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# ON THE DOMINANCE OF CAPITALISTS LEADERSHIP IN A 'FEEDBACK-STACKELBERG' SOLUTION OF A DIFFERENTIAL GAME MODEL OF CAPITALISM *

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This paper deals with a modified version of the Lancaster model of capitalism, where it is assumed that there is a cost jointly borne by the two groups of players (workers versus capitalists) and associated with the bargaining of a larger share of consumption for the workers. It is shown that a Feedback-Stackelberg solution, with the capitalists acting as leaders and announcing their investment policy at the beginning of each period, is a solution dominating the Feedback-Nash solution. The paper is also intended to be a tutorial on the Feedback-Stackelberg solution, a concept not so often used by economic modelers.

## 1. Introduction and motivation

In 1973, Kelvin Lancaster proposed a differential game model which illustrates some basic factors underlying the relationship between distribution and growth in industrial societies [Lancaster (1973)]. In this model, one group (the workers) controls its own consumption, while the other group (the capitalists) controls the allocation of the remainder output between investment and its own consumption. Lancaster studied the model in the framework of open-loop information structure for both players, and showed that the equilibrium of this game of capitalism is inefficient when compared to a social optimum.

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In a recent paper, Pohjola (1983) has studied the same differential game, and obtained the characterization of the so-called open-loop Stackelberg solutions where one player announces in advance his whole course of action, here referred to as an open-loop control. Pohjola was able to show that both Stackelberg solutions, with leadership devoted to the workers or to the capitalists, would lead to an improvement of the situation for both players. However, the Stackelberg game is in a stalemate since neither player wants to act as a leader.

In the present paper, the game of capitalism is slightly modified in order to incorporate a direct cost associated with the bargaining process necessary to obtain a larger share of income for the workers. This cost is modeled as a reduction of output which is more pronounced the higher the share of income claimed by the workers is. Such a reduction in output could, for example, be the result of strikes or even the consequence of a diversion of some productive effort in order to negotiate and gain a larger share for the workers. Technically this assumption destroys the 'trilinear' structure of the differential game [see Clemhout and Wan (1974) for a discussion of this class of differential games to which the Lancaster model pertains]. As a consequence the 'bang-bang' behavior of equilibrium strategies is no more a necessity, at least for the workers.

For our modified game of capitalism, we consider three possible equilibria associated with the following three types of information structures:

(i) the Feedback-Nash information structure where each player knows only the current state and current time.
(ii) the Feedback-Stackelberg information structures where one player, called the follower, knows the state, the time and the control actually chosen by the other player. This other player, called the leader, knows only the state and the time. Depending on who is the leader, there are two possible Feedback-Stackelberg structures.

The Feedback-Stackelberg solution concept was introduced, in the realm of multi-stage games, by Simaan and Cruz (1973). Through an example, they showed that this solution would not necessarily leave the leader better-off than if he had played according to the Feedback-Nash information structure. This fact, and also, probably, the fact that the solution concept was not defined, in this early work, in terms of a game in normal form, are responsible for the concept not to appeal to economic modelers.

Başar and Haurie (1982, 1984) have recently established a link between the Feedback-Stackelberg solution and an equilibrium solution for a game in normal form having an asymmetric information structure. Using the $G(\delta)$-game approach of Avner Friedman (1971), these authors have also extended the concept to the differential game setting. A nice property of these Feedback concepts is that the associated equilibrium solutions can be obtained via a dynaming programming approach. As also shown in Başar and Haurie (1982, 1984), the concepts and the dynamic programming approach are unaffected by the presence of stochastic disturbances defined as a Markov process.

In this paper, we will be able to show, by applying these concepts to the modified Lancaster game of capitalism, that:

(a) the Feedback-Nash solution and the Feedback-Stackelberg solution, with the workers being the leader, are identical.
(b) the Feedback-Stackelberg solution, with the capitalists being the leader, leads to an outcome which dominates the Feedback-Nash solution, in the sense that both players then become better-off. In practical terms this means that the capitalists are better-off by letting the workers know what their current decision on investment share is, before the workers ask for their share of consumption. By doing this they induce the workers to moderate their claims and this results in higher pay-offs for both groups.

The paper is organized as follows. In section 2, the Feedback-Stackelberg solution concept is reviewed, and some directly relevant results proved in Başar and Haurie (1984) are reported. In section 3, a simple two-period model is formulated. The three equilibria are characterized for this simple game in section 4, under the assumption of linearity of the bequest functions used in the second period. It is then shown that properties (a) and (b) stated above are true for this simple two-period game. In section 5, the results are then extended to a multi-stage game setting and finally to the differential game setting. In section 6, conclusions are drawn from these results. An appendix contains details on the derivation of the results concerning differential games.

## 2. Some discussion on the Feedback-Nash and Stackelberg equilibria in multi-stage and differential games

In this section we shall review the main aspects of the theory of Feedback-Stackelberg solution of dynamical games developed in more detail in Başar and Haurie (1982, 1984).

Let $U_1$ and $U_2$ be the decision spaces of two players $P_1$ and $P_2$. Let $J_i$: $U_1 \times U_2 \to R$ be a real valued functional denoting the pay-off to $P_i$, for $i = 1, 2$. Let us assume that the reaction set of $P_2$,

$$T_2\left(u_1^\circ\right) = \left\{u_2^\circ \in U_2\colon\ J_2\left(u_1^\circ, u_2^\circ\right) = \max_{u_2 \in U_2} J_2\left(u_1^\circ, u_2\right)\right\}, \tag{1}$$

is a singleton for every decision $u_1^\circ$ of $P_1$, so that it defines a mapping $T_2$: $U_1 \to U_2$.

Then, we call a pair $(\hat{u}_1, \hat{u}_2) \in U_1 \times U_2$ a Stackelberg solution for the static game defined in normal form by the quadruple $\{U_1, U_2, J_1, J_2\}$, with $P_1$ as the leader, if

$$\hat{u}_1 = \arg\max_{u_1 \in U_1} J_1(u_1, T_2(u_1)), \tag{2}$$

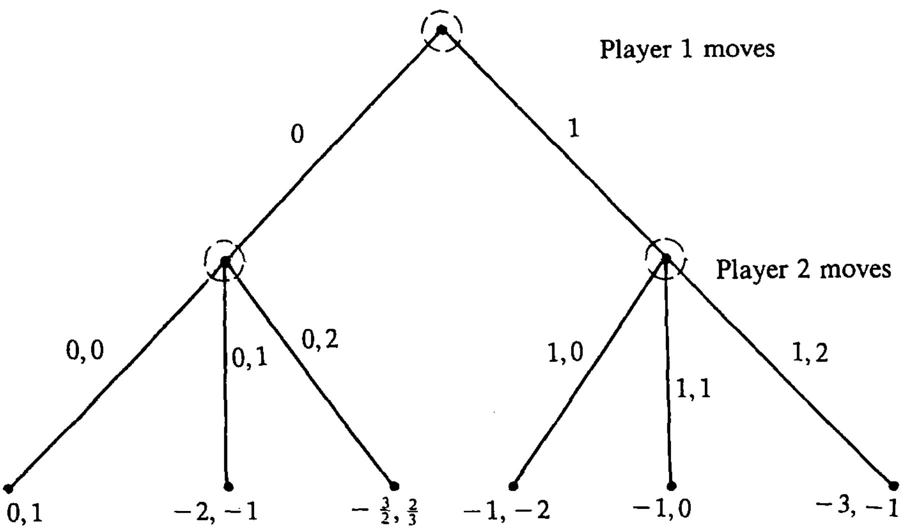
$$\hat{u}_2 = T_2(\hat{u}_1). \tag{3}$$

Consider, as an illustration, the bimatrix game defined below:

| $u_1$ | $u_2$: 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0,1<br>S | −2, −1 | −3/2, 2/3 |
| 1 | −1, −2 | −1,0<br>N | −3, −1 |

