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Informational Properties of the Nash Solutions of Two Stochastic Nonzero-Sum Games*

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Within the framework of stochastic two-person nonzero-sum games, we deal with two commonly used models in engineering and economics—namely, the LQG (Linear-Quadratic-Gaussian) and the duopoly problems. We investigate how variations in information available to either player affect the equilibrium Nash strategies for these two models, whose existence and uniqueness have been proven in the paper. We show that for the LQG model better information for either player results in lower average Nash costs for both players; whereas for the duopoly model better information for one player helps him alone to achieve a higher average Nash profit, and it hurts the other player in the sense that his average Nash profit decreases. We further relate these properties of the Nash solutions for these two games to some of the distinct features of zero-sum games and team problems.

I. INTRODUCTION

Every stochastic two-person nonzero-sum static game involves

(i) x: unknown state of nature; $x \in X$, the set of possible states of nature. Usually a probability distribution is given on X.

(ii) u: the control variable of player 1; $u \in U$, the set of all possible control values that can be picked by player 1. Similarly, v: the control variable of player 2; $v \in V$, the set of all possible control values that can be picked by player 2.

(iii) $J_1(x, u, v)$ and $J_2(x, u, v)$: the objective functions of players 1 and 2, respectively (assumed to be known by both players).

(iv) $z_i = h_i(x, w_i)$: the measurement made by player i about the

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state of nature x. $h_i(\cdot, \cdot)$ is a measurable function in two variables and w_i represents the noise corrupting the measurement; the statistics of w_i are known by both players for $i = 1, 2$. Alternatively, the probability density function $p(z_i \mid x)$ may be specified.

(v) γ_i: the control law of player i; $\gamma_i \in \Gamma_i$, the space of all admissible control laws for player i. More specifically, Γ_1 is the space of all Borel measurable functions mapping the measurement space Z_1 into U. Similarly, Γ_2 is the space of all Borel measurable functions mapping Z_2 into V.

(vi) $\bar{J}_i(\gamma_1, \gamma_2) \triangleq E\{J_i(x, u, v) | \, u = \gamma_1(\cdot), \;\; v = \gamma_2(\cdot)\}, \;\; i = 1, 2$: the expected cost (or payoff) to player i under the assumption that player 1 uses the control law $\gamma_1(\cdot) \in \Gamma_1$ and player 2 uses $\gamma_2(\cdot) \in \Gamma_2$. $E\{\cdot\}$ denotes the expectation operation taken over the prior statistics of the random variables x, w_1, and w_2.

With the assumption that $\bar{J}_i(\gamma_1, \gamma_2)$ stands for the expected cost (payoff) incurred to player i for a given pair of policies $\{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$, the prime objective of player i is to pick a policy $\gamma_i \in \Gamma_i$ so as to minimize (maximize) his average cost(payoff)against some optimal policy of player j, $j \neq i$. Hence, the equilibrium solution for the game will be given by $\{\gamma_1^* \in \Gamma_1, \gamma_2^* \in \Gamma_2\}$, satisfying the inequalities

$$\bar{J}_1(\gamma_1^*(\cdot), \gamma_2^*(\cdot)) \leqslant \bar{J}_1(\gamma_1, \gamma_2^*(\cdot))$$

and (1)

$$\bar{J}_2(\gamma^*(\cdot), \gamma_2^*(\cdot)) \leqslant \bar{J}_2(\gamma_1^*(\cdot), \gamma_2)$$

for all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Such an equilibrium solution is known as the *Nash* (or equivalently *Cournot*) solution for the game considered in this section, and $\bar{J}_i(\gamma_1^*, \gamma_2^*)$ is said to be the *expected* (*average*) *Nash cost* of player.

It is clear from the above that the term "stochastic game" is used here in a rather different sense the well-known "Stochastic game" of Shapley (*Proc. NAS* **39**, 1953). However, the terminology is so natural in this instance and yet the problems are so different that no confusion should result. Also, the adjective "static" is used in the sense that the information z_i is not affected by the control actions u, v taken, as might be the case when "dynamics" are present.

Different measurements $z_i = h_i(x, w_i)$ will, in general, provide player i with different information about the unknown parameter (state of nature) x, and different information about x will, in general, yield different Nash policies for both players, which will in turn give rise to different Nash costs. Hence, within this framework, one of the goals of this paper is an attempt to investigate the effects of the change in information to either player on the Nash costs of both players. We basically work with two scalar models commonly used in the engineering and economics literature—namely, the LQG (Linear-Quadratic-Gaussian) and the duopoly model—and also assume $h_i(\cdot, \cdot)$ to be linear in both variables, with the random variables involved being statistically independent and Gaussian. The information available to each player about the random state x varies with the variance of the additive Gaussian noise corrupting his measurements. We investigate the variations in the Nash costs of both players as a function of the variations in the information available to either player. We show that for the LQG model "better information" (yet to be defined precisely in Section II.1) for player i results in lower average Nash costs for both players but that it sometimes helps player j ($j \neq i$) more than it helps player i. For the duopoly model, however, "better information" for player i definitely helps him in achieving a lower average Nash cost (or higher average Nash profit), but it always hurts the other player in the sense that his average Nash profit decreases. These two distinct properties of the LQG and duopoly models can be related to rather distinct features of zero-sum games and team problems, which are two extreme cases of nonzero-sum two-person games.

A second goal of this paper is to demonstrate an approach to the Nash solution of a certain class of stochastic game-theoretic problems, particularly with respect to its existence and uniqueness. In this respect, the method used is not restricted to the two specific models discussed in the paper and has considerable general applicability. This point will become clear in the sequel.

