies that for each fixed k, the sequence $\{M_{k,K+1}\}K > 0$ is unbounded. This means that given any (sufficiently large) $\alpha > 0$, there exists a $K > 0$, and an initial state $x_1 \in \mathbb{R}^n$, such that the value of the K-stage game exceeds

[14] See [25].

$\alpha |x_1|^2$. Now choose $w_k = 0$ for $k > K$. Then

$$\begin{aligned}\inf_{\mu}\sup_{w} L_\gamma^\infty(\mu, w) &\geq \inf_{\mu}\left\{\sum_{k=K+1}^{\infty}\{|x_{k+1}|_Q^2 + |u_k|^2\}\right.\\ &\quad + \sum_{k=1}^{K}\{|x_{k+1}|_Q^2 + |u_k|^2\\ &\quad \left. - |\nu_k^*(x_k)|^2\} + |x_1|_Q^2\right\}\\ &\geq x_1^T M_{1,K+1} x_1 > \alpha |x_1|^2\end{aligned}$$

which shows that the upper value can be made arbitrarily large. In the above, $\nu_{[1,K]}^*$ in the first inequality is the feedback saddle-point controller for Player 2 in the K-stage game, and the second inequality follows because the summation from $K+1$ to ∞ is nonnegative and hence the quantity is bounded from below by the value of the K-stage game. ◇

Lemmas 5.1–5.4 can now be used to arrive at the following counterpart of Theorem 3.1 in the infinite-horizon case.

Theorem 5.1: Consider the infinite-horizon discrete-time linear-quadratic auxiliary (soft-constraint) dynamic game, with $\gamma > 0$ fixed and $(A, Q^{1/2})$ constituting an observable pair. Then the following hold.

i) The game has equal upper and lower values if and only if the GARE (5.2b) admits a positive definite solution satisfying the condition (5.3);

ii) If the GARE admits a positive definite solution, satisfying (5.3), then it admits a minimal such solution, to be denoted $\overline{M}^+(\gamma)$. Then, the finite value of the game is (5.5).

iii) The upper (minimax) value of the game is finite if and only if the upper and lower values are equal.

iv) If $\overline{M}^+(\gamma) > 0$ exists, as given above, the controller $\bar{\mu}_{[1,\infty)}$ given by (5.1a), with $\overline{M}(\gamma)$ replaced by $\overline{M}^+(\gamma)$, attains the finite upper value, in the sense that

$$\sup_{w_{[1,\infty)}\in M[1,\infty)} L_\gamma^\infty(\bar{\mu}_{[1,\infty)}, w_{[1,\infty)}) = x_1^T\overline{M}^+(\gamma)x_1 \quad (5.6)$$

and the maximizing feedback solution above is given by (5.1b), again with $\overline{M}$ replaced by $\overline{M}^+$.

v) Whenever the upper value is bounded, the feedback matrix

$$F := \left(I - BB^T\overline{M}^+(\gamma)\overline{\Lambda}^+(\gamma)^{-1}\right)A \quad (5.7a)$$

is stable, that is it has all its eigenvalues inside the unit circle. This implies that the linear system

$$x_{k+1} = Fx_k + Dw_k \quad (5.7b)$$

is input–output stable.

Proof: Parts i)–iii) follow from the sequence of Lemmas 5.1–5.4, as also discussed prior to the statement of the theorem. To prove part iv), we first note that the optimization problem in (5.6) is the maximization of

$$\sum_{k=1}^{\infty} |x_{k+1}|_Q^2 + |B^T\overline{M}^+\overline{\Lambda}^{+-1}Ax_k|^2 - \gamma^2|w_k|^2$$

over $w_{[1,\infty)}$, subject to the state equation constraint (5.7b). First, consider the truncated (K-stage) version

$$\max_{w_{[1,K]}}\sum_{k=1}^{K}\{|x_{k+1}|_Q^2 + |B^T\overline{M}^+\overline{\Lambda}^{+-1}Ax_k|^2 - \gamma^2|w_k|^2\}$$
$$\leq \max_{w_{[1,K]}}\left\{|x_{K+1}|_{\overline{M}^+}^2 + \sum_{k=1}^{K}|x_k|_{\tilde{Q}_k}^2 - \gamma^2|w_k|^2\right\}$$