A direct verification shows that the pair $(\hat{u}_1, \hat{u}_2) = (0,0)$ is the unique Stackelberg solution with $P_1$ as the leader, yielding the pay-off pair (0, 1). This matrix game also admits a unique Nash equilibrium solution, which is $(u_1^*, u_2^*) = (1,1)$, leading to a pay-off $(-1,0)$.

Now consider the game in extensive form defined by the following tree, where the information structure is shown by the dotted circles:




This game has been obtained from the previous one by expliciting the fact that $P_1$, in his role as a leader, has to announce his decision first. The normal-form version of this game is given by the following bimatrix game:

| | $\gamma_2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | (0,0) | (0,1) | (0,2) | (1,0) | (1,1) | (1,2) | (2,0) | (2,1) | (2,2) |
| 0 | 0,1 | 0,1 | 0, −1 | −2, −1 | −2, −1 | −2, −1 | $-\frac{3}{2}, \frac{2}{3}$ | $-\frac{3}{2}, \frac{2}{3}$ | $-\frac{3}{2}, \frac{2}{3}$ |
| 1 | −1, −2 | −1,0 | −3, −1 | −1, −2 | −1,0 | −3, −1 | −1, −2 | −1,0 | −3, −1 |

where a strategy $\gamma_2(u_1) = (i, j)$ means that $u_2 = i$ if $u_1 = 0$, $u_2 = j$ if $u_1 = 1$. It is straightforward to check that the pair $\hat{u}_1 = 0$, $\hat{\gamma}_2 = (0,1)$, leading to the pay-off pair (0,1), is a Nash equilibrium for this bimatrix game. Note, however, that the pair $\hat{u}_1 = 1$, $\hat{\gamma}_2 = (1,1)$ is also a Nash equilibrium, leading to the same pay-off pair (−1,0). Only the former, which we call the 'strong equilibrium', corresponds to the Stackelberg solution (together with $P_2$'s optimal response function), whereas the latter remains as a weak equilibrium, corresponding to the Nash solution of the original matrix game. These, in fact, do not constitute the entire class of Nash equilibria of the second matrix game; for example, the pairs $\hat{u}_1 = 0$, $\hat{\gamma}_2 = (0,0)$; $\hat{u}_1 = 0$, $\hat{\gamma}_2 = (0,2)$; and $\hat{u}_1 = 1$, $\hat{\gamma}_2 = (2,1)$ are also Nash equilibria.

In spite of this proliferation of equilibria when we go from a matrix game to a dynamic two-stage game, the Nash and Stackelberg equilibrium pay-off pairs of the original game remain intact, i.e., all Nash equilibria of the new game have pay-off pairs corresponding to one of the equilibria of the original game. In Başar and Haurie (1984), it has been established rigorously that these observations are in fact valid on a much broader scale; more precisely, to every static game admitting a unique Stackelberg solution with $P_k$ as the leader, there corresponds a two-stage dynamic game one of whose equilibria, which we call 'strong equilibrium', corresponds in its outcome to that of the Stackelberg solution. This establishes formally an equivalence between the concept put forth by Von Stackelberg (1934), and the concept of equilibrium in a game having an asymmetrical information structure. Let us note that the information structure shown on the game tree above corresponds to the so-called 'Feedback' type where each player knows only the current state of the game.

This equivalence permits a direct extension of the concept of Stackelberg equilibrium to that of Feedback-Stackelberg equilibrium in a multi-stage game setting.

Consider the $T$-stage dynamic game with state dynamics

$$x(t+1) = f^t(x(t), u_1(t), u_2(t)), \qquad t = 0, 1, \ldots, T-1, \tag{4}$$

$$x(t) \in X, \quad u_i(t) \in U_i, \qquad i = 1, 2, \tag{5}$$

and pay-off functionals

$$J_i = q_i(x(T)) + \sum_{t=0}^{T-1} g_i^t(x(t), u_1(t), u_2(t)), \qquad i = 1,2, \tag{6}$$

where $f^t$, $q_i$ and $g_i^t$ are mappings of appropriate specifications.

Consider a stagewise asymmetric mode of play whereby one of the players (say, $P_1$) announces his action before the other player does, at each stage. The relevant solution concept in this case is the so-called Feedback-Stackelberg solution first introduced by Simaan and Cruz (1973a, b). In Başar and Haurie (1984) it is proved that, associated with the $T$-stage game (4)–(6), there is a $2T$-stage game whose unique Feedback equilibrium solution gives rise to the same outcome as the unique Feedback-Stackelberg solution of (4)–(6), if both exist. Therefore the Feedback-Stackelberg solution of a multi-stage game corresponds to an equilibrium in a game having an asymmetrical information structure.

There are many situations in the economic life where one agent has, either deliberately or by some institutional rule, to announce first his action and let the other agents react. This does not mean that the agent can or will announce in advance his whole course of action (viz. strategy). It only says that there is a built-in asymmetry in the information used at each stage by different agents. We believe that in many dynamic economic situations such an information structure is more reasonable than the one corresponding to the open-loop Stackelberg solution concept. There are two reasons for this belief: first the open-loop Stackelberg solution will not lend itself to a maximum-principle type of decomposition over time. The second reason is that most economic models are simplified paradigms for basically uncertain phenomena; the open-loop concept, with its lack of possibility to adapt, is therefore difficult to implement in practice.

The situation is more attractive for the feedback concepts since they lend themselves to a dynamic programming form of decomposition over time, and they are structurally stable even in the presence of (Markovian) stochastic perturbations [in Başar and Haurie (1982, 1984), stochastic uncertainties on the structure, and even the mode of play of the game, are allowed].

From multi-stage games one can pass to the setting of differential games, by using the $G(\delta)$-game approach developed by Avner Friedman (1971). In Başar and Haurie (1984) this extension is fully developed. In the present paper, to avoid lengthy repetitions, the developments are made in the appendix for the game of capitalism presented in the next sections.

