ELEKTRİK, VOL.1, NO.1, February 1993, ©TÜBİTAK EMO

# Stochastic Differential Games With Weak Spatial and Strong Informational Coupling *†

**Tamer Başar , R. Srikant**
*Coordinated Science Laboratory*
*University of Illinois*
*1101 W. Springfield Avenue*
*Urbana, IL 61801/USA*

**Abstract**

*We formulate a parameterized family of linear quadratic two-person nonzero-sum stochastic differential games where the players are weakly coupled through the state equation and the loss functions, and strongly coupled through the measurements. A small parameter $\epsilon$ characterizes this family, in terms of which the subsystems are coupled (weakly). With $\epsilon = 0$ the problem admits a unique Nash equilibrium solution, while for $\epsilon \neq 0$, no general constructive method is available to obtain the Nash equilibrium solution and to prove existence and uniqueness. In this paper, we first obtain a candidate Nash solution which involves only finite-dimensional controllers, and then show its existence and unicity (for sufficiently small $\epsilon \neq 0$, and within a certain class) by proving the existence and uniqueness of the solution of a pair of coupled nonlinear differential equations. We also show that approximate solutions to this pair of differential equations can be obtained using an iterative technique, and that these approximate solutions result in approximate Nash equilibrium solutions. Further, the equilibrium policies are linear, requiring only finite-dimensional controllers, in spite of the fact that a separation (of estimation and control) result does not hold in the strict sense.*

**Keywords:** *Stochastic differential games, Nash equilibria, Weak coupling, Stochastic measurements.*

## 1. Introduction

We formulate a class of stochastic nonzero-sum differential games where the players are weakly coupled through the state equation while sharing the same source of information (measurement). This latter feature brings in a strong *informational coupling*, the presence of which makes a constructive derivation of Nash equilibria quite a challenging task unless the weak coupling parameter ($\epsilon$) is set to zero. In the approach developed in this paper, we relate the existence of a finite-dimensional Nash equilibrium to the existence of a solution to a pair of coupled, nonlinear differential equations. Then, we prove the existence of a solution to the differential equations by using perturbation theory. Further, we show that the resulting solution is *admissible* in the limit as $\epsilon \to 0$, thus justifying the choice of a finite-dimensional Nash equilibrium, even though it may not be unique.

* This research was supported in part by the U.S. Department of Energy under Grant DE-FG-02-88-ER-13939.

† An earlier version of the paper was presented at the 9th International Conference on Analysis and Optimization, Antibes, France, June 12-15, 1990.

The rest of the paper is organized as follows. A precise formulation of the problem is given in the next section, followed (in section 3) with the derivation of a candidate solution and verification of its existence for $\epsilon \neq 0$ sufficiently small. Section 4 discusses the properties of the Nash equilibrium solution obtained in section 3, and section 5 provides the concluding remarks.

## 2. Problem Statement

The class of linear-quadratic, nonzero-sum, stochastic differential games under consideration, with weak spatial and strong informational coupling between the players, can be defined in precise mathematical terms as follows: Given an underlying probability space, the evolution of the composite state ($x := (x^{1'}, x^{2'})'$) of the game is described by the linear Itô stochastic differential equation:

$$\begin{aligned} dx_t &= A(t;\epsilon)x_t dt + \tilde{B}^1(t)u_t^1 dt + \tilde{B}^2(t)u_t^2 dt + F(t)dw_t, \\ & t_0 \leq t \leq t_f, \qquad x_{t_0} = x_0, \end{aligned} \tag{1}$$

where the initial state $x_0$ is taken to be a Gaussian distributed random vector with mean $\bar{x}_0$ and covariance $\Sigma_0 > 0$, $dim(x^i) = n_i$, $dim(u^i) = r_i$, $i = 1, 2$,

$$A(t;\epsilon) := \begin{pmatrix} A_1 & \epsilon A_{12} \\ \epsilon A_{21} & A_2 \end{pmatrix}(t); \quad \tilde{B}_1(t) := \begin{bmatrix} B_1(t) \\ \dots \\ 0 \end{bmatrix}; \tilde{B}_2(t) := \begin{bmatrix} 0 \\ \dots \\ B_2(t) \end{bmatrix}, \tag{2}$$

$$F(t) = \text{ block diag } (F_1(t), F_2(t)); \quad F_1F_1' > 0, \quad F_2F_2' > 0,$$