where

$$\tilde{Q}_k := \begin{cases} Q + A^T\left(\overline{M}^+\overline{\Lambda}^{+-1}\right)^T BB^T\overline{M}^+\overline{\Lambda}^{+-1}A, & k > 1\\ A^T\left(\overline{M}^+\Lambda^{+-1}\right)^T BB^T\overline{M}^+\overline{\Lambda}^{+-1}A, & k = 1\end{cases}$$

with the inequality following because $\overline{M}^+ \geq Q$. Adopting a dynamic programming approach to solve this problem, we have, for the last stage, after some manipulations the following:

$$\max_{w}\{|Fx_K + Dw|_{\overline{M}^+}^2 + |x_K|_{\tilde{Q}_K}^2 - \gamma^2|w|^2\} = |x_K|_{\overline{M}^+}^2$$

which is uniquely attained by

$$\begin{aligned} w^* &= \left(\gamma^2 I - D^T\overline{M}^+D\right)^{-1}D^T\overline{M}Fx_K\\ &\equiv (1/\gamma^2)D^T\overline{M}^+\overline{\Lambda}^{+-1}Ax_K \qquad (*)\end{aligned}$$

under the strict concavity condition

$$\gamma^2 I - D^T\overline{M}^+D > 0$$

which is (5.3) and is satisfied by hypothesis. Hence, recursively we solve identical optimization problems at each stage, leading to

$$\max_{w_{[1,K]}}\sum_{k=1}^{K}\left\{|x_{k+1}|_Q^2 + |B^T\overline{M}^+\overline{\Lambda}^{+-1}Ax_k|^2 - \gamma^2|w_k|^2\right\} \leq x_1^T(\overline{M}^+ - Q)x_1$$

where the bound is independent of K. Since we already know from part ii) that this bound is the value of the game, it readily follows that the control $\bar{\mu}_{[1,\infty)}$ attains it, and the steady-state controller (*) maximizes $L_\gamma^\infty(\bar{\mu}_{[1,\infty}, w_{[1,\infty)})$.

Now, finally to prove part v), we use an argument similar to that used in linear regulator theory. Toward this end, we first note that boundedness of the upper value implies, with $w_k \equiv 0$, $k \geq 1$, that

$$\begin{aligned} & |x_k|_Q^2 + |\bar{\mu}_k(x_k)|^2 \to 0, \qquad \text{as } k \to \infty\\ \Leftrightarrow\quad & Cx_k \to 0 \text{ and } Sx_k \to 0, \quad S := B^T\overline{M}^+\overline{\Lambda}^{+-1}A;\\ & C^TC := Q\\ & Cx_{k+1} \to 0 \Rightarrow C(A + BS)x_k \to 0 \Rightarrow CAx_k \to 0\\ & \cdots \quad \cdots \quad \cdots\\ & Cx_{k+n-1} \to 0 \Rightarrow \cdots \Rightarrow CA^{(n-1)}x_k \to 0.\end{aligned}$$

But $CA^ix_k \to 0$, $i = 0, \cdots, n-1$, implies by *observability* that $x_k \to 0$, and hence F is stable since x_k is generated here by (5.7b), with $w_k \equiv 0$.

Note that the theorem above does *not* claim that the policies (5.1a) and (5.1b) with $\overline{M} = \overline{M}^+$ are in saddle-point

equilibrium. Part iv) only says that with $\mu_{[1,\infty)}$ fixed at $\bar{\mu}_{[1,\infty)}$, the policy $\bar{\nu}_{[1,\infty)}$ maximizes $L_\gamma^\infty(\bar{\mu}_{[1,\infty)}, w_{[1,\infty)})$, which is only one side (left-hand side) of the saddle point inequality (3.2).[15] This, however, is sufficient for the disturbance attenuation problem under consideration, since our interest lies only in the upper value of the auxiliary game. In view of this, the solution to the infinite-horizon version of the disturbance attenuation problem follows from Theorem 5.1 above, by following the line of reasoning that led from Theorem 3.1 to Theorem 4.1. The result is given below as Theorem 5.2.