II. The Two Models

II.1 *The LQG Model*

The first model to be considered in this paper is the LQG model[1] commonly used in the engineering literature—that is, the nonzero-sum two-person stochastic game defined by the quadratic cost functions

$$J_{1L}(x, u, v) = (x + u + v)^2\, c_1 + u^2 d_1 \tag{2a}$$

and

$$J_{2L}(x, u, v) = (x + u + v)^2\, c_2 + v^2 d_2\,, \tag{2b}$$

[1] This model describes a situation in which two players share a common objective but have possibly different physical constraints and/or costs on their noncooperative actions to achieve that objective.

where c_1, c_2, d_1, and d_2 are positive scalars known to both players, and x is a Gaussian random variable with mean $\bar{x}$ and variance σ; i.e.,

$$x \in N[\bar{x}, \sigma] \ (\sigma > 0).$$

Player i observes the random variable x through noisy observation z_{iL}, where

$$z_{iL} = x + w_{iL}, \quad w_{iL} \in N[0, s_{iL}], \quad s_{iL} > 0, \quad i = 1, 2 \tag{3}$$

with w_{1L}, w_{2L}, and x statistically independent.

For this problem, $U = V = R^1$ and $\Gamma_1 = \Gamma_2 = \Sigma$, which is the space of all Borel measurable functions mapping R^1 into R^1. Hence an admissible control policy for player i is to pick $\gamma_{iL}(\cdot)$ as a measurable function of his observation z_{iL}; i.e., $\gamma_{iL} \in \Sigma$ $(i = 1, 2)$.

DEFINITION 1. Within the setup described above, we say that a measurement z_{iL} provides *better information* about the random variable x than a measurement z'_{iL} if, and only if, $s_{iL} < s'_{iL}$.

The use of the term "better information" can be justified for the problem at hand by observing the following two facts:

(a) Denoting the mean-squared estimation error $E\{(x - E[x \mid z_{iL}])^2\}$ by $e(s_{iL})$, it is not difficult to show that $e(s_{iL} < e(s'_{iL})$ if, and only if, $s_{iL} < s'_{iL}$.

(b) By using entropy analysis, the information flow about x is given by $R(s_{iL}) = H(x) - H(x \mid z_{iL})$, where $H(x)$ denotes the entropy associated with the random variable x; i.e.,

$$H(x) = -\int_{-\infty}^{+\infty} p(x) \log p(x)\, dx,$$

where $p(x)$ is the probability density function of x. $H(x \mid z_{iL})$ denotes the equivocation; i.e.,

$$H(x \mid z_{iL}) = -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, z_{iL}) \log p(x \mid z_{iL})\, dx\, dz_{iL},$$

where $p(x, z_{iL})$ is the joint probability density of the random variables x and z_{iL} and $p(x \mid z_{iL})$ is the conditional probability density of x given z_{iL}. It can be shown that, for the measurement defined by (3), Shannon's information flow $R(s_{iL})$ is given by

$$R(s_{iL}) = \frac{1}{2} \log \left(\frac{\sigma + s_{iL}}{s_{iL}}\right),$$

and hence $R(s_{iL}) > R(s'_{iL})$ if, and only if, $s_{iL} < s'_{iL}$.

It can be shown that, within the framework of Definition 1, our concept of "better information" can be interpreted as the concept of "informativeness" (condition G) introduced by Marschak and Miyasawa (see [3], in particular pp. 149–150). This equivalence can be established by identifying the increase in the variance of the noise as a noise source cascaded onto the original channel.

Under the hypothesis of Definition 1, we now seek a Nash solution pair for the LQG problem posed in this section and want to investigate how better information on part of either player affects the average Nash costs of both players.

II. 2. *The Duopoly Model*

The second model to be considered in this paper is a stochastic Cournot duopoly situation—i.e., the problem of two firms with identical products operating in a market in which the structure of the market demand function is known. To be more precise, we consider a linear demand curve of the form

$$p = \alpha - \beta(q + r), \tag{4}$$

where p denotes the price of the commodity, q is the production level of firm 1, r is the production level of firm 2, and $q + r$ is the total quantity demanded. $\beta > 0$ is a scalar that determines the slope of the demand curve, and it is known to both firms. α is a random variable whose statistics are known by both firms to be Gaussian with mean $\bar{\alpha} > 0$ and variance $\xi > 0$; i.e., $\alpha \in N[\bar{\alpha}, \xi]$. Each firm has access to information about the value of α through noisy channels and, adopting an additive model, we define the observation of firm i by

$$z_{iD} = \alpha + w_{iD}, \qquad w_{iD} \in N[0, s_{iD}], \qquad s_{iD} > 0, \qquad i = 1, 2$$

where α, w_{1D}, and w_{2D} are statistically independent.

Within the framework of the duopoly problem defined above, we again consider a static nonzero-sum stochastic game, and furthermore we assign quadratic cost functions to both firms. Firm i wants to maximize his average profit $\bar{P}_i$ under the assumptions that firm $j\,(j \neq i)$ acts rationally and that the process is noncooperative—that is, there can be no explicit collusion. The profit functions P_i are defined by

$$P_1(\alpha, q, r) = qp - k_1 q^2, \qquad k_1 > 0 \tag{5a}$$

and

$$P_2(\alpha, q, r) = rp - k_2 q^2, \qquad k_2 > 0 \tag{5b}$$

and the average profit function $\bar{P}_i(\gamma_{1D}, \gamma_{2D})$ for the ith firm, for fixed $\gamma_{1D}(\cdot)$ and $\gamma_{2D}(\cdot) \in \Sigma$ is given by

$$\bar{P}_i(\gamma_{1D}, \gamma_{2D}) = E\{P_i(q = \gamma_{1D}(z_{1D}), r = \gamma_{2D}(z_{2D})) \mid \gamma_{1D}(\cdot), \gamma_{2D}(\cdot)\}, \quad i = 1, 2 \tag{6}$$

where the expectation operation is taken over the prior statistics of α, w_{1D}, and w_{2D}.