$\epsilon$ is a small (in magnitude) (coupling) parameter, and the partitions of $A$, $\tilde{B}_1$, $\tilde{B}_2$ and $F$ are compatible with the subsystem structure, so that with $\epsilon = 0$ the system decomposes into two completely decoupled and stochastically independent subsystems, each one controlled by a different player. The functions $u_t^1$ and $u_t^2$, $t \geq t_0$, represent the controls of Players 1 and 2, respectively, which are vector stochastic processes with continuous sample paths, as to be further explained in the sequel. The driving term $w_t := (w_t^{1'}, w_t^{2'})'$, $t \geq t_0$ is a standard vector Wiener process, which is independent of the initial state $x_0$.

The common observation $y$ of the players is described by

$$dy_t = C(t)x_t dt + G(t)dv_t, \quad y_{t_0} = 0; \quad C := (C_1, C_2), \tag{3}$$

where $dim(y) = m$, $C_i$ is $m \times r_i$, $i = 1, 2$, $GG' > 0$, and $v_t$, $t \geq t_0$ is another standard vector Wiener process, independent of $\{w_t\}$ and $x_0$. This common observation constitutes the only strong coupling between the players.

All matrices in the above formulation are taken to be continuous on the time interval $[t_0, t_f]$. Let $C_m = C_m[t_0, t_f]$ denote the space of the continuous functions on $[t_0, t_f]$, with values in $\mathbb{R}^m$. Further let $\mathcal{Y}_t$ be the sigma-field in $C_m$ generated by the cylinder sets $\{y \in C_m, y_s \in B\}$ where $B$ is a Borel set in $\mathbb{R}^m$, and $t_0 \leq s \leq t$. Then, the information gained by each player during the course of the game is completely determined by the information field $\mathcal{Y}_t$, $t \geq t_0$. A permissible strategy for Player i is a mapping $\gamma_i(\cdot, \cdot)$ of $[t_0, t_f] \times C_m$ into $\mathbb{R}^{r_i}$ with the following properties:

(i) $\gamma_i(t, \eta)$ is continuous in $t$ for each $\eta \in C_m$.

(ii) $\gamma_i(t, \eta)$ is uniformly Lipschitz in $\eta$, i.e., $|\gamma_i(t, \eta) - \gamma_i(t, \xi)| \leq k||\eta - \xi||$, $t \in [t_0, t_f]$, $\eta$, $\xi \in C_m$, where $||\cdot||$ is the sup norm on $C_m$.

(iii) $u_t^i = \gamma_i(t, \eta)$ is adapted to the information field $\mathcal{Y}_t$.

Let us denote the collection of all strategies described above, for Player i, by $\Gamma_i$. It is known (see, for example, [7]) that, corresponding to any pair of strategies $\{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$, the stochastic differential equation (1) admits a unique solution that is a sample-path-continuous second-order process. As a result, the observation process $y_t$, $t \geq t_0$ will also have continuous sample paths.

For any pair of strategies $\{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$, we introduce the cost function for Player i, $i = 1, 2$, as

$$J_i(\gamma_1, \gamma_2) = E\left\{x_{t_f}'\tilde{Q}_{if}(\epsilon)x_{t_f} + \int_{t_0}^{t_f}(x_t'\tilde{Q}_i(t;\epsilon)x_t + u_t^{i'}u_t^i)dt\right\}, \tag{4}$$

where all the matrices are nonnegative definite and

$$\tilde{Q}_{1f}(\epsilon) := \text{block diag } (Q_{1f}, \epsilon Q_{12f}); \quad \tilde{Q}_{2f}(\epsilon) := \text{block diag } (\epsilon Q_{21f}, Q_{2f}),$$

$$\tilde{Q}_1(t;\epsilon) := \text{block diag } (Q_1(t), \epsilon Q_{12}(t)); \quad \tilde{Q}_2(t;\epsilon) := \text{block diag } (\epsilon Q_{21}(t), Q_2(t)).$$