Theorem 5.2: For the time-invariant infinite-horizon disturbance attenuation problem of this section, assume that (A, B) is stabilizable and $(A, Q^{1/2})$ is observable. Then the following holds.

i) There exists a scalar $\hat{\gamma} > 0$ such that for every $\gamma > \hat{\gamma}$, the nonlinear algebraic equation (GARE) (5.2a) admits a minimal positive definite solution $\bar{M}^+(\gamma)$ in the class of nonnegative definite matrices which further satisfy the condition (5.3).

ii) Let $\hat{\Gamma}$ be the set of nonnegative $\hat{\gamma}$'s satisfying the condition of i) above. Let $\hat{\gamma}_\infty^* := \inf\{\hat{\gamma} > 0: \hat{\gamma} \in \hat{\Gamma}\}$. Then, the stationary feedback controller[16]

$$\begin{aligned} u_k &= \bar{\mu}^*(x_k) \\ &= -B^T\bar{M}^+(\hat{\gamma}_\infty^*)\bar{\Lambda}^+(\hat{\gamma}_\infty^*)^{-1}Ax_k, \qquad k = 1, 2, \cdots, \end{aligned} \tag{5.8}$$

solves the infinite-horizon disturbance attenuation problem, achieving an attenuation level of $\gamma_\infty^* = \hat{\gamma}_\infty^*$, i.e.,

$$\begin{aligned} (\gamma_\infty^*)^2 &:= \inf_{\mu \in M_{[0,\infty)}} \sup_{w_{[1,\infty)}} T(\mu, w_{[1,\infty)}) \\ &= \sup_{w_{[1,\infty)}} T(\bar{\mu}_{[1,\infty)}^*, w_{[1,\infty)}) = (\hat{\gamma}_\infty^*)^2. \end{aligned}$$

Furthermore, the controller (5.8) leads to an input–output stable system.

Proof: First, we know from the relationship between the disturbance attenuation problem and the related soft-constraint game that γ_∞^* is the optimum attenuation level for the former if and only if the game with the objective function L_γ^∞ has a bounded upper value for all $\gamma > \gamma_\infty^*$, and the upper value is unbounded for $\gamma < \gamma_\infty^*$. This, on the other hand, holds (in view of Theorem 5.1 and under the observability condition) if and only if (5.2a) admits a positive-definite solution, satisfying (5.3), which is also the minimal such solution. Hence, all we need to show is that there exists some finite $\hat{\gamma}$, so that for all $\gamma > \hat{\gamma}$ the upper value is bounded. It is at this point that the first condition (on stabilizability) of the Theorem is used, which ensures that the upper value is bounded for sufficiently large $\hat{\gamma}$, since as $\gamma \uparrow \infty$ the LQ game reduces to the standard linear regulator problem. A continuity argument proves the result. ◇

Remark 5.1: Returning to the earlier illustrative example of Section IV-B where we had found (through a limiting approach) that $\gamma_\infty^* = \sqrt{2}$, let us now directly use Theorem 5.2. First, solving for $\bar{M}(\gamma)$ from (5.2a), we obtain

$$\bar{M}(\gamma) = \frac{1 \pm \sqrt{(5\gamma^2 - 1)/(\gamma^2 - 1)}}{2}$$

which is nonnegative if and only if we choose the positive square root, and (5.3) holds if and only if $\gamma > \sqrt{2}$. This readily leads to $\gamma_\infty^* = \sqrt{2}$ as the attenuation constant. The resulting closed-loop system is $x_{k+1} = w_k$, which is clearly input–output stable.