For the purpose of consistency in the paper, we will henceforth be dealing with the cost function $J_{iD}(\alpha, q, r)$ instead of the profit function $P_{iD}(\alpha, q, r)$, where $J_{iD}(\alpha, q, r) \equiv -P_{iD}(\alpha, q, r)$ $(i = 1, 2)$. We again seek a Nash solution pair γ_{1L}^*, $\gamma_{2L}^* \in \Sigma$ consistent with inequalities (1), and furthermore we investigate how better information (in the sense of Definition 1) on the part of either firm affects the average Nash profits of both firms.

II.3 *Existence and Uniqueness of Affine Nash Policies*

The solutions to the nonzero-sum game problems of Sections II, 1 and II.2 will be obtained by first embedding these two problems within the framework of a more general stochastic game problem and then by deriving the Nash solution to this auxiliary problem. To this end, we consider a nonzero-sum stochastic game defined by the quadratic cost functions

$$J_1(x, u, v) = (x + u + v)^2 \mu_1 + u^2\mu_2 + v^2\mu_3 + vx\mu_4 + x^2\mu_5 \tag{7a}$$

and

$$J_2(x, u, v) = (x + u + v)^2 \mu_6 + v^2\mu_7 + u^2\mu_8 + ux\mu_9 + x^2\mu_{10}, \tag{7b}$$

where μ_i are scalars known to both players, with μ_1, μ_2, μ_6, and μ_7 positive. x is a Gaussian random vector with mean $\bar{x}$ and variance $\sigma > 0$. Player 1 controls u through an observation z_1 and player 2 controls v through an observation z_2, where

$$z_i = h_i x + w_i, \qquad w_i \in N[0, s_i], \qquad s_i > 0, \qquad i = 1, 2 \tag{8}$$

with x, w_1, and w_2 statistically independent.

Defining the average cost function $\bar{J}_i(\gamma_1, \gamma_2)$ for the ith player in a similar way as in item (vi) of Section I, we seek a Nash solution pair $(\gamma_1, \gamma_2 \in \Sigma)$ for the problem defined by (7) and (8).

The LQG model and the duopoly model of Sections II.1 and II.2, respectively, can now be thought of as special cases of this more general problem through the following identifications:

Identification 1. The problem defined by (7) and (8) becomes identical to the LQG model of Section II.1 when $\mu_1 = c_1$, $\mu_2 = d_1$, $\mu_3 = \mu_4 = \mu_5 = 0$, $\mu_6 = c_2$, $\mu_7 = d_2$, $\mu_8 = \mu_9 = \mu_{10} = 0$, $h_1 = h_2 = 1$, $s_1 = s_{1L}$, and $s_2 = s_{2L}$.

Identification 2. The problem defined by (7) and (8) becomes identical to the duopoly model of Section II.2 when

$$\mu_1 = \frac{\beta}{2}, \quad \mu_2 = \frac{\beta}{2} + k_1, \quad \mu_3 = -\frac{\beta}{2}, \quad \mu_4 = -\beta, \quad \mu_5 = -\frac{\beta}{2},$$

$$\mu_6 = \frac{\beta}{2}, \quad \mu_7 = \frac{\beta}{2} + k_2, \quad \mu_8 = -\frac{\beta}{2}, \quad \mu_9 = -\beta, \quad \mu_{10} = -\frac{\beta}{2},$$

$$x \equiv -\frac{\alpha}{\beta}, \quad \bar{x} = -\frac{\bar{\alpha}}{\beta}, \quad h_1 = h_2 = -\beta, \quad \sigma = \frac{\xi}{\beta^2}, \quad s_1 = s_{1D},$$

$$s_2 = s_{2D}, \quad u \equiv q, \quad \text{and} \quad v \equiv r.$$

The following two propositions will establish the existence and uniqueness of affine Nash strategies for this general problem.

PROPOSITION 1. *There exists a Nash solution pair for the stochastic nonzero-sum game defined by* (7) *and* (8) *if, and only if, there exists a person-by-person optimal solution* (*i.e., minimizing control laws*) *for the team problem defined by the cost function*

$$J(u, v) = (x + u + v)^2 \mu_1\mu_6 + u^2\mu_2\mu_6 + v^2\mu_7\mu_1, \tag{9}$$

and the observations (8), *where u and v belong to the same class of admissible controls as in the original game problem. Furthermore, the optimal solution to this team problem is equivalent to the Nash solution of the original game problem.*

Proof. We observe that a pair $[u = \gamma_1^*(z_1), v = \gamma_2^*(z_2)]$ is a Nash solution to the nonzero-sum game defined by (7) and (8) if, and only if, the same pair provides a Nash solution for the nonzero-sum game defined by the cost functions

$$\tilde{J}_1(u, v) = \eta_1 J_1(x, u, v) + f_1(v, x) \tag{10a}$$

and

$$\tilde{J}_2(u, v) = \eta_2 J_2(x, u, v) + f_2(u, x) \tag{10b}$$

and the measurements (8). In (10a) and (10b), $\eta_1 > 0$ and $\eta_2 > 0$ are scalars and $f_i(\cdot, \cdot)$ is any scalar measurable function in two variables for $i = 1, 2$. This observation follows directly from the definition of a Nash solution [see relations (1)] and the static property of the information structure (8)—i.e., the independence of the information variables z_i from the control variables u, v.

Now picking

$$\eta_1 = \mu_6, \qquad \eta_2 = \mu_1,$$

$$f_1(v, x) = v^2(\mu_7\mu_1 - \mu_3\mu_6) - vx\mu_4\mu_6 - x^2\mu_5\mu_6,$$

and

$$f_2(u, x) = u^2(\mu_2\mu_6 - \mu_8\mu_1) - ux\mu_9\mu_1 - x^2\mu_{10}\mu_1,$$

we see that

$$\tilde{J}_1(u, v) \equiv \tilde{J}_2(u, v) \equiv J(u, v),$$

which is defined by (9).[2] Q.E.D.

