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SOME THOUGHTS ON RATIONAL EXPECTATIONS MODELS, AND ALTERNATE FORMULATIONS†

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Abstract—We present, in this paper, an alternative, optimization based formulation for "forward looking" models in economics, more commonly known as rational expectations models. For one basic scalar model, we study both finite and infinite horizon formulations under two different information patterns, and in each case we obtain explicit expressions for the unique solution without making any *a priori* assumptions on its structure. We then compare these results with other possible solutions obtainable using the traditional approach to rational expectations. The approach introduced here can handle higher order models as well as nonlinear ones, and also those where there is an additional exogeneous input controlled by a different set of agents. Some of these possible extensions are briefly discussed in the latter part of the paper; others are left as challenging but highly promising problems for future research.

1. INTRODUCTION

There is a large body of papers in the economics literature, which deal with the questions of well-posedness of, and existence and uniqueness of solutions to, "forward looking" dynamic models, more commonly known as *rational expectations* models. We cite, as a few representative papers on this topic, the works of Lucas (1975), Sargent and Wallace (1975), Barro (1976), Taylor (1977), Shiller (1978), Blanchard (1979) and Blanchard and Kahn (1980). The underlying dynamic models are "forward looking", because the future behavior depends explicitly on the expectations the agents have on the future itself; and they are called "rational expectations models", because the expectations on the future outcomes are (or should be) formed on some rational basis. One such (nontrivial) model which will primarily be the focus of our attention in the sequel, is given by the scalar difference equation

$$y_t = ay_{t-1} + bE_{t-1}y_{t+1} + \epsilon_t. \tag{1}$$

Here, a and $b \neq 0$ are constant parameters, $\{\epsilon_t\}$ is a sequence of independent zero-mean random variables with finite variance, and $E_{t-1}y_{t+1} := E\{y_{t+1}|\eta_t\}$ is the conditional expectation of y_{t+1} based on some information, η_t, available to the agents at time t. The subscript $t-1$ is used to capture the assumption that this information η_t is based on the *past* values of the relevant state of the economy, that is $\{y_{t-1}, y_{t-2}, \ldots\} =: y^{t-1}$. A common assumption is to let $\eta_t = y^{t-1}$; but other formulations are also possible, such as $\eta_t = z^{t-1}$, where z_t denotes some "noisy" measurement on y_t:

$$z_t = y_t + \xi_t, \tag{2}$$

with $\{\xi_t\}$ being another sequence of independent, zero-mean random variables with finite variance.

The basic question addressed in the literature over the years, rephrased in the above context, is whether there exists a (unique) stochastic process $\{y_t\}$ that satisfies (1) for all t *of interest*. A common assumption, made primarily for the reason of tractability, is to let the time interval be infinite (on both sides), so that the stochastic process sought could be restricted to the class of stationary (or, most of the time, wide sense stationary) processes. Even in this class, the solution will, in general, be nonunique. For a simple illustration of this nonuniqueness, consider the model (1) with $\eta_t = y^{t-1}$, and under the parametric restriction $ab < 1/4$. Introduce two scalars d_+ and d_- (compactly written as $d_\pm$):

$$d_\pm = \frac{1}{b^2}[\tfrac{1}{2} - ab \pm (1/4 - ab)^{1/2}], \tag{3a}$$

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which are the two solutions to the second order equation $(a + bd)^2 = d$. Now it is not difficult to see that the stochastic process $\{y_t\}$ generated by

$$y_t = (a + bd_\pm)y_{t-1} + \epsilon_t \tag{3b}$$

solves (1), for either value of $d_\pm$, since $E_{t-1}y_{t+1} = d_\pm y_{t-1}$. Furthermore, by appropriately choosing a and b, one can ensure that $|a + bd_\pm| < 1$, so that the process $\{y_t\}$ is wide sense stationary whenever $\{\epsilon_t\}$ is. It turns out, as we will see later, that there are also other solutions to (1), for which $E_{t-1}y_{t+1}$ depends not only on y_{t-1} but also on y^{t-2}, possibly infinite past. On the other hand, if we had to choose only between the two given above, then there is reason to believe (as also argued in Taylor, 1977) that the preference would go towards the one with lower error variance. Under this additional criterion we would eliminate the solution corresponding to d_+ since

$$\operatorname{var}(y_{t+1} - E_{t-1}y_{t+1}) = d_\pm \operatorname{var}(\epsilon_t) + \operatorname{var}(\epsilon_{t+1}), \tag{3c}$$

and $d_+ > d_-$. One appealing feature of the resulting solution (over others which are given by higher order ARMA processes) is that it is valid even if the initial and final times are *finite*. For example, the process satisfying (1) could start at $t = 0$, with a given value for y_0, in which case the process (3b) will again solve (1). Such a finite horizon formulation would also allow for nonstationary shocks (ϵ_t), and possibly nonconstant values for a and b in (1).

This prelude now brings us to the two fundamental questions that we raise (and resolve) in this paper. The first is whether one can come up with a finite-horizon formulation corresponding to (1) whose possibly unique solution would yield in the limit (as the time interval becomes infinite) a stationary solution for (1). This would provide a natural selection criterion among a large number of solutions to (1), and also form a natural basis for generating a dynamic decision process which would be compatible with the available information. Of course, the finite horizon model could also provide a better (more realistic) description of the cause–effect relationship of the agents' decisions, and therefore could be of independent interest. Our answer, in the paper, to this first question is in the affirmative.

The second question we raise is a more subtle one, which involves a philosophical deviation from the model (1) without necessarily departing conceptually from the initial *raison d'être* for formulating "forward looking" rational expectations models. To make our point, let us go back to (1) and reflect a little on the real meaning of the second (forward looking) conditional expectations term on its right hand side. This term, in fact, represents the aggregate decision of the economic agents as to their best estimate of some future state of the economy. In this sense, it is probably a better reflection of the reality to rewrite (1) as

$$y_t = ay_{t-1} + bv_t + \epsilon_t, \tag{4}$$

where $\{v_t\}$ represents the aforementioned aggretate decision variable, chosen under the informational restriction that $v_t = \gamma_t(\eta_t)$, for some (general-measurable) function γ_t. Now, the main rationale behind choosing $\gamma_t(\eta_t)$ as $E\{y_{t+1}|\eta_t\}$ in (1) is to make v_t as close as possible (in a certain sense) to y_{t+1}, with one criterion leading to such a (unique) choice being, *with t isolated*,

$$\min_{\gamma_t} E\{(\gamma_t(\eta_t) - y_{t+1})^2\}. \tag{5}$$

We maintain that the starting point for any rational choice for v_t should be a criterion such as (5), and *not* directly the conditional expectation $E_{t-1}y_{t+1}$. In fact, since the problem involves multiple stages, the economic agents are not interested in minimizing the variance in (5) *for a particular t*, but rather its cumulative over *all t* of interest. Hence, in a realistic scenario, (5) should be replaced by

$$J_s^T := \min_{\{\gamma_t\}} \sum_{t=s}^{T} E\{(\gamma_t(\eta_t) - y_{t+1})^2\}\rho^{t-s}, \tag{6}$$

where the minimization is subject to the dynamics (4), with $v_t = \gamma_t(\eta_t)$. Here $[s, T]$ is the time horizon, which could also be infinite, and ρ denotes a positive discount factor $(0 < \rho \leqslant 1)$, with $\rho = 1$ implying that prediction errors at all stages are given equal weight and emphasis. We submit

the policy optimization problem (6), together with (4), as a strong alternative to (1). It is perhaps similar to (1) at a conceptual level, but departs drastically from it at the technical level along with the ensuing economic interpretation. For example, because of the spillover terms across stages, it is generally *not* true (see Section 2 for justification) that the input of (1), $v_t = E_{t-1}y_{t+1}$, solve (6). A further advantage of dealing with the policy optimization problem (6) is that it can also handle problems with time-varying parameters and finite horizon, which may better model realistic situations.