Remark 5.2: It is possible to obtain the infinite-horizon version of Theorem 4.2 for the time invariant problem, by essentially using the result of Theorem 5.2. First, we invoke the two conditions (on stabilizability and observability) of Theorem 5.2, under which for $\gamma > \gamma_\infty^*$ we have the steady-state controller (from (4.27))

$$\mu_\gamma^\circ(\xi_k) = -B^T\bar{M}^+(\gamma)\bar{\Lambda}^+(\gamma)^{-1}A\xi_k \tag{5.9a}$$

where ξ_k is now generated by

$$\xi_{k+1} = -(I - \bar{\Lambda}^+(\gamma)^{-1})A\xi_k + Ax_k. \tag{5.9b}$$

One then has to bring in additional conditions, for the stability of the delayed linear system[17], so that (5.9b) along with the system

$$x_{k+1} = Ax_k - BB^T\bar{M}^+(\gamma)\bar{\Lambda}^+(\gamma)^{-1}A\xi_k + Dw_k$$

is stable, and the solution of (3.22a), $\hat{S}_{k,K+1}$, converges to a well-defined limit as $K \to \infty$(say, $\bar{S}(\gamma)$). Let

$$\Gamma_\infty^\circ := \{\gamma \geq \gamma_\infty^*: \gamma^2 I - \hat{D}^T\bar{S}(\gamma)\hat{D} > 0\}. \tag{5.10a}$$

Then, the minimax attenuation level is

$$\gamma_\infty^\circ := \inf\{\gamma: \gamma \in \Gamma_\infty^\circ\}. \tag{5.10b}$$

V. Some Extensions

In this section, we discuss some extensions of the results of the previous sections in three directions, to accommodate *stochastic disturbances with unknown means*, *general costs functions of the type* (2.3), and *nonzero initial states*.

A. Stochastic Disturbances

Consider the formulation of Section II with the performance index (2.8), but with the disturbance Σ_k, $k \in \boldsymbol{K}$, taken as a Gaussian white noise sequence with covariance $\times_k$, $k \in \boldsymbol{K}$, and unknown mean m_k, $k \in \boldsymbol{K}$. Also, both the norm bound on the disturbance and the performance index now involve expectation (E) operations, under the statistics of the

[15] See [26] for a numerical example in the continuous time, which shows that the limit (as the time interval becomes infinite) of the maximizer's saddle-point policy in the finite-horizon game may not constitute a saddle-point policy for the infinite-horizon game.

[16] Here the gain coefficient of the controller remains bounded as $\gamma \downarrow \gamma_\infty^*$ by (the infinite-horizon version of) Lemma 4.1.

[17] One such sufficient condition is $Q > 0$, under which boundedness of upper value implies input-output stability under controller (5.9a).

Gaussian disturbance. In this set-up, the minimax controller and the disturbance attenuation constant can be introduced exactly as in Section II, which make the general approach of Sections II–IV equally applicable here, with one major difference: we now have a (partially known) stochastic disturbance, instead of a deterministic one. However, since the disturbance can be decomposed as

$$w_k = m_k + (w_k - m_k), \qquad k \in \boldsymbol{K}$$

where $w_k - m_k$, $k \in \boldsymbol{K}$, is a zero-mean white Gaussian noise process, we can introduce an auxiliary (stochastic) game as in Remark 3.4, with the θ_k introduced there being $D_k(w_k - m_k)$, and w_k being m_k. If m_k, $k \in \boldsymbol{K}$, is allowed to depend on the state, the minimax controller for the auxiliary game is precisely the one given in Theorem 3.1, by (3.6), for every γ satisfying (3.5). The only difference here, as stated in Remark 3.4, is in the saddle-point value (3.9), which now has an additional term

$$E[L^*] = x_1^T M_1 x_1 + \sum_{k=1}^{K} Tr[D_k^T M_{k+1} D_k \Sigma_k]. \quad (6.1)$$

Mimicking the proof of Theorem 4.1, we first write down the counterpart of (4.1b)

$$E[J(\mu^*_{[1,K]}, w_{[1,K]})] \le E[L^*] + \gamma^2 \sum_{k=1}^{K} |m_k|^2$$

and as $|x_1| \to 0$, we have, for all $m_{[1,K]}$

$$E[J(\mu^*_{[1,K]}, w_{[1,K]})] \le \gamma^2 \sum_{k=1}^{K} |m_k|^2 + \sum_{k=1}^{K} Tr[D_k^T M_{k+1} D_k \Sigma_k]. \quad (6.2)$$