PROPOSITION 2. *The optimal solution to the team problem* (9) *with the information structure* (8) [*and consequently to the game problem* (7) *with the information structure* (8)] *is* uniquely *defined by*

$$u^* = \gamma_1^*(z_1) = a_1^*(z_1 - h_1\bar{x}) + a_2^*\bar{x} \tag{11a}$$

and

$$v^* = \gamma_2^*(z_2) = b_1^*(z_2 - h_2\bar{x}) + b_2^*\bar{x}, \tag{11b}$$

where

$$a_1^* = -\frac{\mu_1}{\mu_1 + \mu_2}\left(\frac{h_1\sigma}{h_1^2\sigma + s_1}\right)\left[1 - \left(\frac{\mu_6}{\mu_6 + \mu_7}\right)\left(\frac{h_2^2\sigma}{h_2^2\sigma + s_2}\right)\right]$$
$$\times \left[1 - \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)\left(\frac{\mu_6}{\mu_6 + \mu_7}\right)\left(\frac{h_1h_2\sigma}{h_1^2\sigma + s_2}\right)\left(\frac{h_1h_2\sigma}{h_2^2\sigma + s_2}\right)\right]^{-1}, \tag{11c}$$

$$a_2^* = -\frac{\mu_1}{\mu_1 + \mu_2}\left(1 - \frac{\mu_6}{\mu_6 + \mu_7}\right)\left[1 - \frac{\mu_1}{\mu_1 + \mu_2}\left(\frac{\mu_6}{\mu_6 + \mu_7}\right)\right]^{-1}, \tag{11d}$$

$$b_1^* = -\frac{\mu_6}{\mu_6 + \mu_7}\left(\frac{h_2\sigma}{h_2^2\sigma + s_2}\right)\left[1 - \frac{\mu_1}{\mu_1 + \mu_2}\left(\frac{h_1^2\sigma}{h_1^2\sigma + s_1}\right)\right]$$
$$\times \left[1 - \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)\left(\frac{\mu_6}{\mu_6 + \mu_7}\right)\left(\frac{h_1h_2\sigma}{h_1^2\sigma + s_1}\right)\left(\frac{h_1h_2\sigma}{h_2^2\sigma + s_2}\right)\right]^{-1}, \tag{11e}$$

[2] As one of the referees has also pointed out, Proposition 1 can be made more general to show equivalence for a much broader class of cost functions and similarly structured n-person games ($n > 2$). This, undoubtedly, will increase the potential application of our result, but for the purpose of the present paper we have found it satisfactory to leave the proposition in its present less general form.

and

$$b_2{}^* = -\frac{\mu_6}{\mu_6 + \mu_7}\left(1 - \frac{\mu_1}{\mu_1 + \mu_2}\right)\left[1 - \frac{\mu_1}{\mu_1 + \mu_2}\left(\frac{\mu_6}{\mu_6 + \mu_7}\right)\right]^{-1}. \tag{11f}$$

Proof. Since $J(u, v)$ is strictly convex and quadratic in both u and v, together with the information structure (8) the team problem admits a unique person-by-person optimal solution which is affine in the measurements of each player; this affine solution is also globally optimal (See Radner [1], Theorems 4 and 5). Furthermore, since the mean values of the random variables w_1 and w_2 are zero, we can restrict ourselves to policies of the form

$$\gamma_1(z_1) = a_1(z_1 - h_1\bar{x}) + a_2\bar{x} \tag{12a}$$

and

$$\gamma_2(z_2) = b_1(z_2 - h_2\bar{x}) + b_2\bar{x}. \tag{12b}$$

Now, minimizing $\bar{J}(\gamma_1, \gamma_2)$ over $\gamma_1(\cdot) \in \Sigma$ for fixed $\gamma_2(\cdot) \in \Sigma$, we obtain

$$\begin{aligned}\gamma_1(z_1) &= -\frac{\mu_1}{\mu_1 + \mu_2} E[x + \gamma_2(z_2) \mid z_1] \\ &= -\frac{\mu_1}{\mu_1 + \mu_2}\left[\bar{x} + \frac{h_1\sigma}{h_1{}^2\sigma + s_1}(z_1 - h_1\bar{x})\right] \\ &\quad - \frac{\mu_1}{\mu_1 + \mu_2} E[\gamma_2(z_2) \mid z_1],\end{aligned} \tag{13a}$$

and, minimizing $\bar{J}(\gamma_1, \gamma_2)$ over $\gamma_2(\cdot) \in \Sigma$ for fixed $\gamma_1(\cdot) \in \Sigma$, we similarly obtain

$$\begin{aligned}\gamma_2(z_2) &= -\frac{\mu_6}{\mu_6 + \mu_7}\left(\bar{x} + \frac{h_2\sigma}{h_2{}^2\sigma + s_2}(z_2 - h_2\bar{x})\right) \\ &\quad - \frac{\mu_6}{\mu_6 + \mu_7} E[\gamma_1(z_1) \mid z_2].\end{aligned} \tag{13b}$$

Substituting (12a), and (12b) into (13a) and (13b) and requiring them to hold for all z_1 and z_2, we have

$$\begin{aligned} -\frac{\mu_1}{\mu_1 + \mu_2}(1 + b_2) &= a_2, \\ -\left(\frac{\mu_1}{\mu_1 + \mu_2}\right)\frac{(1 + b_1h_2)\, h_1\sigma}{h_1{}^2\sigma + s_1} &= a_1, \\ -\frac{\mu_6}{\mu_6 + \mu_7}(1 + a_2) &= b_2, \end{aligned} \tag{14}$$

and

$$-\left(\frac{\mu_6}{\mu_6+\mu_7}\right)\frac{(1+a_1h_1)\,h_2\sigma}{h_2{}^2\sigma+s_2}=b_1\,,$$

and the solution quadruple to (14) determines the solution given by (11a)–(11f). Q.E.D.

Now, the following two lemmas follow directly from Proposition 2 and Identifications 1 and 2, and hence their proofs will be omitted.