It is this alternative formulation for rational expectations that we will study in the main balance of this paper, under two different types of measurement processes, $\eta_t = y^{t-1}$ and $\eta_t = z^{t-1}$. Existence and uniqueness of the solution to the finite horizon dynamic policy optimization problem (6) will be established, and properties of the solution, including its limiting behavior as the time horizon becomes sufficiently large will be analyzed (Sections 2 and 3). Furthermore, a finite horizon version of (1) will be formulated, and it will be shown to admit a unique solution, the limit of which (as time horizon goes to infinity) captures the minimum variance solution (3b) with $d = d_-$ (Section 4). Discussions in Section 5 on extensions to other forward looking models, as well as to models with additional exogeneous control inputs will conclude the paper.

2. THE SOLUTION TO THE POLICY OPTIMIZATION PROBLEM: THE PERFECT MEASUREMENT CASE

In this and the next section, we provide a complete solution to the policy optimization problem (6), (4), with $v_t = \gamma_t(\eta_t)$, first for $\eta_t = y^{t-1}$ (perfect measurements) and then (in Section 3) for $\eta_t = z^{t-1}$ (noisy measurements). Initially we take T and s finite; set $s = 1$, without any loss of generality, and leave T an arbitrary integer. Furthermore, we take y_0 as a second-order random variable, independent of the sequence $\{\epsilon_t\}_{t=1}^{T+1}$, as well as $\{\xi_t\}_{t=0}^{T}$ for the noisy case. To complete the description, we have to specify the terminal conditions on the problem. Consistent with the criterion (6), we assume that the prediction process by the economic agents ends at $t = T$, and that there is no effort to predict (or any interest in forecasting) the value of y_{T+2}; accordingly we set $v_{T+1} = 0$.

Under the above specified side conditions, the unique solution to the problem with perfect measurements is presented in Theorem 2.1 below, after giving a condition (Condition 2.1) which is generally satisfied. We should note (as an advance warning to the reader) that, in the statement of the theorem the information $\eta_t = y^{t-1}$ has been replaced by $\tilde{\eta}_t = (y^{t-1}, v^{t-1})$, without any loss of generality, since both generate the same sigma field—v_s being measurable with respect to the sigma field generated by y^{s-1}, for all $s < t$.

Condition 2.1

Let $\{x_t\}_{t=T}^{1}$ be a sequence generated by

$$x_{t-1} = 1 - \frac{ab}{x_t}, \quad x_T = 1 - ab \neq 0. \tag{7}$$

Assume that x_t does not vanish for any $t > 1$. □

Theorem 2.1

Let Condition 2.1 be satisfied. Then, the dynamic policy optimization problem (4), (6), with perfect measurements and $s = 1$, admits the unique solution

$$\begin{aligned} v_t^* = \gamma_t^*(\tilde{\eta}_t) &= \alpha_t y_{t-1} + \beta_t v_{t-1}, \quad 2 \leqslant t \leqslant T; \\ &= m_1 m_0 y_0, \qquad\qquad t = 1; \end{aligned} \tag{8}$$

where α_t and β_t are given by

$$\alpha_t = -[ab - \rho m_t n_t x_t a]/[b^2 + \rho n_t x_t^2], \tag{9a}$$

$$\beta_t = b/[b^2 + \rho n_t x_t^2], \tag{9b}$$

and n_t, m_t are defined recursively by

$$n_{t-1} = 1 - b\beta_t \equiv \rho n_t x_t^2/[b^2 + \rho n_t x_t^2]; \quad n_T = 1, \tag{10a}$$

$$m_{t-1} = (1 - x_{t-1})/b \equiv a/[1 - bm_t]; \quad m_T = a. \tag{10b}$$

The minimum value in (6), that is the lowest possible discounted cumulative prediction error variance is

$$J_1^T = k_1 + n_1 m_1^2 \operatorname{var}(\epsilon_1), \tag{11a}$$

where k_1 is the last step in the backwards iteration

$$k_{t-1} = \rho k_t + (1 + \rho n_t m_t^2)\operatorname{var}(\epsilon_t);$$
$$k_T = \operatorname{var}(\epsilon_{T+1}). \tag{11b}$$

Proof. See Appendix A. □

An interesting feature of the solution given in Theorem 2.1 is that the best predictor policy for the economic agents is a linear function of the most recent measurement and the most recent policy input, and thus has a finite dimensional representation even if $s \to -\infty$, instead of $s = 1$. Furthermore, because of the recursive computation of the coefficients α_t and β_t in (8), the optimum policy is *time-consistent*, for all $t > s + 1$, not only *on* but also *off* the optimum path. A mathematically precise statement for this time-consistency can be given as follows: If $\{\gamma_t^*\}_{t=1}^T$ is the solution to the problem defined on the time interval $[1, T]$, then for all $t > 1$, the truncated version $\{\gamma_p\}_{p=t}^T$ solves J_{t-1}^T for arbitrary but fixed v_{t-1} and y_{t-1}, regardless of the past decisions. (A justification for this statement follows from the proof of Theorem 2.1 given in Appendix A.) Yet another appealing feature of the solution is that the minimum cumulative error variance is independent of the statistics of the initial state y_0; it depends only (and linearly) on the variances of the shocks (ϵ_t) into the system. A consequence of this observation is that as the shock variances go to zero, J_1^T also goes to zero, thus leading to *perfect foresight* in the limit.

One natural question to ask at this point is whether the equality $v_t^* = E_{t-1}y_{t+1}^*$ holds, where $\{y_t^*\}$ represents the corresponding path satisfying

$$y_t^* = (a + b\alpha_t)y_{t-1}^* + b\beta_t v_{t-1}^* + \epsilon_t, \quad y_0^* = y_0; \tag{12}$$

$$v_t^* = \alpha_t y_{t-1}^* + \beta_t v_{t-1}^*, \qquad v_0^* = 0. \tag{13}$$

As to be expected, the equality does not hold (in a sense, v_t^* "does better" than $E_{t-1}y_{t+1}$). We postpone an indirect verification of this to Section 3 where we obtain a unique solution to (4) on the time interval $[1, T]$, under the restriction $v_t = E_{t-1}y_{t+1}$, which is different from $\{y_t^*\}$ generated by (13). What we show below, however, is that the equality holds asymptotically (as the length of the time horizon goes to infinity) and hence the limit of (8) (in the sense to be clarified below) provides a solution to (1) in the absence of any side conditions. Toward this end, it will first be useful to analyze Condition 2.1 and the limiting properties of the sequence $\{x_t\}$.