Now, let us assume that there is a fixed bound on the l^2-norm of $m_{[1,K]}$ (which also translates to a norm bound on $w_{[1,K]}$)

$$\sum_{k=1}^{K} |m_k|^2 \le b^2 \quad (6.3)$$

under which the bound (6.2) becomes

$$E[J(\mu^*_{[1,K]}, w_{[1,K]})] \le \gamma^2 b^2 + \sum_{k=1}^{K} Tr[D_k^T M_{k+1} D_k \Sigma_k]. \quad (6.4)$$

Hence, given the class of controls (3.6), parameterized by γ which satisfies (3.5), the lowest bound on

$$\max\left\{E[J(\mu_{[1,K]}, w_{[1,K]})] : \sum_{k=1}^{K} |m_k|^2 \le b^2\right\}$$

is given by the minimum of the R.H.S. of (6.4) subject to

$$\gamma^2 I - D_k^T M_{k+1} D_k \ge 0, \qquad k \in \boldsymbol{K}. \quad (6.5)$$

This minimizing γ is not necessarily equal to the γ^* of Theorem 4.1, for all b, because the first term on the R.H.S. of (6.4) is increasing with γ while the second term is decreasing (because M_{k+1} decreases with increasing γ, in the matrix sense). Since this involves the minimization of a continuous function over a closed and bounded interval, there exists, necessarily, an optimal solution—to be denoted by $\tilde{\gamma}$.[18] Then, the controller $\mu^*_{\tilde{\gamma}}$ (given by (3.6)) is the optimum one in the given class; we are not claiming, however, optimality over the class of all controllers, as in Theorem 4.1.

To explore the dependence of $\tilde{\gamma}$ on b, let us return to the example of Section III, in the presence of the additional stochastic input, and with $K = 2$. If we take $\Sigma_2 = \Sigma_1 = 1$, and leave b as a variable, the minimization problem leads to

$$\tilde{\gamma}^2 = \begin{cases} \frac{1}{2}\left(1 + \frac{1}{b}\right), & b^2 < 3 - 2\sqrt{2} \\ 1 + \frac{1}{\sqrt{2}}, & \text{else}. \end{cases}$$

Hence, if b is greater than a threshold value, the minimax solution of Theorem 4.1 also provides a solution to the stochastic problem.

It is also possible to extend this analysis to the (stochastic) case where m_k is not allowed to depend on the state. But in this case the counterpart of Theorem 3.1 (for the auxiliary game problem) is much more complicated (as partially explained in Remark 3.4) and because of space limitations we will not be discussing it here.

B. More General Plants and Cost Functions

Another extension of the results of the previous sections would be the original class of disturbance attenuation problems formulated by (2.1a), (2.1b), and (2.3), where there is a direct link to the output from both the disturbance and the control in a strictly causal sense. This extension is, in fact, fairly straightforward, if we use the usual trick of first expanding and then contracting the state space.[19] Toward this end, let us consider z_k, $k \in \boldsymbol{K}$, as an additional state variable, and introduce the state dynamics

$$t_{k+1} = \tilde{A}_k t_k + \tilde{B}_k u_k + \tilde{D}_k w_k$$

where

$$t_k := (z_k^T, x_k^T)^T$$

and

$$\tilde{A}_k := \begin{pmatrix} 0 & H_{k+1} A_k \\ 0 & A_k \end{pmatrix}, \qquad \tilde{B}_k := \begin{pmatrix} H_{k+1} B_k + F_k \\ B_k \end{pmatrix},$$

$$\tilde{D}_k := \begin{pmatrix} H_{k+1} D_k + G_k \\ D_k \end{pmatrix}.$$

Then, the performance index (2.3) can be written as

$$J' = \sum_{k=1}^{K} \{|t_{k+1}|^2_{\tilde{Q}_{k+1}} + |u_k|^2\}$$

where $\tilde{Q}_k := \text{block diag}(Q_k, 0)$. Thus, the problem is cast in

[18] Note that, contrary to the case of the purely deterministic disturbance, here $\tilde{\gamma}$ will in general depend on b; see the example to follow.