We conclude this section by mentioning that we do not allow memory in the information structure used by the players. The use of memory strategies would induce informational non-uniqueness of the equilibria in deterministic games as shown in Başar and Olsder (1982). [For a theory of differential games with Stackelberg solutions in the class of memory strategies, see Tolwinski (1983).] Such strategies allow for the formulation of threats or incentives by the leader. In a stochastic context Radner (1982) considers also such strategies. The analysis of the Lancaster model under such information structures would lead to completely different results, which we do not consider in this paper.

## 3. A two-period Lancaster type model

We consider a one-sector single-technique economy whose output in period $t = 0$ may be consumed or invested. It is assumed that the workers can control their share of consumption in total output of period 0, $u_1$, within given institutional limits: $c \leq u_1 \leq b$, where $0 < c < b < 1$. The capitalists' control variable, $u_2$, is the share of output, non-consumed by the workers, which is invested in the production sector.

In Lancaster's original model (1973), or in the more recent version by Pohjola (1983), it is assumed that workers can obtain their desired share of output at no direct cost. Clearly this does not take into account the cost of bargaining, which is often equivalent to a loss of production (e.g. strikes). We therefore introduce a new term in the model, which is a function $\psi(u_1)$ giving the part of the potential output which is available in the economy, given the share of consumption $u_1$ obtained by the workers. Of course $\psi(u_1) \in [0, 1]$, for all $u_1 \in [c, b]$. Furthermore we will assume that $\psi$ is differentiable and satisfies for all $u_1 \in (0, 1)$:

$$\psi'(u_1) < 0, \quad (u_1\psi(u_1))' > 0, \quad \psi''(u_1) < 0, \tag{7}$$

which means that $\psi$ is concave and there is always some short-term incentive for the workers to ask for a larger share of consumption. Let $k$ represent the stock of capital at the beginning of period $t = 0$; then, given the controls chosen by the two groups of players (viz. the workers and the capitalists), the stock of capital available at the beginning of period 1 will be

$$K = ak\psi(u_1)(1 - u_1)u_2, \tag{8}$$

where $a$ is the given output–capital ratio.

Player 1 (the workers) valorizes this capital stock according to a bequest function $V_1(K)$. Similarly, player 2 (the capitalists) uses a function $V_2(K)$ for valorizing this capital stock. Thus, given $k$, the pay-offs resulting from the control pair $(u_1, u_2)$ are, respectively,

$$J_1(k; u_1, u_2) = ak\psi(u_1)u_1 + V_1(K), \tag{9}$$

$$J_2(k; u_1, u_2) = ak\psi(u_1)(1 - u_1)(1 - u_2) + V_2(K), \tag{10}$$

where $K$ is given by (8), and $u_1$ and $u_2$ satisfy

$$0 < c \leq u_1 \leq b < 1, \tag{11}$$

$$0 \leq u_2 \leq 1. \tag{12}$$

The model specified by (7)–(12) captures the essential structure of the dynamic conflict between capitalists and workers put forth by Lancaster (1973). It has this added feature that bargaining over the consumption share going to the workers is now costly to both players.

As it will be shown in section 5, this simple two-period model will allow us also to study $T$-period models and eventually a differential game version of the model.

## 4. Analysis of Nash and Stackelberg equilibria when the bequest functions are linear

In this section we will study and compare the Nash and Stackelberg equilibria for the game defined by (7)–(12) when the functions $V_i(K)$ are taken to be linear with respect to the inherited capital stock $K$:

$$V_i(K) = \beta_i K, \qquad i = 1,2. \tag{13}$$

The game, in its normal form, is defined by the pay-off functions

$$J_1(k; u_1, u_2) = ak\psi(u_1)u_1 + \beta_1 ak\psi(u_1)(1 - u_1)u_2, \tag{14}$$

$$J_2(k; u_1, u_2) = ak\psi(u_1)(1 - u_1)(1 - u_2) + \beta_2 ak\psi(u_1)(1 - u_1)u_2, \tag{15}$$

and the constraints (11)–(12).

The gradients of the pay-offs are defined by

$$\partial J_1/\partial u_1 = ak(u_1\psi(u_1))'(1 - \beta_1 u_2) + \beta_1 ak\psi'(u_1)u_2, \tag{16}$$

$$\partial J_1/\partial u_2 = \beta_1 ak\psi(u_1)(1 - u_1), \tag{17}$$

and

$$\partial J_2/\partial u_1 = ak((1 - u_1)\psi(u_1))'(1 + (\beta_2 - 1)u_2), \tag{18}$$

$$\partial J_2/\partial u_2 = ak(1 - u_1)\psi(u_1)(\beta_2 - 1). \tag{19}$$

We can now characterize the three possible equilibria.

### 4.1. Nash equilibrium

The pair $(u_1^*, u_2^*)$ constitutes a Nash equilibrium at $t = 0$ and for the capital stock $k$ if $c \le u_1^* \le b$, $0 \le u_2^* \le 1$, and

$$\forall u_1 \in [c, b], \qquad J_1(k; u_1^*, u_2^*) \ge J_1(k; u_1, u_2^*), \tag{20}$$

$$\forall u_2 \in [0,1], \qquad J_2(k; u_1^*, u_2^*) \ge J_2(k; u_1^*, u_2). \tag{21}$$

Studying the expression for $\partial J_2/\partial u_2$ in (19) we see that

$$\begin{aligned} \beta_2 &< 1 \Rightarrow \partial J_2/\partial u_2 < 0 \\ &= 1 \Rightarrow \partial J_2/\partial u_2 \equiv 0 \\ &> 1 \Rightarrow \partial J_2/\partial u_2 > 0. \end{aligned} \tag{22}$$

Therefore the optimal response of player 2 is easy to determine: it will be

$$\begin{aligned} u_2^* &= 0 && \text{if} \quad \beta_2 < 1 \\ &\in [0,1] && \text{if} \quad \beta_2 = 1^1 \\ &= 1 && \text{if} \quad \beta_2 > 1. \end{aligned} \tag{23}$$

Notice that this response policy is independent of the action $u_1$ of the other player.

Now, if we turn our attention to the expression for $\partial J_1/\partial u_1$, given in (16), we can readily see that

$$u_2^* = 0 \Rightarrow \partial J_1/\partial u_1 = ak(u_1\psi(u_1))' > 0 \Rightarrow u_1^* = b, \tag{24}$$

while

$$u_2^* = 1 \Rightarrow \partial J_1/\partial u_1 = ak(u_1\psi(u_1))'(1 - \beta_1) + \beta_1 ak\psi'(u_1), \tag{25}$$

and thus, in view of (7),

$$\beta_1 \ge 1 \Rightarrow \partial J_1/\partial u_1 < 0 \Rightarrow u_1^* = c. \tag{26}$$

However, for $\beta_1 < 1$ ($u_2^* = 1$ still prevailing), the sign definiteness of $\partial J_1/\partial u_1$ depends to a great extent on the structure of $\psi$ and hence not much can be

[1] This is a globally optimal response, because when $\beta_2 = 1$, $J_2$ is independent of $u_2$.