LEMMA 1. *The Nash equilibrium solution to the LQG problem of Section* II.1 *is uniquely defined by*

$$u^*=\gamma_{1L}^*(z_{1L})=(z_{1L}-\bar{x})\,a_{1L}^*+a_{2L}^*\bar{x} \tag{15a}$$

and

$$v^*=\gamma_{2L}^*(z_{2L})=(z_{2L}-\bar{x})\,b_{1L}^*+b_{2L}^*\bar{x}, \tag{15b}$$

where

$$\begin{aligned}
a_{1L}^*&=-\frac{c_1}{c_1+d_1}\left(\sigma\frac{d_2}{c_2+d_2}+s_{2L}\right)\eta_L^{-1},\\
a_{2L}^*&=-\frac{c_1}{c_1+d_1}\frac{d_2}{c_2+d_2}\left(1-\frac{c_1}{c_1+d_1}\frac{c_2}{c_2+d_2}\right)^{-1},\\
b_{1L}^*&=-\frac{c_2}{c_2+d_2}\left(\sigma\frac{d_1}{c_1+d_1}+s_{1L}\right)\eta_L^{-1},\\
b_{2L}^*&=-\frac{c_2}{c_2+d_2}\frac{d_1}{c_1+d_1}\left(1-\frac{c_1}{c_1+d_1}\frac{c_2}{c_2+d_2}\right)^{-1},
\end{aligned} \tag{15c}$$

and

$$\eta_L=\frac{1}{\sigma}\left((\sigma+s_{1L})(\sigma+s_{2L})-\sigma^2\frac{c_1}{c_1+d_1}\frac{c_2}{c_2+d_2}\right).$$

LEMMA 2. *The Nash* (*Cournot*) *equilibrium solution to the duopoly problem of Section* II.2 *is uniquely defined by*

$$q^*=\gamma_{1D}^*(z_{1D})=(z_{1D}-\bar{\alpha})\,a_{1D}^*+a_{2D}^*\bar{\alpha} \tag{16a}$$

and

$$r^*=\gamma_{2D}^*(z_{2D})=(z_{2D}-\bar{\alpha})\,b_{1D}^*+b_{2D}^*\bar{\alpha}, \tag{16b}$$

where

$$\begin{aligned} a_{1D}^* &= \frac{\xi}{2\beta_1}\left(\frac{2\beta_2-\beta}{2\beta_2}\,\xi + s_{2D}\right)\eta_D^{-1}, \\ a_{2D}^* &= \left(1-\frac{\beta}{2\beta_2}\right)\left[2\beta_1\left(1-\frac{\beta^2}{4\beta_1\beta_2}\right)\right]^{-1}, \\ b_{1D}^* &= \frac{\xi}{2\beta_2}\left(\frac{2\beta_1-\beta}{2\beta_1}\,\xi + s_{1D}\right)\eta_D^{-1}, \\ b_{2D}^* &= \left(1-\frac{\beta}{2\beta_1}\right)\left[2\beta_2\left(1-\frac{\beta^2}{4\beta_1\beta_2}\right)\right]^{-1}, \\ \eta_D &= (\xi+s_{1D})(\xi+s_{2D}) - \frac{\beta^2}{4\beta_1\beta_2}\,\xi^2, \\ \beta_1 &= \beta + k_1, \end{aligned} \tag{16c}$$

and

$$\beta_2 = \beta + k_2 .$$

III. Some Properties of the Nash Solutions for the Two Models

We now investigate the variations in the average Nash costs of both players due to changes in the information available to either player for both the LQG and the duopoly models. Variations in information will be caused by changes in the variance of the additive noise corrupting the measurements, which is consistent with Definition 1 of Section II.1.

Property 1. The Nash equilibrium solution pair $(\gamma_{1L}^*, \gamma_{2L}^*)$ defined in Lemma 1 for the LQG model has the property that better information on the part of either player lowers the average Nash cost of both players. That is, in mathematical terms,

$$\frac{\partial \bar{J}_{iL}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{jL}} > 0 \qquad \text{for} \quad i = 1, 2 \quad \text{and} \quad j = 1, 2. \tag{17}$$

Proof. By using Lemma 1, the average Nash cost for player 1 can be witten as

$$\begin{aligned} \bar{J}_{1L}(\gamma_{1L}^*, \gamma_{2L}^*) = {} & (1 + a_{2L}^* + b_{2L}^*)^2\,\bar{x}^2 c_1 + (1 + a_{1L}^* + b_{1L}^*)^2\,\sigma c_1 \\ & + a_{1L}^{*2} s_{1L} c_1 + b_{1L}^{*2} s_{2L} c_1 + a_{2L}^{*2}\bar{x}^2 d_1 + a_{1L}^{*2} d_1(\sigma + s_{1L}). \end{aligned}$$

Hence,

$$\frac{\partial \bar{J}_{1L}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{1L}}$$

$$= 2c_1(\bar{x}^2 + \sigma)(1 + a_{2L}^* + b_{2L}^*)\left(\frac{\partial a_{2L}^*}{\partial s_{1L}} + \frac{\partial b_{2L}^*}{\partial s_{1L}}\right)$$

$$+ 2s_{1L}c_1a_{1L}^* \frac{\partial a_{1L}^*}{\partial s_1} + 2s_{2L}c_1b_{1L}^* \frac{\partial b_{1L}^*}{\partial s_{1L}}$$

$$+ 2\bar{x}^2 d_1 a_{2L}^* \frac{\partial a_{2L}^*}{\partial s_{1L}} + 2d_1(\sigma + s_{1L})\, a_{1L}^* \frac{\partial a_{1L}^*}{\partial s_{1L}}$$

$$+ a_{1L}^{*2}(c_1 + d_1) + 2\sigma c_1(1 + a_{1L}^* + b_{1L}^*)\left(\frac{\partial a_{1L}^*}{\partial s_{1L}} + \frac{\partial b_{1L}^*}{\partial s_{1L}}\right), \quad (18a)$$