[19] For a related transformation in the time-invariant continuous-time case, see [27] which uses a loop-shifting method.

the framework of the "state-output" problem of the previous sections, making the developed theory applicable here as well, provided that we replace all the system matrices by the new ones defined above. Note that since

$$\tilde{A}_k t_k = \begin{pmatrix} H_{k+1} \\ I \end{pmatrix} A_k x_k$$

the unique FB saddle-point policies depend only on x_k (and not also on z_k), thus making the solution compatible with the underlying information pattern. Perhaps another useful observation here is that even though the main recursion (3.3) now appears (at the outset) to be of dimension $2n \times 2n$, M_k is in fact block diagonal, with the first block being equal to Q_k. Hence, one needs to iterate only on the second $(n \times n)$ block M_k, thus making the recursion only of dimension $n \times n$ as in (3.3).

C. Nonzero Initial State

If the initial state x_1 is a known nonzero quantity, the formulations (2.4) and (2.5) are not equivalent, even if the controllers are restricted to the linear class. In this case, the former is more meaningful and its solution can be shown to satisfy a threshold property, with the threshold determined by some norm of x_1; details of the derivation of the minimax controller in this case are quite involved and lengthy to be included here (see [28]).

If on the other hand, the initial state is an unknown quantity, one approach would be to consider it also as part of the disturbance, and to obtain the minimax controller under "worst" possible values of x_1. We then replace the norm of $w_{[1,K]}$ in (2.4) by the quantity

$$|x_1|^2_{Q_0} + \sum_{k=1}^{K} |w_k|^2$$

which really amounts to adding an extra stage to the problem, with

$$w_0 := Q_0^{1/2} x_1 \Rightarrow x_1 = x_0 + w_0 \qquad x_0 = 0.$$

Hence, we now have a design problem that has $K + 1$ stages instead of K, again with *zero* initial state, thus making the earlier results directly applicable. Note that in this reformulation we could take the disturbance at stage *zero* to have the same dimension as the state, because for the finite-horizon problem no restrictions were imposed on the dimensions of the disturbance across stages.

VII. Conclusions

A main objective of this paper has been to demonstrate the close relationship between the theory of linear-quadratic dynamic games and that of controller design for linear plants with optimum (or a prescribed degree of) disturbance attenuation. We have shown that some of the available results from the former (most of which predate the development of H^∞ optimal-control theory) can effectively be used to obtain a clean derivation of minimax controllers in the discrete-time disturbance rejection problem for both finite- and infinite-horizon formulations. An important byproduct of this analysis is the *existence* of an *optimal* (minimax) controller, which is known not to be the case in the continuous-time H^∞ optimal-control problem. The approach of this paper also enables us (as discussed in Section VI-A) to formulate and solve a class of minimax design problems where the disturbance is taken at the outset as a stochastic process whose statistics are known, other than the mean which is an l^2-bounded function. Furthermore, it is possible to use this approach, and particularly Theorem 3.1, to obtain the minimax controller when only a subset of the state variables is available to the controller, through a disturbance-corrupted channel—the so-called "four-block $H\infty$ optimal-control problem" in discrete time). The counterpart of these results in the continuous time (and under different information patterns) could also be obtained, using essentially the same approach, and results from [17, Chapt. 6].[20]

An issue of importance in such design problems, which we have not addressed in this paper, is *robustness*. A question of interest in this context is the performance of the chosen (minimax) controller outside the set of values where the disturbance achieves the maximum, and an ordering of different minimax controllers according to this "neighborhood improvement" criterion. For some initial work in this area, we refer the reader to [16].

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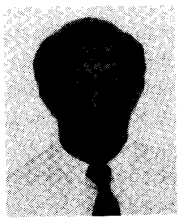

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