Table 1

Nash equilibrium solution of the two-period game.

| $\beta_1, \beta_2$ | $u_1^*$ | $u_2^*$ |
|---|---|---|
| $\beta_2 < 1$, $\beta_1$ arbitrary | $b$ | 0 |
| $\beta_2 > 1$, $\beta_1 \geq 1$ | $c$ | 1 |
| $\beta_2 > 1$, $\beta_1 < 1$ | a *unique* point on $[c, b]$, but cannot be determined explicitly | 1 |
| | $c$ | any point in $[0,1] \cap [1/\beta_1, \infty)$ |
| $\beta_2 = 1$, $\beta_1$ arbitrary | a *unique* point on $[c, b]$, but cannot be determined explicitly | any point in $[0,1] \cap [0, 1/\beta_1)$ |

said regarding the equilibrium value of $u_1$ without knowing more on $\psi$. By setting $\partial J_1/\partial u_1 = 0$, we obtain the equation

$$u_1 = \{\psi(u_1) - \beta_1[\psi(u_1) - \psi'(u_1)]\}/[(\beta - 1)\psi'(u_1)], \tag{27}$$

whose solution determines the nature of the equilibrium solution for player 1. More specifically, if (27) does not admit a solution in $(c, b)$, then the equilibrium solution is an extreme point, either $u_1^* = c$ or $u_1^* = b$, depending on the sign of $\partial J_1/\partial u_1$, whereas if (27) admits a solution in $(c, b)$, it is indeed the unique equilibrium solution for player 1, because $J_1$ is strictly concave in $u_1$ for every fixed $u_2$ (for $\beta_1 < 1$):

$$\partial^2 J_1/\partial u_1^2 = ak(u_1\psi(u_1))''(1 - \beta_1) + \beta_1 ak\psi''(u_1) < 0.^2$$

Hence, for $\beta_1 < 1$ and with $u_2^* = 1$, the Nash equilibrium solution for player 1 is *unique*, but it could be either an inner point or a boundary point of $[c, b]$, depending on the specific structure of $\psi$.

Now, for the final remaining case, $u_2^* \in (c, b)$, we obtain from (15) that

$$\beta_1 \geq 1/u_2^* \Rightarrow \partial J_2/\partial u_1 < 0 \Rightarrow u_1^* = c.$$

For $\beta_1 < 1/u_2^*$, however, the equilibrium solution for player 1 cannot be determined explicitly (it lies somewhere in $[c, b]$, depending on $\psi$), but it is still unique due to strict concavity of $J_1$ with respect to $u_1$ ($\partial^2 J_1/\partial u_1^2 < 0$, $\forall u_1 \in [c, b]$, $u_2^* \in (0, 1)$, with $\beta_1 < 1/u_2^*$). We summarize these results in table 1.

[2] This follows from (7) along with the identity $(u_1\psi(u_1))'' \equiv 2\psi'(u_1) + u_1\psi''(u_1) < 0$.

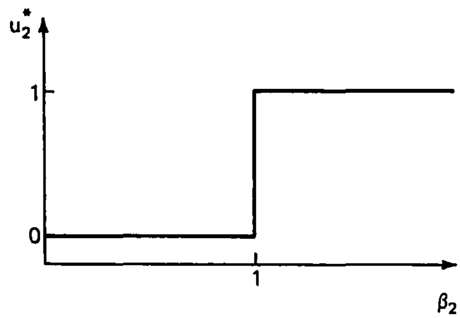



Fig. 1. Dependence of $u_2^*$ on the parameter $\beta_2$.

Therefore a Nash-equilibrium still has, as in the original Lancaster model, the 'bang-bang' structure for player 2, while, for player 1, the cost of bargaining may induce a balanced trade-off between the future advantage obtained from the investment by player 2, and the immediate advantage obtained from a higher consumption share, depending on the specific choice of $\psi$. This behavior can be seen clearly in the depictions of figs. 1–3, which are based on the results tabulated in table 1.

Note that in fig. 3 on the next page, we have drawn the curve as a non-increasing continuously differentiable function of $\beta_1$, starting at $b$ when $\beta_1 = 0$ and reaching the point $c$ for $\beta_1 = 1$. The starting point can be shown to be $b$ by applying a limiting argument on (25) as $\beta_1 \to 0$, in which case $\partial J_1/\partial u_1 \to ak(u_1\psi(u_1))' > 0$. An important further property of the curve is that there exists a point, say $\beta_1^\circ$, in (0,1) such that $u_1(\beta_1) = b$, $\forall\beta_1 \leq \beta_1^\circ$, and $c < u_1(\beta_1) < b$, $\forall\beta_1 \in (\beta_1^\circ, 1)$. The segment of the curve in this open interval is

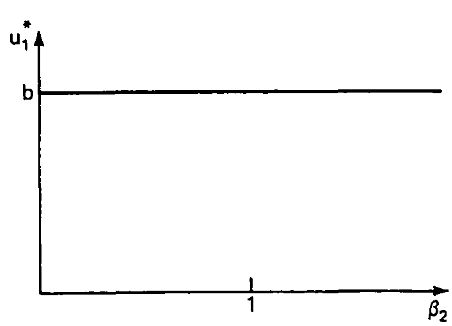



Fig. 2. Dependence of $u_1^*$ on the parameter $\beta_1$, when $\beta_2 < 1$.

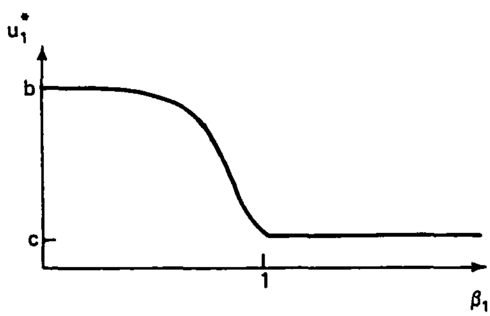



Fig. 3. Dependence of $u_1^*$ on the parameter $\beta_1$, when $\beta_2 > 1$.

in fact strictly decreasing, because from the expression for $\partial J_1/\partial u_1$ as given in (25), one can readily obtain

$$\mathrm{d}u_1/\mathrm{d}\beta_1 = \frac{ak\left[(u_1\psi(u_1))' - \psi'(u_1)\right]}{(1-\beta_1)(u_1\psi(u_1))'' + \beta_1 ak\psi''(u_1)},$$

which is negative $\forall u_1 \in [c, b]$ and $\forall \beta_1 \in [0, 1)$.

It is useful to note at this point that, if there is no cost associated with the claim of consumption shares, i.e. if $\psi(u_1) = 1$, then the equilibrium solution for player 1 for the case $\beta_2 > 1$ would simply be

$$\begin{aligned} \beta_1 &> 1 \Rightarrow u_1^* = c, \\ &= 1 \Rightarrow u_1^* \in [c, b], \text{ any value,} \\ &< 1 \Rightarrow u_1^* = b. \end{aligned} \tag{28}$$

We now conclude this subsection by stating a property of the Nash equilibrium solution of the two-period game, which will be most useful in section 5 when we seek extensions to multi-period settings.