Lemma 2.1

(i) Condition 2.1 is satisfied whenever $ab \leqslant 1/4$. For $ab > 1/4$, the set of values of the product ab for which Condition 2.1 is not satisfied is finite. (Hence, the condition is not satisfied only on a set of (Lebesque) measure zero.)

(ii) Let $\{x_t^T\}_{t=1}^T$ be the sequence generated by (7), where T is also considered as a variable. For $ab \leqslant 1/4$, and every finite t,

$$\lim_{T\to\infty} x_t^T = \bar{x} = \tfrac{1}{2}[1 + (1 - 4ab)^{1/2}]. \tag{14}$$

For $ab > 1/4$, the sequence does not converge.

Proof. Introduce the new sequence $\{\phi_t\}_{t=0}^\infty$ where $\phi_t = x_{T-t}^T - \bar{x}$. This new sequence is generated by

$$\phi_{t+1} = \left(\frac{1 - \bar{x}}{\phi_t + \bar{x}}\right)\phi_t; \quad \phi_0 = 1 - ab - \bar{x}.$$

For $0 < ab \leqslant 1/4$, the sequence $\{\phi_t\}$ is positive and strictly decreasing, since $\phi_0 > 0$ and $0 < 1 - \bar{x} < \bar{x}$. Hence,

$$\lim_t \phi_t = 0,$$

implying that

$$\lim_t x_{T-t}^T = \bar{x},$$

which is equivalent to (14). Furthermore, for $ab = 0$, $\phi_t = 0$ for all $t \geqslant 0$; hence (14) stands proven for $0 \leqslant ab \leqslant 1/4$. As a by-product we obtain the result that $x_t^T > \bar{x} > 0$ for all $t \leqslant T$, thus proving part (i) of the Lemma for $0 \leqslant ab \leqslant 1/4$.

For the range of values $ab < 0$, we introduce a subsequence of $\{\phi_t\}$, denoted $\{\tilde{\phi}_t\}$, where $\tilde{\phi}_t = \phi_{2t}$. This new sequence is generated by

$$\tilde{\phi}_{t+1} = \left(\frac{1 - \bar{x}^2}{\tilde{\phi}_t + \bar{x}^2}\right)\tilde{\phi}_t; \quad \tilde{\phi}_0 = 1 - ab - \bar{x},$$

which is again a positive, strictly decreasing sequence since $(1 - \bar{x})^2 < \bar{x}^2$ and $\phi_0 > 0$. Thus,

$$\lim_t \phi_{2t} = \lim_t \tilde{\phi}_t = 0,$$

which further implies that

$$\lim_t \phi_{2t+1} = 0,$$

since $[(1 - \bar{x})/(\phi_{2t} + \bar{x})]$ is bounded. (Note that $\{\phi_t\}$ itself is not monotonic, but $\{\phi_{2t}\}$ is, which makes $\{\phi_{2t+1}\}$ bounded and have a decreasing envelope.) Hence,

$$\lim_t x_{T-t}^T = \bar{x},$$

for $ab < 0$.

To complete the proof of part (ii), it will be sufficient to observe that the difference equation (7) does not have an equilibrium point for $ab > 1/4$, and hence the sequence cannot converge to a limit for those values of the parameters. To complete the proof of part (i), on the other hand, we first note that for $ab < 0$ the sequence generated by (7) is positive and hence satisfies Condition 2.1 automatically. Second, we note that in region $ab > 1/4$, each x_t can be solved recursively as a rational function of ab (i.e. as a fraction with both numerator and denominator being polynomial functions of ab). Since there is only a finite number of x_t's ($T - 1$ of them), and each numerator polynomial has only a finite number of zeroes, the final statement of part (i) readily follows. This completes the proof of the lemma. □

In view of Lemma 2.1, we have to restrict the parameter space to $ab \leqslant 1/4$, if we are interested in the limiting behavior of the solution presented in Theorem 2.1. Under this restriction, and using the same interpretation for $m_t^T, n_t^T, \alpha_t^T, \beta_t^T$ as in x_t^T introduced in Lemma 2.1, it almost immediately follows that

$$\lim_T m_t^T = (1 - \bar{x})/b := \bar{m};$$

$$\lim_T n_t^T = 0;$$

$$\lim_T \alpha_t^T = -a/b; \quad \lim_T \beta_t^T = 1/b.$$

In the above the second limit may require some explanation; all others are by mere substitution. To obtain the result, we let $\tilde{b}_t^2 = b^2/\rho x_t^2$ in (10a) (note that $x_t \neq 0$ by Lemma 2.1), and rewrite the recursion as

$$n_{t-1} = n_t/[n_t + \tilde{b}_t^2]; \quad n_T = 1,$$

which clearly generates a positive strictly decreasing sequence, converging to zero.

The above now implies that, as the time horizon becomes sufficiently long, the optimum policy (8) converges to the *stationary* policy

$$v_t = \gamma^*(\tilde{\eta}_t) = -(a/b)y_{t-1} + (1/b)v_{t-1}. \tag{15a}$$

When substituted in (10), it yields

$$y_t^* = v_{t-1} + \epsilon_t \tag{15b}$$

as the equilibrium path generating equation, along with (15a).

Two observations are in place here. The first is the validity of the equality

$$E_{t-1}y_{t+1}^* = E_{t-1}(v_t + \epsilon_{t+1}) = v_t, \tag{16a}$$

which shows that the stochastic process generated by (15) is in fact a solution to (1). The second observation is that the per-stage steady-state prediction error variance is

$$\mathrm{var}(y_{t+1}^* - E_{t-1}y_{t+1}^*) = \mathrm{var}(\epsilon_{t+1}), \tag{16b}$$

which is lower than (3c). This is in fact the best one can do if the sole purpose is minimization of the steady-state prediction error variance, since

$$\begin{aligned}\mathrm{var}(y_{t+1} - E_{t-1}y_{t+1}) &= \mathrm{var}(\epsilon_{t+1}) + \mathrm{var}(ay_t + bv_{t+1} - E_{t-1}y_{t+1}) \\ &\geqslant \mathrm{var}(\epsilon_{t+1}).\end{aligned}$$

We now summarize these results in the following theorem.