but

$$\frac{\partial a_{2L}^*}{\partial s_{1L}} = \frac{\partial b_{2L}^*}{\partial s_{1L}} = 0, \quad (18b)$$

$$\frac{\partial a_{1L}^*}{\partial s_{1L}} = \frac{1}{\eta_L{}^2\sigma} \frac{c_1}{c_1 + d_1} (\sigma + s_{2L}) \left(\sigma \frac{d_2}{c_2 + d_2} + s_2\right) > 0, \quad (18c)$$

and

$$\frac{\partial b_{1L}^*}{\partial s_{1L}} = -\frac{1}{\eta_L{}^2} \frac{c_2}{c_2 + d_2} \left(\sigma \frac{c_1}{c_1 + d_1} \frac{d_2}{c_2 + d_2} + s_{2L} \frac{c_1}{c_1 + d_1}\right) < 0. \quad (18d)$$

By substituting (18b)–(18d) into (18a), some rather extensive but straightforward manipulations yield

$$\frac{\partial \bar{J}_{1L}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{1L}} = \sigma s_{1L} + \sigma s_{2L} + s_{1L}s_{2L} + \sigma^2\left(1 - \frac{c_1}{c_1 + d_1} \frac{c_2}{c_2 + d_2}\right)$$
$$+ 2s_{1L}c_1b_{1L}^* \frac{\partial b_{1L}^*}{\partial s_{1L}}. \quad (18e)$$

But since

$$b_{1L}^* \frac{\partial b_{1L}^*}{\partial s_{1L}} > 0,$$

(18e) proves Property 1 for $i = j = 1$. It directly follows from a symmetry property of the LQG model that (18e) proves Property 1 also for $i = j = 2$. For this case, an expression for

$$\frac{\partial \bar{J}_{2L}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{2L}}$$

can be obtained from (18e) by interchanging s_{iL}, c_i, d_i, and b_{iL} with s_{jL}, c_j, d_j, and b_{jL}, respectively, with $i \neq j$, $i = 1, 2$, and $j = 1, 2$.

In order to prove inequality (17) for $i \neq j$, consider

$$\frac{\partial \bar{J}_{1L}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{2L}} = 2\sigma c_1(1 + a_{1L}^* + b_{1L}^*)\left(\frac{\partial a_{1L}^*}{\partial s_{2L}} + \frac{\partial b_{1L}^*}{\partial s_{2L}}\right) + b_{1L}^* c_1 \left(b_{1L}^* + 2s_{2L}\frac{\partial b_{1L}^*}{\partial s_{2L}}\right) + 2a_{1L}^* \frac{\partial a_{1L}^*}{\partial s_{2L}}(c_1 + d_1)\left(\sigma\frac{d_1}{c_1 + d_1} + s_{1L}\right). \tag{19a}$$

But,

$$\frac{\partial a_{2L}^*}{\partial s_{2L}} = \frac{\partial b_{2L}^*}{\partial s_{2L}} = 0, \tag{19b}$$

$$\frac{\partial a_{1L}^*}{\partial s_{2L}} = -\frac{1}{\eta_L{}^2}\frac{c_1}{c_1 + d_1}\frac{c_2}{c_2 + d_2}\left(\sigma\frac{d_1}{c_1 + d_1} + s_{1L}\right) < 0, \tag{19c}$$

and

$$\frac{\partial b_{1L}^*}{\partial s_{2L}} = \frac{1}{\eta_L{}^2\sigma}\frac{c_2}{c_2 + d_2}(\sigma + s_{1L})\left(\sigma\frac{d_1}{c_1 + d_1} + s_{1L}\right) > 0. \tag{19d}$$

Substitution of (19b)–(19d) into (19a) and cancellation of some common terms yield

$$\frac{\partial \bar{J}_{1L}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{2L}} = \frac{c_1 c_2}{(c_2 + d_2)\,\sigma\eta_L{}^3}\left(\sigma\frac{d_1}{c_1 + d_1} + s_{1L}\right)^2 \times \left[\left(1 + \frac{d_2}{c_2 + d_2}\right)(\sigma + s_{1L})(\sigma + s_{2L}) - \left(\frac{c_2}{c_2 + d_2}\right)^2\left(\frac{c_1}{c_1 + d_1}\right)\sigma^2\right] > 0, \tag{19e}$$

which proves Property 1 for $i = 1$ and $j = 2$. Similarly, when $i = 2$ and $j = 1$, Property 1 follows from (19e) by interchanging s_{iL}, c_i, and d_i with s_{jL}, c_j, and d_j, respectively, with $i \neq j$, $i = 1, 2$, and $j = 1, 2$.

One natural question to ask at this stage is, since both players benefit from either player having better information, which player profits more from the ith player's better measurements. It has been shown in [2] that for the complete symmetrical game for both players—i.e., when $c_1 = c_2$, $d_1 = d_2$, and $s_{1L} = s_{2L}$—an incremental increase in information for the

ith player lowers the jth player's average Nash cost more than it does his. In other words,

$$\frac{\partial \bar{J}_{iL}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{iL}} < \frac{\partial \bar{J}_{jL}(\gamma_{1L}^*, \gamma_{2L}^*)}{\partial s_{iL}}$$

for $i \neq j$ and $c_1 = c_2$, $d_1 = d_2$, and $s_{1L} = s_{2L}$. This says that worse information on part of one player hurts the other player more than it does him. We have also shown in [2] that $s_{iL} < s_{jL}$ and $i \neq j$ imply for all $c_1 = c_2$ and $d_1 = d_2$ the strict inequality $\bar{J}_{iL}(\gamma_{1L}^*, \gamma_{2L}^*) > \bar{J}_{jL}(\gamma_{1L}^*, \gamma_{2L}^*)$. Motivated by the stage-TV comedy produced by Neil Simon, we call this rather queer property of the Nash solution the "odd couple" effect.