*Property 1.* The pay-offs for the two players, corresponding to the Nash equilibrium solution determined above (and tabulated in table 1) are non-decreasing functions of the parameters $\beta_1$ and $\beta_2$, except perhaps at the point $\beta_2 = 1$.

*Verification.* Firstly note that, except possibly at $\beta_2 = 1$, $u_2^*$ does not decrease with increasing $\beta_2$ (it is in fact piecewise constant, cf. fig. 1). For fixed $\beta_1 > 0$,

$u_1^*$ does not increase with increasing $\beta_2$ (again it is piecewise constant, cf. table 1). Then, it is easy to see that for fixed $\beta_1 > 0$, both $J_1(k; u_1^*, u_2^*)$ and $J_2(k; u_1^*, u_2^*)$ are non-decreasing for increasing $\beta_2$,[3] and hence the players do not lose as $\beta_2$ increases.

Now, for fixed $\beta_2 \neq 1$, $u_2^*$ does not depend on $\beta_1$, and $u_1^*$ decreases with increasing $\beta_1$ as shown earlier. Hence, following the foregoing argument, $J_1(k; u_1^*, u_2^*)$ and $J_2(k; u_1^*, u_2^*)$ in fact increase with increasing $\beta_1$ – leading to the conclusion that both players are better off as $\beta_1$ increases. ■

### 4.2. *Stackelberg solution with player 1 as the leader*

From the previous analysis, it is immediately clear that, since the decision of player 2 is independent of the action $u_1$ taken by player 1, the reaction function of player 2 is a constant. Therefore the Stackelberg equilibrium with player 1 as the leader will be equivalent to the Nash solution.

### 4.3. *Stackelberg solution with player 2 as the leader*

First we determine the reaction function of player 1. We already know that

$$u_2 = 0 \Rightarrow u_1 = b,$$

and

$$\begin{aligned} u_2 = 1 \Rightarrow u_1 &= c, \quad \beta_1 \geq 1, \\ &= \eta, \quad \beta_1 < 1, \end{aligned}$$

where $\eta$ is a fixed (unique) point in $[c, b]$.

Finally, for $u_2 \in (c, b)$,

$$\begin{aligned} u_1 &= c, \qquad u_2 \geq 1/\beta_1, \\ &= \alpha, \qquad 0 < u_2 < 1/\beta_1, \end{aligned}$$

where $\alpha$ is a fixed (unique) point in $[c, b]$. Here both $u_1 = \alpha$ and $u_1 = \eta$ satisfy the equation

$$(u_1\psi(u_1))'(1 - \beta_1 u_2) + \beta_1 \psi'(u_1) u_2 = 0, \tag{29}$$

[3] The best way to see this is to rewrite $J_1$ and $J_2$ as

$$J_1 = ak\psi(u_1^*)(\beta_1 u_2^* - 1)(-u_1^*) + \beta_1 ak\psi(u_1^*)u_2^*,$$

$$J_2 = ak\psi(u_1^*)(1 - u_1^*) - (1 - \beta_2)ak\psi(u_1^*)(1 - u_1^*)u_2^*,$$

and observe that all four terms are non-decreasing with increasing $\beta_2$.

the former for a general $u_2 < 1/\beta_1$, and the latter for $u_2 = 1$ and $\beta_1 < 1$. To summarize, the reaction function of player 1 to player 2's announced decision $u_2 \in [0,1]$ is defined in two separate regions defined by $\beta_1$ as

$$\begin{aligned}
\beta_1 > 1: \quad u_1 = T(u_2) &= b, && u_2 = 0, \\
&= \alpha \in [c,b], && 0 < u_2 < 1/\beta, \\
&= c, && 1/\beta_1 \leq u_2 \leq 1, \\
\beta_1 \leq 1: \quad u_1 = T(u_2) &= b, && u_2 = 0, \\
&= \eta \in [c,b], && 0 < u_2 < 1, \\
&= c, && u_2 = 1,
\end{aligned} \tag{30}$$

where $\alpha$ and $\eta$ are as defined above.

Now, it is not difficult to see that $T$ is a differentiable mapping with range in $(c, b)$, and that, by differentiating (29),

$$\begin{aligned}
\mathrm{d}u_1/\mathrm{d}u_2 &= T'(u_2) \\
&= \beta_1 \frac{(u_1\psi(u_1))' - \psi'(u_1)}{(u_1\psi(u_1))''(1-\beta_1 u_2) + \beta_1 u_2 \psi''(u_1)}.
\end{aligned}$$

Since, whenever $T_1(u_2) \in (c,b)$, $1 - \beta_1 u_2 > 0$ [by (29)], we have, also as a consequence of

$$(u_1\psi(u_1))' > 0, \quad \psi''(u_1) < 0,$$

$$(u_1\psi(u_1))'' < 0,$$

the strict inequality

$$\mathrm{d}u_1/\mathrm{d}u_2 = T'(u_2) < 0, \quad \text{for range of } T \text{ in } (c,b). \tag{31}$$

Note also that $T(0) > T(1)$ for $\beta_1 > 0$. Thus, the response of the workers to an announced increase of investment by the capitalists is a reduction of their demand of consumption share, as depicted in fig. 4.

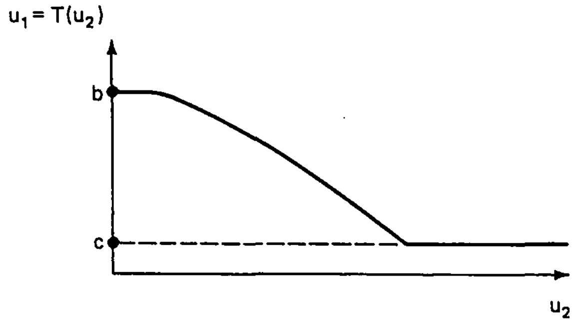



Fig. 4. A typical reaction function $T(u_2)$.

A pair $(\hat{u}_1, \hat{u}_2)$ is a Stackelberg equilibrium with player 2 as the leader if

$$c \leq \hat{u}_1 \leq b, \qquad 0 \leq \hat{u}_2 \leq 1,$$

and

$$\forall u_1 \in [c, b], \quad J_1(k; \hat{u}_1, \hat{u}_2) \geq J_1(k; u_1, \hat{u}_2), \tag{32}$$

$$\forall u_2 \in [0, 1], \quad J_2(k; \hat{u}_1, \hat{u}_2) \geq J_2(k; T(u_2), u_2)), \tag{33}$$

where $T(\cdot)$: $u_2 \to u_1 = T(u_2)$ is the unique optimal reaction function defined above.

Then, whenever it is a boundary point, the decision of the leader (player 2) under the Stackelberg equilibrium concept will be determined by the sign of the total derivative

$$\begin{aligned}\frac{\partial J_2}{\partial u_2} + \frac{\partial J_2}{\partial u_1}\frac{\mathrm{d}u_1}{\mathrm{d}u_2} &= ak(1-u_1)\psi(u_1)(\beta_2 - 1)\\ &\quad + ak((1-u_1)\psi(u_1))'(1-u_2(1-\beta_2))\frac{\mathrm{d}u_1}{\mathrm{d}u_2},\end{aligned} \tag{34}$$

where $u_1 = T(u_2)$ as given by (30).