Theorem 2.2

Assume that $ab \leqslant 1/4$. Then, the infinite horizon version of the dynamic policy optimization problem (4), (6) admits a unique optimum stationary policy, given by (15a). Furthermore, the corresponding path, generated by (15), constitutes a minimum prediction error variance solution to (1). □

Remark. Note that the latter part of the statement of the theorem is the strongest possible, since minimality is against all possible linear or nonlinear structural forms of $E_{t-1}y_{t+1}$ in (1). The literature, heretofore, has exclusively dealt with linear structural forms. A second point that is worth making here is that even though for the finite horizon problem the restriction $v_t = E_{t-1}y_{t+1}$ is an unnatural one (because of the spillover across stages), we find (at least within the domain of the model adopted here) that the effect of the correlation across stages dies out as the time horizon grows, thus making $v_t = E_{t-1}y_{t+1}$ a natural choice. In spite of this, the advantage of viewing the infinite horizon problem as the limit of a finite horizon dynamic optimization problem should be apparent here, since it produces the best possible solution (in the minimum variance sense) to (1). □

The sequence of finite horizon problems used in the construction of the limiting solution in Theorem 2.2 were those where no prediction was made at the last stage, i.e. $v_{T+1} \equiv 0$. Even though this is a natural side condition to impose, since y_{T+2} does not enter the optimization problem, one might still be interested in finding out how the result of Theorem 2.2 would be affected if v_{T+1} were not taken to be zero, say an arbitrary linear function in the same form as (8): $v_{T+1} = \hat{\alpha}y_T + \hat{\beta}v_T$. To explore this somewhat, we first note that the statement of Theorem 2.1 would then remain intact, with only the boundary conditions (10a) and (10b) changed to

$$n_T = (1 - b\hat{\beta})^2; \quad m_T = a + b\hat{\alpha}. \tag{17a}$$

This also leads to a corresponding change in the terminal condition of (7):

$$x_T = 1 - ab - b^2\hat{\alpha}. \tag{17b}$$

The following counterpart of Lemma 2.1 (ii) now follows from an almost identical proof (to that of Lemma 2.1):

Lemma 2.2

Let $\{x_t^T\}_{t=1}^T$ be the sequence generated by (7), where the terminal condition is now (17b). Let $ab \leqslant 1/4$, and $\hat{\alpha}$ be such that $1 - ab - \hat{x} > \hat{a}b^2$. (Note that $\hat{\alpha} \leqslant 0$ always satisfies this condition.) Then

$$\lim_T x_t^T = \bar{x} = \tfrac{1}{2}[1 + (1 - 4ab)^{1/2}].$$

For $ab > 1/4$, the sequence does not converge. □

Hence, as long as $\hat{\alpha}$ satisfies the condition of Lemma 2.2, the limiting value for $\{x_t^T\}$, and consequently those of $\{m_t^T\}$, $\{n_t^T\}$, $\{\alpha_t^T\}$, $\beta_t^T\}$ remain the same (and independent of $\hat{\alpha}$), implying that even if v_{T+1} is not chosen identically zero, the statement of Theorem 2.2 could be true. There is therefore, a large class of finite horizon problems (larger than the class initially formulated) which yield in the limit the stationary policy (15a).

3. THE NOISY MEASUREMENT CASE

We now address the problem of obtaining the optimum solution to (6) when the information available at time t is $\eta_t = z^{t-1}$, where z_t is a noise corrupted version of y_t, as given by (2). We take all the random variables to be Gaussian, with the variances for y_0, ϵ_t and ξ_t denoted by σ_0, p_t and q_t, respectively. The main result to be developed below is that the results obtained for the perfect measurement case can directly be used here, that is the problem features a "certainty equivalence" property.

Towards showing this equivalence we first introduce the notation $\hat{y}_t := E\{y_t | \tilde{\eta}_{t+1}\} =: E_t y_t$ where $\tilde{\eta}_{t+1} := (z^{t-1}, v_t)$. It is a standard result (see, for example, Anderson and Moore, 1979) that $\hat{y}_t$ is given recursively by the Kalman filter equations:

$$\hat{y}_t = a\hat{y}_{t-1} + bv_t + [\hat{\sigma}_t/(\hat{\sigma}_t + q_t)]r_t; \quad \hat{y}_{-1} = 0, \tag{18a}$$

$$r_t := z_t - a\hat{y}_{t-1} - bv_t; \quad v_0 := 0, \tag{18b}$$

$$\hat{\sigma}_{t+1} = [a^2 q_t/(\hat{\sigma}_t + q_t)]\hat{\sigma}_t + p_{t+1}; \quad \hat{\sigma}_0 = \sigma_0. \tag{18c}$$

Here $\{r_t\}$ is the *innovations process* which has zero mean, is independent from stage to stage, and has variance $\mathrm{var}(r_t) = q_t + \hat{\sigma}_t$. Furthermore, $\hat{\sigma}_t$ admits the interpretation that

$$\hat{\sigma}_t := \mathrm{var}(y_t - E_{t-1}y_t) \equiv \min_{\mu} \mathrm{var}(y_t - \mu(\tilde{\eta}_t)),$$

where the minimization is over arbitrary (not necessarily linear) maps.

Now, by following an argument similar to that used in LQG stochastic control (Bertsekas, 1976), we obtain the following sequence of equalities, where $\tilde{\eta}_t := (z^{t-1}, v^{t-1})$:

$$\begin{aligned} E\{(\gamma_t(\tilde{\eta}_t) - y_{t+1})^2\} &= E\{E_{t+1}(v_t - \hat{y}_{t+1} + \hat{y}_{t+1} - y_{t+1})^2\} \\ &= E\{E_{t+1}(v_t - \hat{y}_{t+1})^2\} + E\{E_{t+1}(\hat{y}_{t+1} - y_{t+1})^2\} \\ &= E\{(\gamma_t(\tilde{\eta}_t) - \hat{y}_{t+1})^2\} + \hat{\sigma}_{t+1}q_{t+1}/(\hat{\sigma}_{t+1} + q_{t+1}). \end{aligned}$$

Here the third equality has followed from the fact that the estimation error $y_{t+1} - \hat{y}_{t+1}$ has zero mean, and is orthogonal to any function of $\tilde{\eta}_{t+2}$. Finally, using the above identity in (6) by noting that η_t could be replaced with $\tilde{\eta}_t$ without affecting the underlying stochastic control problem, we arrive at the minimization problem

$$J_s^T = \min_{\{\gamma_t\}} \sum_{t=s}^{T} E\{(\gamma_t(\eta_t) - \hat{y}_{t+1})^2\}\rho^{t-s} + \sum_{t=s}^{T} \{q_{t+1}\hat{\sigma}_{t+1}/(\hat{\sigma}_{t+1} + q_{t+1})\}\rho^{t-s}, \tag{19}$$

where the dynamic constraint is now (18a). Since $\{r_t\}$ is an independent process, this is exactly the problem covered by Theorem 2.1, with only $\{y_t\}$ replaced by $\{\hat{y}_t\}$. Hence the certainty equivalence principle of stochastic control holds here, and the following counterpart of Theorem 2.1 readily follows:

Theorem 3.1

(i) Let Condition 2.1 be satisfied. Then, the dynamic policy optimization problem (4), (6) with noisy measurements ($\eta_t = z^{t-1}$) and with Gaussian distribution for all the random variables, admits the *unique* solution

$$\begin{aligned} v_t^* = \gamma_t^*(\tilde{\eta}_t) &= \alpha_t \hat{y}_{t-1} + \beta_t v_{t-1}, \quad 2 \leqslant t \leqslant T; \\ &= m_1 m_0 \hat{y}_0, \qquad\qquad t = 1; \end{aligned} \tag{20}$$

where α_t, β_t, m_t are as defined in Theorem 2.1 and $\{\hat{y}_t\}$ is the sequence of estimates generated recursively by the Kalman filter (18).