Property 2. The Nash (Cournot) equilibrium solution pair $(\gamma_{1D}^*, \gamma_{2D}^*)$ defined in Lemma 2 for the duopoly model has the property that better information on the part of one player lowers the average Nash cost of that player but increases the average Nash cost of the other player. That is,

$$\frac{\partial \bar{J}_{iD}(\gamma_{1D}^*, \gamma_{2D}^*)}{\partial s_{jD}} \begin{cases} > 0, & i = j \\ < 0, & i \neq j. \end{cases} \tag{20}$$

Proof. By using Lemma 2, the average Nash cost for player 1 can be written as

$$\begin{aligned}\bar{J}_{1D}(\gamma_{1D}^*, \gamma_{2D}^*) = {} & a_{2D}^*(\beta_1 a_{2D}^* + \beta b_{2D}^* - 1)\,\bar{\alpha}^2 - a_{1D}^*\xi \\ & + \beta_1 a_{1D}^{*2}(\xi + s_{1D}) + \beta a_{1D}^* b_{1D}^* \xi.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial \bar{J}_{1D}(\gamma_{1D}^*, \gamma_{2D}^*)}{\partial s_{1D}} = {} & \frac{\partial}{\partial s_{1D}}\, a_{2D}^*(\beta_1 a_{2D}^* + \beta b_{2D}^* - 1)\,\bar{\alpha}^2 \\ & - \frac{\partial a_{1D}^*}{\partial s_{1D}}\,\xi + 2\beta_1 a_{1D}^* \frac{\partial a_{1D}^*}{\partial s_{1D}}(\xi + s_{1D}) \\ & + \beta_1 a_{1D}^{*2} + \beta\xi \frac{\partial a_{1D}^*}{\partial s_{1D}}\, b_{1D}^* + \beta\xi \frac{\partial b_{1D}^*}{\partial s_{1D}}\, a_{1D}^*.\end{aligned} \tag{21a}$$

But,

$$\frac{\partial a_{2D}^*}{\partial s_{1D}} = \frac{\partial b_{2D}^*}{\partial s_{1D}} = 0, \tag{21b}$$

$$\frac{\partial a_{1D}^*}{\partial s_{1D}} = -\frac{\xi}{2\beta_1 \eta_D{}^2}\left(\frac{2\beta_2 - \beta}{2\beta_2}\,\xi + s_{2D}\right)(\xi + s_{2D}) < 0, \tag{21c}$$

and

$$\frac{\partial b_{1D}^*}{\partial s_{1D}} = \frac{\xi^2 \beta}{4\beta_2 \beta_1 \eta_D{}^2} \left[s_{2D} + \xi\left(1 - \frac{\beta}{2\beta_2}\right)\right] > 0. \tag{21d}$$

By substituting (21b)–(21d) into (21a), simplification yields

$$\frac{\partial \bar{J}_{1D}(\gamma_{1D}^*, \gamma_{2D}^*)}{\partial s_{1D}} = \left(\beta_1 + \frac{\beta^2 \xi^2}{2\beta_2 \eta_D}\right) a_{1D}^{*2} > 0,$$

which proves Property 2 for $i = j = 1$. By using a symmetry property of the duopoly model, it can further be shown that

$$\frac{\partial \bar{J}_{2D}(\gamma_{1D}^*, \gamma_{2D}^*)}{\partial s_{2D}} = \left(\beta_2 + \frac{\beta^2 \xi^2}{2\beta_1 \eta_D}\right) b_{1D}^{*2} > 0.$$

To prove inequality (20) for $i \neq j$, we consider

$$\begin{aligned}\frac{\partial \bar{J}_{1D}(\gamma_{1D}^*, \gamma_{2D}^*)}{\partial s_{2D}} = {} & \frac{\partial}{\partial s_{2D}} a_{2D}^*(\beta_1 a_{2D}^* + \beta b_{2D}^* - 1)\,\bar{\alpha}^2 \\ & - \frac{\partial a_{1D}^*}{\partial s_{2D}}\,\xi + 2\beta_1(\xi + s_{1D})\frac{\partial a_{1D}^*}{\partial s_{2D}}\,a_{1D}^* \\ & + \beta\frac{\partial a_{1D}^*}{\partial s_{2D}}\,b_{1D}^*\xi + \beta a_{1D}^*\frac{\partial b_{1D}^*}{\partial s_{2D}}\,\xi.\end{aligned} \tag{22a}$$

But,

$$\frac{\partial a_{2D}^*}{\partial s_{2D}} = \frac{\partial b_{2D}^*}{\partial s_{2D}} = 0, \tag{22b}$$

$$\frac{\partial a_{1D}^*}{\partial s_{2D}} = \frac{\xi^2 \beta}{4\beta_1 \beta_2 \eta_D{}^2}\left[s_{1D} + \xi\left(1 - \frac{\beta}{2\beta_1}\right)\right] > 0, \tag{22c}$$

and

$$\frac{\partial b_{1D}^*}{\partial s_{2D}} = -\frac{\xi(\xi + s_{1D})}{2\beta_2 \eta_D{}^2}\left[s_{1D} + \xi\left(1 - \frac{\beta}{2\beta_1}\right)\right] < 0. \tag{22d}$$

Substituting (22b)–(22d) into (22a), we obtain

$$\begin{aligned}\frac{\partial \bar{J}_{1D}(\gamma_{1D}^*, \gamma_{2D}^*)}{\partial s_{2D}} = {} & -\frac{\xi^3 \beta(\xi + s_{1D})}{4\beta_1 \beta_2 \eta_D{}^3}\left[s_{1D} + \xi\left(1 - \frac{\beta}{2\beta_1}\right)\right] \\ & \times \left[s_{2D} + \xi\left(1 - \frac{\beta}{2\beta_2}\right)\right] < 0,\end{aligned}$$

which proves Property 2 for $i = 1$ and $j = 2$. Similarly, when $i = 2$ and

$j = 1$, Property 2 follows from (22e) with s_{iD} and β_i interchanged with s_{jD} and β_j, respectively, for $i \neq j$, $i = 1, 2$, and $j = 1, 2$. Q.E.D.