As $\mathrm{d}u_1/\mathrm{d}u_2$ is always non-positive, and is actually equal to 0 when $u_1 = b$ or $u_1 = c$, we see from (7), (31) and (32) that

$$\beta_2 \geq 1 \Rightarrow \partial J_2/\partial u_2 > 0 \Rightarrow \hat{u}_2 = 1. \tag{35}$$

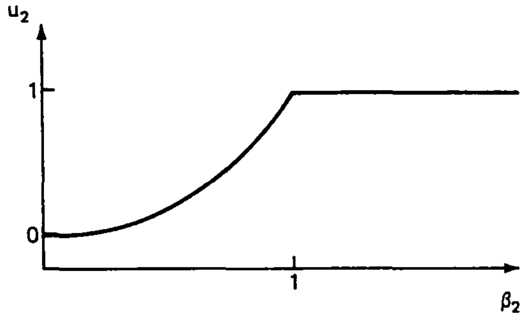



Fig. 5. $\hat{u}_2$ as a function of $\beta_2$.

However, when $\beta_2 < 1$, there can exist a value $u_2$ for which

$$\frac{\partial J_2}{\partial u_2} + \frac{\partial J_1}{\partial u_1}\frac{\mathrm{d}u_1}{\mathrm{d}u_2} = 0.$$

Therefore we can make a conjecture on the shape of the graph of the optimal decision $\hat{u}_2$ as a function of $\beta_2$, as shown in fig. 5, which should be compared with fig. 1. It appears that, being the leader, player 2 will never invest less than the amount dictated by the Nash equilibrium. This can easily be understood since, by investing more the capitalists can induce the workers to moderate their demand of consumption share.

As it is well known, the leader will always be better-off compared with what he would receive in the Nash solution.

For the follower, player 1, the situation is the following. The reaction function is either constant ($u_1 = c, u_1 = b$) or differentiable with a negative slope, as depicted in fig. 4.

The marginal effect of a variation in $u_2$, $\mathrm{d}u_2$, on the pay-off to player 1, behaving as a follower, is thus given by

$$\frac{\partial J_1}{\partial u_1}\mathrm{d}u_1 + \frac{\partial J_1}{\partial u_2}\mathrm{d}u_2 = \frac{\partial J_1}{\partial u_2}\mathrm{d}u_2,$$

since, either $T_1'(u_2) = 0$ and $\mathrm{d}u_1 = 0$, or $T'(u_2)$ is negative and then $\partial J_1/\partial u_1 = 0$. According to (17), $\partial J_1/\partial u_2 > 0$, except when $u_2 = 0$ in which case it is zero.

Therefore, since the leadership of player 2 can only raise the value of $\hat{u}_2$ compared with the Nash equilibrium value $u_2^*$, this also generates a better pay-off for player 1.

As a last remark, we should note that the optimal decisions for both players, either in the Nash or the Stackelberg equilibrium, are independent of the value of the stock of capital $k$ available at the beginning of period 0. This indicates that the following holds:

$$J_i(k; u_1^*, u_2^*) = \beta_i^* k, \qquad i = 1,2,$$

$$J_i(k; \hat{u}_1, \hat{u}_2) = \hat{\beta}_i k, \qquad i = 1,2,$$

where, according to what has been said before,

$$\hat{\beta}_1 \geq \beta_i^* \quad \text{for} \quad i = 1,2.$$

We can summarize the findings of this section in the following:

*Proposition 1. In the two-period model (7)–(12), if the bequest functions are linear functions of the final stock of capital $K$, then the following hold*:

(*i*) *The value of the pay-offs associated with either one of the three possible equilibria are again linear functions of the initial capital stock $k$.*
(*ii*) *The Nash equilibrium pay-offs are non-decreasing functions of the slopes $\beta_1$ and $\beta_2$ of the bequest functions.*
(*iii*) *The Nash solution coincides with the Stackelberg solution with player 1 as the leader.*
(*iv*) *The Stackelberg solution with player 2 (the capitalists) as the leader dominates the Nash solution, i.e., the pay-offs under the Stackelberg solution are no worse, and could even be better than the Nash pay-offs for both players. In the Stackelberg solution the capitalists tend to invest more in order to induce the workers to moderate their demand of a bigger share of income.*

## 5. Extensions to multi-stage and differential game settings

In this section we show that the analysis of the previous section can be directly extended to a multi-stage or a differential game setting, which will enable us to compare the associated feedback Nash and Stackelberg equilibria discussed in more general terms in section 2.

### 5.1. A multi-stage game of capitalism

The state of the economy at stage $t$ is described by the stock of capital $k(t)$ at the beginning of period $t$. This state evolves according to the equation

$$k(t+1) = k(t) + ak(t)\psi(u_1(t))(1 - u_1(t))u_2(t), \tag{36}$$

where $u_1(t) \in [c, b]$ is the consumption share asked by the workers in period $t$

and $u_2(t) \in [0,1]$ is the share of non-consumed output invested by the capitalists in period $t$.

We consider the game played over a sequence of $T$ periods, with respective pay-offs

$$J_1(k^\circ; u_1(\cdot), u_2(\cdot)) = \sum_{t=0}^{T-1} ak(t)\psi(u_1(t))u_1(t), \tag{37}$$

$$J_2(k^\circ; u_1(\cdot), u_2(\cdot)) = \sum_{t=0}^{T-1} ak(t)\psi(u_1(t))(1-u_1(t))(1-u_2(t)), \tag{38}$$

where $k^\circ$ is the initial capital stock at $t=0$, and $u_i(\cdot) = (u_i(0), u_i(1), \ldots, u_i(T-1))$ is the control sequence of player $i$.

At stage $T$, as in the original Lancaster model, there is no bequest function. It should be clear in what follows that the analysis could be conducted in exactly the same way with a pair of bequest functions linear with respect to the terminal state $k(T)$ reached.

The way this game is played depends on the information structure. As discussed in section 2 we will consider the three possible feedback equilibria, corresponding to the following three information structures:

(a) no leader, i.e., each player has only access to the current state and stage information,

(b) one of the players is the leader, i.e., the other player (follower) has access to the current state, the control chosen by the leader, and the stage. The leader has only access to the current state and stage.