(ii) The minimum value in (6) is given, in view of (19), and using the variance for r_t developed earlier, by

$$J_1^T = \hat{k}_1 + n_1 m_1^2 \hat{\sigma}_1^2/(\hat{\sigma}_1 + q_1), \tag{21a}$$

where $\hat{k}_1$ is the last step of the backwards iteration

$$\begin{aligned} \hat{k}_{t-1} &= \rho \hat{k}_t + \rho n_t m_t^2 \hat{\sigma}_t^2/(\sigma_t + q_t) + \hat{\sigma}_t; \\ \hat{k}_T &= \hat{\sigma}_{T+1}. \ \square \end{aligned} \tag{21b}$$

For the infinite horizon version of the problem, the analysis preceding Theorem 2.2 equally applies here (because of certainty equivalence), subject to some obvious modifications. The optimum stationary policy, replacing (15a), would be

$$v_t = \gamma^*(\tilde{\eta}_t) = -(a/b)\hat{y}_{t-1} + (1/b)v_{t-1}, \tag{22}$$

again provided that $ab \leqslant 1/4$. The corresponding equilibrium path $\{y_t^*\}$ is generated by

$$y_t^* = a(y_{t-1}^* - \hat{y}_{t-1}^*) + v_{t-1} + \epsilon_t, \tag{23a}$$

along with

$$\hat{y}_t^* = v_{t-1} + [\hat{\sigma}_t/(\hat{\sigma}_t + q_t)]r_t. \tag{23b}$$

Now, the important conclusion is that a relationship similar to (16a) also holds here:

$$\begin{aligned} E_{t-1} y_{t+1}^* &= a E_{t-1}(y_t^* - \hat{y}_t^*) + v_{t-1} \\ &= v_{t-1}. \end{aligned} \tag{24}$$

(Note that the last line follows because of the nestedness property of conditional expectations: $E_{t-1}\hat{y}_t^* \equiv E_{t-1}E_t y_t^* = E_{t-1}y_t^*$.) Hence, a solution to (1), when E_{t-1} is interpreted as conditional expectation under noisy measurements, is provided by (23), along with (22). Note that this time we have a "three-dimensional" representation for the solution, as opposed to the "two-dimensional" representation in the perfect measurement case.

For t sufficiently large, and when q_t and p_t are constants, it is a standard result of linear filtering theory (see Anderson and Moore, 1979) that $\{\hat{\sigma}_t\}$ generated by (18c) converges to the positive root of the quadratic equation

$$\sigma^2 + (q - a^2 q - p)\sigma - qp = 0, \tag{25}$$

say $\bar{\sigma}$. Then, if we are sufficiently far away from either end, the per-stage steady-state prediction error variance yielded by the optimum stationary policy is

$$\mathrm{var}(y_{t+1}^* - E_{t-1} y_{t+1}^*) = p + a^2 \bar{\sigma} q/(\bar{\sigma} + q),$$

which can again be shown [by a modified version of the argument used following (16b)] to be the minimum possible error variance. Hence, we conclude this section with the following counterpart of Theorem 2.2:

Theorem 3.2

Let all parameters be time-invariant, and invoke the condition $ab \leqslant 1/4$. Then, the infinite horizon version of the dynamic policy optimization problem (4), (6) with noisy measurements,

admits a unique optimum stationary policy, given by (22) where $\{\hat{y}_t\}$ is generated by (23b) with $\hat{\sigma}_t = \bar{\sigma}$ [the positive root of (25)] and $r_t = r$. Furthermore, the corresponding path, generated by (23a), along with (the time-invariant version of) (23b) and (22), constitutes a minimum prediction error variance solution to (1). □

4. A DIRECT SOLUTION TO THE RATIONAL EXPECTATIONS MODEL

In this section we work directly with the more traditional rational expectations model (1), which is obtained from (4) by restricting the policy variable to $v_t = E_{t-1}y_{t+1}$, for all t. Our objective here is *first* to show that the finite horizon version of (1) admits a unique solution under a general information pattern, *second* to obtain the limit of that solution as the time horizon becomes infinite, and *finally* to compare these results with those presented in the two previous sections.

Let η_t stand for any one of the two information patterns $\eta_t = y^{t-1}$ or $\eta_t = z^{t-1}$, where z_t was defined by (2). Let $E_{t-1}y_{t+1}$ be the conditional expectation for y_{t+1}, given η_t of the perfect or noisy measurement type. We now seek a solution to the finite horizon rational expectations model:

$$\begin{aligned} y_{T+1} &= ay_T + \epsilon_{T+1} \\ y_t &= ay_{t-1} + bE_{t-1}y_{t-1} + \epsilon_t, \quad t \leqslant T. \end{aligned} \tag{26}$$

Let $v_t := E_{t-1}y_{t+1}$, and note that

$$\begin{aligned} v_T &= E_{T-1}(ay_T + \epsilon_{T+1}) = aE_{T-1}y_T \\ &= aE_{T-1}(ay_{T-1} + bv_T + \epsilon_T) = a^2E_{T-1}y_{T-1} + abv_T, \end{aligned}$$

leading to

$$v_T = [a^2/(1-ab)]E_{T-1}y_{T-1} =: c_T E_{T-1}y_{T-1},$$

provided that $ab \neq 1$. In view of this we now claim that every solution of (26) must have the property that

$$v_t = c_t E_{t-1}y_{t+1}, \quad t \leqslant T,$$

for some unique sequence $\{c_t\}$. The proof is by induction: assume that the claim is true for $t+1$ (it is clearly valid for T, as shown above) and show that this implies its validity for t. Towards this end, first note the following sequence of equalities:

$$\begin{aligned} v_t \equiv E_{t-1}y_{t+1} &= E_{t-1}(ay_t + bv_{t+1} + \epsilon_{t+1}) \\ &= E_{t-1}(ay_t + bc_{t+1}E_t y_t) \\ &= (a + bc_{t+1})E_{t-1}y_t \\ &= (a + bc_{t+1})E_{t-1}(ay_{t-1} + bv_t + \epsilon_t) \\ &= (a + bc_{t+1})(aE_{t-1}y_{t-1} + bv_t), \end{aligned}$$

where the second line follows by substitution of the asserted form for v_{t+1} and using the independence of ϵ_{t+1} from the past history of the process. The third line follows from the nestedness property of conditional expectations ($E_{t-1}E_t \equiv E_{t-1}$), and the fourth is a consequence of the measurability of v_t with respect to the sigma field generated by η_t. Now, solving for v_t from the above we obtain