Hence, Property 2 implies that for the duopoly model the firm that has better information will end up with a higher share of the profit in the market and an incremental increase in information for one firm will lower the net profit of the other.

IV. Some Related Properties of Zero-Sum Games and Team Problems

In this section we are going to relate Properties 1 and 2 derived in Section III for the duopoly and the LQG models to two distinct features of the solutions of zero-sum games and team problems, respectively. Zero-sum games and team problems can be considered as two extreme cases of nonzero-sum games, or, by looking at it in another way, every zero-sum game and every two-person team problem can be treated within the framework of nonzero-sum two-person games by properly defining the objective functions of the two players. In particular, a zero-sum game is a nonzero-sum game in which $J_1(x, u, v) \equiv -J_2(x, u, v)$, and a two-person team problem is a nonzero-sum game with $J_1(x, u, v) \equiv J_2(x, u, v)$.

By assuming that the information available to the ith player about the state of nature x varies with respect to changes in a parameter s_i which partially defines the information structure (e.g., the variance of the additive observation noise, as in the previous sections), then the saddle-point solution $(\gamma_{1s}^*, \gamma_{2s}^*)$ to any zero-sum game has the property that, if a change in the information available to one player lowers the optimal average cost of one of the players, it definitely increases the optimal average cost of the other player. Mathematically speaking,

$$\left(\frac{\partial \bar{J}_i(\gamma_{1s}^*, \gamma_{2s}^*)}{\partial s_i}\right)\left(\frac{\partial \bar{J}_j(\gamma_{1s}^*, \gamma_{2s}^*)}{\partial s_i}\right)^{-1} < 0, \qquad i \neq j. \tag{23}$$

We note parenthetically that this property follows directly from the definition of a zero-sum game and is quite independent of the form and structure of the game involved. This salient feature of the class of zero-sum games considered as a subclass of nonzero-sum games motivates us to give the following definition.

Definition 2. We say that a Nash solution (γ_1^*, γ_2^*) to a given stochastic two-person nonzero-sum game is *zero sum dominated with respect to parameter* s_i if

$$\left(\frac{\partial \bar{J}_i(\gamma_1{}^*, \gamma_2{}^*)}{\partial s_i}\right)\left(\frac{\partial J_j(\gamma_1{}^*, \gamma_2{}^*)}{\partial s_i}\right)^{-1} < 0, \qquad i \neq j. \tag{24}$$

We now note that the Nash solution, $(\gamma_{1D}^*, \gamma_{2D}^*)$, to the duopoly problem considered in this report is zero sum dominated with respect to the variances of the observation noises of both players (i.e., s_{1D} and s_{2D} ; see Proposition 2). In this sense, the duopoly model has a zero-sum property even though it is not a zero-sum game.

The equilibrium solution $(\gamma_{1T}^*, \gamma_{2T}^*)$ to any team problem, however, has the property that—considered as a nonzero-sum game—if a change in the information available to one player lowers the optimal average cost of one of the players, it also definitely lowers the optimal average cost of the other player, simply because they are equivalent. Mathematically speaking,

$$\left(\frac{\partial \bar{J}_i(\gamma_{1T}^*, \gamma_{2T}^*)}{\partial s_i}\right)\left(\frac{\partial \bar{J}_j(\gamma_{1T}^*, \gamma_{2T}^*)}{\partial s_i}\right)^{-1} = 1 > 0. \tag{25}$$

Some nonzero-sum games have this "team" property even though $J_1(x, u, v) \neq J_2(x, u, v)$, which leads us to the following definition—the team counterpart of Definition 2.

DEFINITION 3. We say that a Nash solution $(\gamma_1{}^*, \gamma_2{}^*)$ to a given stochastic two-person nonzero-sum game is *team dominated with respect to parameter* s_i if

$$\left(\frac{\partial \bar{J}_i(\gamma_1{}^*, \gamma_2{}^*)}{\partial s_i}\right)\left(\frac{\partial J_j(\gamma_1{}^*, \gamma_2{}^*)}{\partial s_i}\right)^{-1} > 0, \qquad i \neq j. \tag{26}$$

Now it follows from Property 1 that the Nash solution $(\gamma_{1L}^*, \gamma_{2L}^*)$ to the LQG model is team dominated with respect to the variances of the observation noises of both players (i.e., s_{1L} and s_{2L}). Hence, the LQG model has that cooperative nature even though it is a noncooperative game.

V. Concluding Remarks

In this paper, we have investigated some properties of the Nash solutions of nonzero-sum stochastic games by using two different models (LQG and duopoly) with respect to a certain parameter of the information structure. The results obtained and the conclusions drawn for these two models are valid for the entire range of the parameters defining these games. Still, we are aware that these models are too specific for any concrete conclusions to be drawn for more general payoff functions and information structures. Even though our general method of approach is applicable to general quadratic cases and multidimensional models, obtaining specific concrete results for these general models requires extensive and cumbersome manipulations which are sometimes far beyond even the symbolic manipulative capacity of today's computing technology.

This paper constitutes the first report on an investigation which might branch out in two directions: (i) general formulation of information structure in which a change in information can be modeled by a change in the information function (i.e., the function that maps the space of all variables of the game into the observation space) as opposed to a variation in the distribution function of a relevant random variable; (ii) investigation of quality of information as related to the payoff structure, which might lead us to some counterintuitive results. One such result is given in [4], where it is shown via a simple example that in nonzero-sum games mutual ignorance might lead to mutual bliss. Our investigation indicates that such counterintuitive conclusions are more a rule rather than an exception for some specific models.

At the present, very few results of sufficient generality are known in this area. We hope that this paper will initiate and stimulate some further research along the lines discussed in this section. Our further results in that direction will be reported in a forthcoming paper.

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