As it is well known [e.g. see Başar and Olsder (1982)], the Feedback-Nash equilibrium solution can be obtained through the dynamic programming equations

$$\begin{aligned} V_1^*(k(t),t) &= ak(t)\psi_1(u_1^*(t))u_1^*(t) \\ &\quad + V_1^*(ak(t)(1-u_1^*(t))u_2^*(t), t+1), \\ V_2^*(k(t),t) &= ak(t)\psi_1(u_1^*(t))(1-u_1^*))(1-u_2^*(t)) \\ &\quad + V_2^*(ak(t)(1-u_1^*(t))u_2^*(t), t+1), \end{aligned} \tag{39}$$

where $(u_1^*(t), u_2^*(t))$ satisfies

$$V_1^*(k(t),t) \geq ak(t)\psi(u_1)u_1 + V_1^*\big(ak(t)(1-u_1)u_2^*(t), t+1\big), \tag{40}$$

$$V_2^*(k(t),t) \geq ak\psi\big(u_1^*(t)\big)\big(1-u_1^*(t)\big)(1-u_2) + V_2^*\big(ak(t)\big(1-u_1^*(t)\big)u_2, t+1\big), \tag{41}$$

$\forall\ u_1, u_2,\ c \leq u_1 < b,\ 0 \leq u_2 \leq 1$, and with the terminal conditions

$$V_i^*(k(T),T) = 0, \qquad i = 1,2. \tag{42}$$

Therefore, at each stage $t$, given the current state $k(t)$, the two players have to solve a game which is exactly the two-stage game defined in section 3.

At stage $T$ the value functions $V_i^*(k(T),T)$ are linear w.r.t. $k(T)$. Therefore, by Proposition 1, at any stage $t \in \{0,1,\ldots,T-1\}$ the value function $V_i^*(k(t),t)$ will be linear in $k(t)$:

$$V_i^*(k(t),t) = \beta_i^*(t)k(t), \qquad i = 1,2. \tag{43}$$

The qualitative study undertaken in section 4.1 is still valid and the equilibrium strategies will be such that (cf. table 1)

$$\begin{aligned} \beta_2^*(t) &< 1 \Rightarrow u_2^*(t) = 0, \\ &= 1 \Rightarrow u_2^*(t) \text{ is a unique point in } [0,1], \\ &> 1 \Rightarrow u_2^*(t) = 1, \end{aligned} \tag{44}$$

$$\begin{aligned} \beta_2^*(t) &< 1 \Rightarrow u_1^*(t) = b, \\ &\geq 1 \text{ and } \beta_1^*(t) \geq 1 \Rightarrow u_1^*(t) = c, \\ &\geq 1 \text{ and } \beta_1^*(t) < 1 \Rightarrow u_1^*(t) \text{ is a unique point in } [c,b]. \end{aligned} \tag{45}$$

Clearly these strategies are only dependent on the stage $t$ and not on the state $k(t)$. This is due to the linearity w.r.t. the state variable of both the state equation (36) and the pay-offs (37). Therefore the Feedback-Nash equilibrium happens to be also an open-loop Nash equilibrium.

Let us now consider the Feedback-Stackelberg solution with player 2 as the leader. As stated in section 2, the solution can be obtained through the dynamic programming equations

$$\hat{V}_1(k(t),t) = ak(t)\psi(\hat{u}_1(t))\hat{u}_1(t) + V_1(ak(t)\psi(\hat{u}_1(t))(1-\hat{u}_1(t))\hat{u}_2(t), t+1), \tag{46}$$

$$\hat{V}_2(k(t),t) = ak(t)\psi(\hat{u}_1(t))(1-\hat{u}_1(t))(1-\hat{u}_2(t)) + V_2(ak(t)\psi(\hat{u}_1(t))(1-\hat{u}_1(t))\hat{u}_2(t), t+1), \tag{47}$$

where the pair $(\hat{u}_1(t), \hat{u}_2(t))$ satisfies

$$\hat{u}_1(t) = T_1(k(t), t; \hat{u}_2(t)), \tag{48}$$

$$\hat{V}_2(k(t),t) \geq ak(t)\psi(T_1(k(t),t;u_2))(1-T_1(k(t),t;u_2))(1-u_2) + \hat{V}_2(ak(t)\psi(T_1(k(t),t;u_2)) \times (1-T_1(k(t),t;u_2))u_2, t+1), \tag{49}$$

$\forall u_2 \in [0,1]$, and where $T_1(k(t), t; u_2)$ is the reaction function of player 1 at $(k(t), t)$ defined by

$$T_1(k(t),t;u_2) = \underset{c \leq u_1 \leq b}{\arg\max}\{ak(t)\psi(u_1)u_1 + \hat{V}_1(ak(t)\psi(u_1)(1-u_1)u_2, t+1)\}. \tag{50}$$

Here again the analysis conducted in section 4.3 permits us to conclude that

$$\hat{V}_i(k(t),t) = \hat{\beta}_i(t)k(t), \qquad i=1,2, \quad t=0,1,\ldots,T-1. \tag{51}$$

It is also possible to describe the strategy pair by the conditions

$$\begin{aligned}\hat{\beta}_2(t) &> 1 \Rightarrow \hat{u}_2(t) = 1,\\ &\leq 1 \Rightarrow \hat{u}_2(t) \text{ is a unique point in } [0,1],\end{aligned} \tag{52}$$

while $\hat{u}_1(t) = T_1(t, \hat{u}_2(t))$. Notice that the reaction function does not depend on the value of the current state.

Finally, starting from stage $T$ where

$$0 = \hat{V}_i(k(T),T) = V_i^*(k(T),T), \qquad i=1,2, \tag{53}$$

Proposition 1(iii) implies that

$$\hat{\beta}_i(T-1) \geq \beta_i^*(T-1), \qquad i=1,2. \tag{54}$$

Now, if at any stage $t+1$ one has

$$\hat{\beta}_i(t+1) \geq \beta_i^*(t+1), \qquad i=1,2, \tag{55}$$

then, by Proposition 1(ii) and 1(iii), one has also at stage $t$

$$\hat{\beta}_i(t) \geq \beta_i^*(t), \qquad i=1,2. \tag{56}$$

This is the main conclusion which we can summarize in:

*Proposition 2. (i) In the multi-stage model (36)–(38), the optimal value functions associated with any equilibrium solution is linear w.r.t. to the state variable, at any stage t.*

*(ii) The Feedback-Stackelberg equilibrium with player 2 as the leader gives rise to a pair of value functions $(\hat{V}_1(k(t),t),\ \hat{V}_2(k(t),t))$ which dominates the pair of value functions $(V_1^*(k(t),t),\ V_2^*(k(t),t))$ associated with the Feedback-Nash equilibrium solution.*

*(iii) The Feedback-Stackelberg solution with player 1 as the leader coincides with the Feedback-Nash solution.*

## 5.2. A differential game model of capitalism

A differential game version of the model discussed above is given by the state equation:

$$\dot{k}(t) = ak(t)\psi(u_1(t))(1-u_1(t))u_2(t), \tag{57}$$

$$0 < c \leq u_1(t) \leq b < 1, \tag{58}$$

$$0 \leq u_2(t) \leq 1, \tag{59}$$

and the pay-offs

$$J_1(k^\circ; u_1(\cdot), u_2(\cdot)) = \int_0^T ak(t)\psi(u_1(t))u_1(t)\,\mathrm{d}t, \tag{60}$$

$$J_2(k^\circ; u_1(\cdot), u_2(\cdot)) = \int_0^T ak(t)\psi(u_1(t))(1-u_1(t))(1-u_2(t))\,\mathrm{d}t, \tag{61}$$

associated with any piecewise continuous control pair

$$u_1(\cdot): [0,T] \to [c,b],$$

$$u_2(\cdot): [0,T] \to [0,1].$$

In section 3 we have discussed in more general terms the relationship between the two paradigms of differential and multi-stage games.