$$v_t = c_t E_{t-1}y_{t-1}, \tag{27}$$

which c_t satisfies

$$c_t = a(a + bc_{t+1})/[1 - b(a + bc_{t+1})]; \quad c_{T+1} = 0, \tag{28}$$

provided that the denominator is nonsingular for all t. Hence the claim stands proven under this nonsingularity condition on the denominator, leading to the unique solution $\{y_t^*\}$ for (26), satisfying for $t \leqslant T$

$$y_t^* = ay_{t-1}^* + bc_t E_{t-1}y_{t-1}^* + \epsilon_t. \tag{29}$$

It is interesting to note that the nonsingularity condition above is precisely Condition 2.1, since for all $t \leqslant T$

$$c_t = \frac{1 - x_t}{b^2} - \frac{a}{b}, \tag{30}$$

where $\{x_t\}$ was defined by (7). Hence the dynamic policy optimization problems of Sections 2 and 3, and the problem of obtaining a unique equilibrium path from (26) require the same existence condition, even though the resulting paths are quite different. To see this latter point more clearly, let us first consider the perfect information case, when $E_{t-1}y_{t-1} = y_{t-1}$ and hence (29) becomes

$$\begin{aligned} y_t^* &= (a + bc_t)y_{t-1}^* + \epsilon_t \\ &\equiv m_t y_{t-1}^* + \epsilon_t, \end{aligned} \tag{31}$$

which should be contrasted with (13). In the noisy measurement case, with all the underlying distributions being Gaussian as in Section 3, we have $E_{t-1}y_{t-1} = \hat{y}_{t-1}$, where $\{\hat{y}_t\}$ is generated by the Kalman filter (28) with $v_t = c_t\hat{y}_{t-1}$:

$$\hat{y}_t = (a + bc_t)\hat{y}_{t-1} + [\hat{\sigma}_t/(\hat{\sigma}_t + q_t)]r_t; \quad \hat{y}_{-1} = 0, \tag{32a}$$

$$r_t = z_t - a\hat{y}_{t-1} - bc_t\hat{y}_{t-1}, \tag{32b}$$

$$\hat{\sigma}_{t+1} = [a^2 q_t/(\hat{\sigma}_t + q_t)]\hat{\sigma}_t + p_{t+1}; \quad \hat{\sigma}_0 = \sigma_0. \tag{32c}$$

In view of this, (29) becomes

$$y_t^* = ay_{t-1}^* + bc_t\hat{y}_{t-1} + \epsilon_t, \tag{33}$$

which is again quite different from the path resulting from a substitution of the optimum policy (28) into (4) and (28).

For the infinite horizon case, one has to study the asymptotic behavior of the sequence generated by (28). This, however, is equivalent to studying the asymptotic behavior of the sequence $\{x_t\}$, because of the relationship (30). Hence the results of Section 3 (in particular, Lemma 3.1) can directly be used here to lead to the conclusion that for $\eta := ab \leqslant 1/4$, $\{c_t^T\}_{t \leqslant T}$ converges as $T \to \infty$, with the limit being

$$\lim_T c_t^T = \bar{c} = \frac{1 - 2\eta - (1 - 4\eta)^{1/2}}{2b^2} \equiv \bar{m}^2 \tag{34}$$

where $\bar{m}$ is the limit of the sequence $\{m_t^T\}_{t \leqslant T}$ generated by (10b). It is also interesting to note that $\bar{c}$ above is identical with d_- introduced in Section 1.

All the above are now summarized in the following theorem which, because of the foregoing discussion, does not require a proof.

Theorem 4.1

(i) Under Condition 2.1, and with the information pattern $\eta_t = y^{t-1}$ or $\eta_t = z^{t-1}$, the fixed-terminal-time rational expectations model (26) admits the unique solution $\{y_t^*\}$ which is generated by (29), where the gain coefficient $\{c_t\}$ is determined recursively by (28).

(ii) Under perfect measurements ($\eta_t = y^{t-1}$), the unique solution of (26) satisfies (31), and under noisy measurements ($\eta_t = z^{t-1}$) and with Gaussian distribution for all the random variables, the unique solution of (35) is generated by (22)–(23).

(iii) As the time horizon becomes infinite, the unique solution $\{y_t^*\}$ above converges to $\{\bar{y}_t\}$ generated by

$$\bar{y}_t = a\bar{y}_{t-1} + b\bar{c}E_{t-1}\bar{y}_{t-1} + \epsilon_t, \tag{35}$$

provided that $ab \leqslant 1/4$. [Here $\bar{c}$ is given by (34).]

(iv) For the case $\eta_t = y^{t-1}$, and the stationary $\{\epsilon_t\}$, the unique stationary limiting process is generated by

$$\bar{y}_t = \bar{m}y_{t-1} + \epsilon_t; \quad \bar{m} := a + b\bar{c}. \tag{36}$$

(v) For the case $\eta_t = z^{t-1}$, and with Gaussian stationary distribution for $\{\epsilon_t\}$ and $\{\xi_t\}$, the unique stationary limiting process is generated by

$$\bar{y}_t = a\bar{y}_{t-1} - b\bar{c}\hat{y}_{t-1} + \epsilon_t, \tag{37a}$$

$$\hat{y}_t = \bar{m}\hat{y}_{t-1} + [\bar{\sigma}/(\bar{\sigma} + q)][z_t - \bar{m}\hat{y}_{t-1}], \tag{37b}$$

provided that we are also sufficiently far away from the initial (starting) time. [Here $\bar{\sigma}$ is the unique positive root of the quadratic equation (25).] □

A comparison of Theorem 4.1 above, and Theorems 2.1, 2.2 and 3.1, 3.2 presented earlier, clearly reveals a strong parallelism between the existence and convergence conditions of the underlying two seemingly different formulations, in spite of the fact that the solutions are significantly different. This provides yet another strong reason for the adoption of the dynamic policy optimization formulation, in place of the standard rational expectations model (8), since the former yields uniformly lowest stagewise prediction error variance for both the finite and the infinite horizon cases, and the conditions under which Theorems 2.1, 2.2 and 3.1, 3.2 are valid cannot be relaxed further, even if we restrict v_t to be $E_{t-1}y_{t+1}$.

5. EXTENSIONS

A major advantage of the dynamic policy optimization formulation for rational expectations models, as introduced in the preceding sections, is that it can handle also higher order models, and those with nonlinearities. We have not dealt with such models in this paper mainly not to obscure the main message we wish to transmit; furthermore, though feasible, the infinite horizon convergence analysis of Lemma 2.1 would have been considerably more complicated for these more general models. Nevertheless, it would be of interest to consider here, for illustration purposes, the vector version of (1):

$$y_t = Ay_{t-1} + BE_{t-1}y_{t+1} + \epsilon_t, \tag{38}$$

where y_t, ϵ_t are of dimension n, and A, B are $n \times n$ matrices. The sequence $\{\epsilon_t\}$ would be a vector-valued stochastic process, independent from stage to stage. The stochastic control formulation corresponding to this rational expectations model would be

$$\min_{\{\gamma_t\}} \sum_{t=s}^{T} E\{(\gamma_t(\eta_t) - y_{t+1})'(\gamma_t(\eta_t) - y_{t+1})\}\rho^{t-s} =: J_s^T,$$

subject to

$$y_t = Ay_{t-1} + Bv_t + \epsilon_t; \quad v_t = \gamma_t(\eta_t), t \leqslant T,$$

$$y_{T+1} = Ay_T + \epsilon_{T+1},$$

where prime denotes the transpose of a vector (or a matrix). One can show (by mimicking the derivation given in Appendix A for Theorem 2.1) that when $\eta_t = y^{t-1}$, and under a condition (replacing Condition 2.1) which is generically true, the problem admits a unique solution in the form

$$\begin{aligned} v_t^* = \gamma_t^*(y^{t-1}, v^{t-1}) &= \Pi_t y_{t-1} + \chi_t v_{t-1}, \quad s < t \leqslant T, \\ &= \Pi_s y_{s-1}, \qquad\qquad t = s, \end{aligned}$$

where $\{\Pi_t\}, \{\chi_t\}$ are $n \times n$ dimensional matrix sequences which are determined recursively in retrograde time. The relevant expressions for these have been given in Appendix B. Again conditions on the parameters of the problem can be developed so that

$$\lim_{T\to\infty} \Pi_t^T = \bar{\Pi}; \quad \lim_{T\to\infty} \chi_t^T = \bar{\chi},$$

and hence a stationary policy would be

$$v_t^* = \bar{\Pi}y_{t-1} + \bar{\chi}v_{t-1},$$

leading to the solution path

$$y_t^* = (A + B\bar{\Pi})y_{t-1}^* + B\bar{\chi}v_{t-1}^* + \epsilon_t,$$
$$v_t^* = \bar{\Pi}y_{t-1}^* + \bar{\chi}v_{t-1}^*.$$

It will, however, generally not be true that $A + B\bar{\Pi} = 0$ (as in the scalar case) unless B is nonsingular (i.e. invertible). Thus, in general, $E_{t-1}y_{t+1}^* \neq v_t^*$.

The noisy measurement case, $\eta_t = z^{t-1}$, with

$$z_t = Hy_t + \xi_t, \tag{39}$$

where H is some (not necessarily square) matrix, and $\{\xi_t\}$ a Gaussian vector process independent from stage to stage, can also be accommodated into our approach without much difficulty. By following the reasoning given prior to the statement of Theorem 3.1, it is not difficult to see that the problem features certainty equivalence. Hence the unique optimal solution to the noisy measurement case will be given by

$$v_t^* = \gamma_t^*(z^{t-1}, v^{t-1}) = \Pi_t\hat{y}_{t-1} + \chi_t v_{t-1}; \quad s < t \leqslant T,$$
$$= \Pi_s\hat{y}_{s-1}; \qquad t = s,$$

when the time horizon is $[s, T]$. Here $\hat{y}_t := E\{y_t | z^t, v^t\}$ is the minimum mean square error estimator for y_t, generated recursively by the Kalman filter equations, in forward time (see Appendix B for the corresponding expressions).

Another class of models to which our alternative formulation would be applicable are those models that involve not only two-step ahead but also one-step ahead prediction. In the scalar linear class, these could be written as

$$y_t = ay_{t-1} + b_1v_{1t} + b_2v_{2t} + \epsilon_t, \tag{40}$$

where v_{1t} and v_{2t} use the same information $\eta_t = y^{t-1}$ (perfect measurement case) or $\eta_t = z^{t-1}$ (noisy measurement case), to predict the values of y_{t+1} and y_t, respectively. This includes, for example, the asset market model considered by Blanchard and Kahn (1980). Towards a reasonable formulation for this problem, we adopt two dynamic criteria; one being J_s^T introduced in (6) with v_t replaced by v_{1t}, and the other one being

$$\min_{\{v_{2t}\}} \sum_{t=s}^{T} E\{(v_{2t} - y_t)^2\}\rho^{t-s}, \tag{41}$$

where in each case the minimization is subject to (38). We can view this as a two-level optimization problem where we first consider the solution of (41) for all $\{v_{1t}\}$, and then substitute this solution into (40) and also use it in J_s^T to obtain the best $\{v_{1t}\}$ again as a solution of an optimization problem.† Now, it is not difficult to see that for every $\{v_{1t}\}$ the unique solution to (41) under (40) is

$$v_{2t} = E_{t-1}y_t, \quad t \leqslant T. \tag{42}$$

There are no spillover terms as in the two-step ahead prediction problem, and the error at each stage is independent of the past values of $\{v_{1t}\}$, $\{v_{2t}\}$ and $\{y_t\}$. Substituting (42) into (40) we obtain

$$y_t = ay_{t-1} + b_2E_{t-1}y_t + b_1v_{1t} + \epsilon_t,$$

and taking the conditional expectation of both sides under the information pattern η_t, we arrive at the equation

$$E_{t-1}y_t = ay_{t-1} + b_2E_{t-1}y_t + b_1v_{1t},$$

from which $E_{t-1}y_t$ can be solved uniquely, provided that $b_2 \neq 1$:

$$E_{t-1}y_t = \frac{1}{1-b_2}[ay_{t-1} + b_1v_{1t}].$$

†The approach adopted here is therefore one of Stackelberg equilibrium (see Başar and Olsder, 1982).

Hence, the reduced version of (38) is

$$y_t = \tilde{a}y_{t-1} + \tilde{b}v_{1t} + \epsilon_t, \tag{43}$$

where

$$\tilde{a} := a/(1-b_2); \quad \tilde{b} := b_1/(1-b_2).$$

This shows that the second-level problem [i.e. J_s^T subject to (43)] is identical with the ones solved in Sections 2 and 3 (for the two information patterns), and thus the results of Theorems 2.1, 2.2 and 3.1, 3.2 would be directly applicable here.

Finally, the dynamic policy optimization approach of this paper would provide a natural set-up for the formulation of target tracking problems with forward looking models. As a simple illustration, consider the linear model

$$y_t = ay_{t-1} + bv_t + cw_t + \epsilon_t,$$

with the same type of information, as earlier, available at every point in time. Here, v_t and w_t are both control variables, which are controlled by two different sets of agents, say A and B, respectively. A wishes to form an accurate (to the extent possible) two-step ahead predictor for y_t, i.e. choose $\{v_t\}$ under a performance index of the type (6). B, on the other hand, wants to keep the trajectory $\{y_t\}$ as close to a desired target as possible. Letting $\{\bar{y}_t\}$ denote this target trajectory, B's optimization problem could be formulated as

$$\min_{\{w_t\}} \sum_{t=s}^{T} E\{(y_{t+1} - \bar{y}_{t+1})^2 + kw_t^2\}\rho^{t-s},$$

where $k > 0$ is a measure of the tradeoff between target achievement and control energy. This is clearly a multi-objective optimization problem, better handled in the framework of (stochastic) dynamic game theory. Various solution concepts, such as *Nash equilibrium*, *Stackelberg equilibrium* and *Pareto equilibrium*, could be adopted here, depending on the particulars of the economic scenario leading to such a model. The results of this paper would not directly be applicable to this class of "forward looking" target tracking problems, but they indeed provide us with considerable insight into the solution process. Details of policy optimization and game theoretic techniques applicable to such models will be developed and presented in future publications. For one recent publication on these extensions, using game theoretic techniques, see Başar (1989).