For the model (57)–(61) we can consider the three possible information structures treated before and the three associated feedback equilibria.

A very similar analysis, developed in the appendix, leads to the following result:

*Proposition 3.* (*i*) *In the model* (*57*)–(*61*), *the Feedback-Stackelberg solution, defined in section 2, with player 2 as the leader, dominates the Feedback-Nash equilibrium.*

(*ii*) *The value functions are linear in the state variable.*

(*iii*) *The Feedback-Nash equilibrium strategies are also open-loop strategies as they do not depend on the state variable.*

## 6. Conclusion

In his (1973) paper Kelvin Lancaster analysed the causes of social welfare loss from the capitalist program, as being essentially due to the fact that any attempt by one group to stick to socially optimal behavior would leave the other group with a strong temptation to take advantage of this by using its optimal reaction. By a converging process 'à la Cournot' the unique stable outcome was shown to be the equilibrium.

Matti Pohjola (1983) showed that, by playing a non-equilibrium solution, viz. an open-loop Stackelberg solution, the situation of both players would improve. However, the open-loop structure of the strategies as well as the stalemate concerning which group would take the leadership still left an unstable situation.

In the present paper we have formulated, for a slightly different version of the model which we believe to be more realistic, an equilibrium solution which dominates the Feedback-Nash solution. This equilibrium corresponds to the Feedback-Stackelberg concept where player 2, viz. the capitalists, announces at each stage what his investment share decision is. Furthermore, in this situation, there is no possibility of improvement if the leadership is left to the workers. There is therefore no stalemate.

As the Feedback-Stackelberg solution is, in fact, an equilibrium, it is a stable situation where no group has any temptation to change unilaterally its course of action. The important conclusion is that the capitalists, by being prompt to announce their investment eagerness, give to the workers an incentive to moderate their consumption share claims. This drives the solution towards a better outcome in terms of social welfare.

Another aspect of this solution is that it is structurally stable, even in the presence of stochastic disturbances. For example we could include randomness in the true cost of bargaining and get the same result [for this we would have to use the stochastic version of the dynamic programming equations as established in Başar and Haurie (1982, 1984)].

One could argue that these dynamic game models are highly abstract since some parameters like the cost of bargaining or the output–capital ratio are not known with equal precision by the two players, and therefore the exact computation of the solutions of the dynamic programming equations would not be possible in practice. In section 4 we have considered a two-period model which is likely to be the type of game actually played during each round of negotiation. Even if the value functions, thereby called bequest functions, are subjectively defined instead of being precisely computed, the advantage of having the capitalists acting first remains. This is a strong case in favor of Feedback-Stackelberg leadership of the capitalists group.

## Appendix 1

*Definition and properties of the Feedback-Stackelberg solution for a differential game of capitalism*

The theory of differential games involves delicate mathematical intricacies mainly due to the continuous-time differential equation description of state dynamics. One of the most rigorous definitions for differential games is given by Avner Friedman (1971) where the equilibrium solution of the game is obtained as the limit of equilibria for $G(\delta)$-games which are discretized (in time) versions of the original differential game. This approach is also used in Başar and Haurie (1984) to show that such a limiting process could very well lead to an asymmetrical equilibrium where player 2, for example, would have the leadership and where, using the notations of section 5.2, the strategies would be described as mappings

$$\hat{\gamma}_1\colon (t,k,u_2) \to u_1(t) \in [c,b], \tag{A.1}$$

$$\hat{\gamma}_2\colon (t,k) \to u_2(t) \in [0,1], \tag{A.2}$$

for player 1 and player 2, respectively. Then, if $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are continuously differentiable in $k$, the continuous-time Feedback-Stackelberg solution can be obtained via the dynamic programming equations

$$-\frac{\partial \hat{V}_1(t,k)}{\partial t} = \max_{u_1 \in [c,b]} \left\{ ak\psi(u_1)u_1 + \frac{\partial \hat{V}_1(t,k)}{\partial k} ak\psi(u_1)(1-u_1)\hat{\gamma}_2(t,k) \right\}, \tag{A.3}$$

$$-\frac{\partial \hat{V}_2(t,k)}{\partial t} = \max_{u_2 \in [0,1]} \left\{ ak\psi(\hat{\gamma}_1(t,k,u_2))(1-\hat{\gamma}_1(t,k,u_2))(1-u_2) + \frac{\partial \hat{V}_2(t,k)}{\partial k} ak\psi(\hat{\gamma}_1(t,k,u_2))(1-\hat{\gamma}_1(t,k,u_2))u_2 \right\}, \tag{A.4}$$

under the additional condition that, $\forall u_1 \in [c,b]$, $\forall u_2 \in [0,1]$,

$$ak\psi(\hat{\gamma}_1(t,k,u_2))\hat{\gamma}_1(t,k,u_2) + \frac{\partial \hat{V}_1(t,k)}{\partial k} ak\psi(\hat{\gamma}_1(t,k,u_2))(1-\hat{\gamma}_1(t,k,u_2))u_2$$
$$\geq ak\psi(u_1)u_1 + \frac{\partial \hat{V}_1(t,k)}{\partial k} ak\psi(u_1)(1-u_1)u_2, \tag{A.5}$$

$$\hat{V}_i(T,k) = 0. \tag{A.6}$$

Therefore, by (A.5), the strategy $\hat{\gamma}_1(t,k,u_2)$ is a local reaction function of player 1 to the action announced by player 2.

There is a strong similarity between the continuous-time dynamic programming equations and the discrete-time analog used in section 5.1.

It would be straightforward to check that the solution of (A.3)–(A.6) would yield linear value functions

$$\hat{V}_i(t,k) = \hat{\beta}_i(t)k \quad \text{with} \quad \hat{\beta}_i(T) = 0. \tag{A.7}$$

In the case of Feedback-Nash solutions the standard approach [see Friedman (1971)] would also yield linear value functions

$$V_i^*(t,k) = \beta_i^*(t)k, \quad i = 1,2, \quad \text{with} \quad \beta_i^*(T) = 0. \tag{A.8}$$

A repetition of the same arguments as before would show that, $\forall t \in [0, T]$,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\beta_i^*(t) \leq -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\beta}_i(t), \qquad i = 1, 2, \tag{A.9}$$

and hence, as a consequence of the equalities

$$\beta_i^*(T) = \hat{\beta}_i(T) = 0, \qquad i = 1, 2,$$

one will have, $\forall t \in [0, T]$,

$$\beta_i^*(t) \leq \hat{\beta}_i(t), \qquad i = 1, 2.$$

Therefore the Feedback-Stackelberg solution dominates the Feedback-Nash solution.

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