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APPENDIX A

Proof of Theorem 2.1

The proof follows from a standard dynamic programming argument, and therefore involves an induction on the time-to-go $T-t$. Towards this end, first consider the truncated objective functional (6) for any $s, s < T$. We assert that

$$J_s^T = \min_{\gamma_s, \gamma_{s+1}} E\{\rho n_{s+1}(v_{s+1} - m_{s+1}y_{s+1})^2 + (v_s - y_{s+1})^2\} + \rho k_{s+1}, \tag{A.1}$$

where n_{s+1}, m_{s+1} and k_{s+1} are as defined in the theorem. The assertion is clearly true for $s = T-1$, with $n_T = 1, m_T = a, k_T = \mathrm{var}(\epsilon_{T+1})$. Let us therefore assume its validity for general s, and prove it for $s-1$. Since J^T_{s-1} can be written as†

$$J^T_{s-1} = \min_{\gamma_s, \gamma_{s-1}} E_{/\eta_{s-1}}\left\{(v_{s-1} - y_s)^2 + \min_{\{\gamma_t\}_{t=s+1}^T} E_{/\eta_s}\left\{\sum_{t=s}^T \rho^{t+1-s}(v_t - y_{t+1})^2\right\}\right\},$$

we clearly have, using the assertion for s,

$$J^T_{s-1} = \min_{\gamma_s, \gamma_{s-1}} E_{/\eta_{s-1}}\{(v_{s-1} - y_s)^2 + \min_{\gamma_{s+1}} E_{/\eta_s}\{\rho^2 n_{s+1}(v_{s+1} - m_{s+1}y_{s+1})^2 + \rho(v_s - y_{s+1})^2\}\} + \rho^2 k_{s+1}.$$

The inner minimization can be rewritten as

$$E_{/\eta_s}\{\min_{v_{s+1}} E_{/\eta_{s+1}}\{\rho^2 n_{s+1}(v_{s+1} - m_{s+1}ay_s - m_{s+1}bv_{s+1})^2 + \rho(v_s - ay_s - bv_{s+1})^2\}\}$$
$$+ \rho^2 n_{s+1} m^2_{s+1}\,\mathrm{var}(\epsilon_{s+1}) + \rho\,\mathrm{var}(\epsilon_{s+1}), \quad \text{(A.2)}$$

which is a quadratic minimization problem in v_{s+1}. Being strictly convex, it admits the unique solution (by simple differentiation with respect to v_{s+1})

$$v_{s+1} = \{[\rho n_{s+1} m_{s+1} a(1 - bm_{s+1}) - ab]y_s + bv_s\}/[\rho n_{s+1}(1 - bm_{s+1})^2 + b^2]$$
$$\equiv \alpha_{s+1} y_s + \beta_{s+1} v_s.$$

This then justifies (7)–(9). Now, substitution of this solution into (A.2) yields an expression that is a complete square in v_s and y_s:

$$\rho n_s(v_s - m_s y_s)^2 + \rho[\rho n_{s+1} m^2_{s+1} + 1]\mathrm{var}(\epsilon_{s+1}). \quad \text{(A.3)}$$

Here

$$n_s = 1 - b\beta_{s+1},$$
$$m_s = a/(1 - bm_{s+1}),$$

which are well defined under Condition 2.1. Finally, using (A.3) in J^T_{s-1}, we have

$$J^T_{s-1} = \min_{\gamma_s, \gamma_{s+1}} E_{/\eta_{s-1}}\{\rho n_s(v_s - m_s y_s)^2 + (v_{s-1} - y_s)^2\} + \rho k_s,$$

which is in the same form as J^T_s, thus completing the induction argument. Note that at the last step of the iteration, $s = 1$, the second term in (A.2) would not be there, and hence the optimum solution would simply be the one annihilating the first squared term, clearly a linear function of only y_0. □

APPENDIX B

In this appendix we provide expressions for the gain coefficients $\{\Pi_t\}, \{\chi_t\}$, as well as the Kalman filter equations for $\{\hat{y}_t\}$, both introduced in Section 5. In what follows A' denotes the transpose of a generic matrix A, and I denotes the identity matrix of an appropriate dimension.

Expressions for $\{\Pi_t\}, \{\chi_t\}$

$\{\Pi_t\}_{t=1}^T$ and $\{\chi_t\}_{t=1}^T$ are uniquely generated by

$$\Pi_t = -[\rho(I - M_t B)' N_t (I - M_t B) + B'B]^{-1}[B' - \rho(I - M_t B)' N_t M_t]A$$
$$\chi_t = [\rho(I - M_t B)' N_t (I - M_t B) + B'B]^{-1} B'$$

where $\{M_t\}$ and $\{N_t\}$ are generated recursively by

$$N_{t-1} = I - B\chi_t; \quad N_T = I,$$
$$M_{t-1} = [I + \rho(I - B\chi_t)^{-1}\chi_t'(I - M_t B)' N_t M_t]A; \quad M_T = A.$$

In the scalar case, Π_t, χ_t, N_t and M_t were denoted by α_t, β_t, n_t and m_t, respectively, in Theorem 2.1. Note that what replaces Condition 2.1 in the vector case is the nonsingularity of $\{N_t\}$, assuming that $B'B > 0$.

Kalman filter equation for $\{\hat{y}_t\}$

Let the time horizon be $[0, T]$, and y_0 be a Gaussian random vector with mean zero and covariance Σ_0. Let $\{y_t\}$ be the vector process generated by (38), and $\{z_t\}$ be the measurement process given by (39). Then $\hat{y}_t := E[y_t | z^t, v^t]$ is generated by

$$\hat{y}_t = A\hat{y}_{t-1} + Bv_t + \hat{\Sigma}_t H'(H\hat{\Sigma}_t H' + R)^{-1} r_t; \quad \hat{y}_{-1} = 0,$$
$$r_t = z_t - HA\hat{y}_{t-1} - HBv_t; \quad v_0 := 0,$$
$$\hat{\Sigma}_{t+1} = A[\hat{\Sigma}_t - \hat{\Sigma}_t H'(H\hat{\Sigma}_t H' + R)^{-1} H\hat{\Sigma}_t]A' + P,$$
$$\hat{\Sigma}_0 = \Sigma_0,$$

where

$$R := \mathrm{cov}(\xi_t), \quad P := \mathrm{cov}(\epsilon_t).$$

---

†$E_{/\eta_s}$ denotes the conditional expectation, with respect to the sigma-field generated by $\eta_s = y^{s-1}$.