Note that the players' costs are also coupled *weakly*, so that if $\epsilon = 0$ each cost function involves only that player's state vector and control function. Of course, even with $\epsilon = 0$ there is still an "informational coupling" through the common observation, which implicitly couples the cost functions under any equilibrium solution concept.

Adopting the Nash equilibrium solution concept, we seek a pair of strategies $(\gamma_1^* \in \Gamma_1, \gamma_2^* \in \Gamma_2)$ satisfying the pair of inequalities

$$J_1(\gamma_1^*, \gamma_2^*) \leq J_1(\gamma_1, \gamma_2^*); \quad J_2(\gamma_1^*, \gamma_2^*) \leq J_2(\gamma_1^*, \gamma_2), \tag{5}$$

for all $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$. To show the explicit dependence of the Nash policies on the available information and the coupling parameter $\epsilon$, we will sometimes use the notation $\gamma_i^*(t, y; \epsilon)$.

Let us first list a few known facts on the Nash solution of this stochastic differential game, when $\epsilon$ is not necessarily a small parameter (in absolute value).

1. Conditions under which a Nash equilibrium exists are not known. What is known, however, is that the solution (whenever it exists) will not satisfy any separation principle (between estimation and control), which is in some sense true even for the zero-sum version of the problem [3].

2. The discrete-time version of the problem, but with private measurements for the players that are shared with a delay of one time unit, has been considered before in [2] where it has been shown that the Nash equilibrium solution is unique and linear in the available information for each player, whenever it exists. The procedure developed there can readily be used to derive a similar result for the common information case (in discrete-time), when even though a separation result does not apply, the Nash controllers for the players have finite dimensional representations (i.e., the controller dimension does not grow with the number of stages in the problem) [4].

3. For the continuous-time problem, however, the procedure of [2] does not apply, and consequently a proof of existence and uniqueness of linear Nash equilibria has been quite elusive for the past decade. For the zero-sum version (of the continuous-time problem), it is possible to prove existence of a unique linear saddle-point equilibrium, though using an indirect approach that employs explicitly the interchangeability property saddle-points [3].

In view of this past experience, we adopt in this paper a different approach for the nonzero-sum stochastic differential game, that exploits the weakness of the coupling between the two subsystems. Our results are presented in the following sections.

## 3. A Finite-dimensional Nash Equilibrium

Our first result is a characterization of a finite-dimensional Nash equilibrium for the nonzero-sum differential game of section 2, when $\epsilon$ is not necessarily a small parameter. Here, though, we assume the existence of a solution to a nonlinear matrix differential equation.

Toward this end, let $P^1$ and $P^2$ be $n := n_1 + n_2$ dimensional symmetric matrix functions defined on the time interval $[t_0, t_f]$, satisfying the coupled nonlinear differential equations

$$\dot{P}^1 + \tilde{A}^{1'} P^1 + P^1 \tilde{A}_1 - P^1 \tilde{\tilde{B}}_1 \tilde{\tilde{B}}_1' P^1 + \tilde{\tilde{Q}}_1 = 0; \quad P^1(t_f) = \tilde{\tilde{Q}}_{1f}, \tag{6}$$

$$\dot{P}^2 + \tilde{A}^{2'} P^2 + P^2 \tilde{A}^2 - P^2 \tilde{\tilde{B}}_2 \tilde{\tilde{B}}'_2 P^2 + \tilde{\tilde{Q}}_2 = 0; \quad P^2(t_f) = \tilde{\tilde{Q}}_{2f}, \tag{7}$$

where

$$\tilde{\tilde{B}}_i(t) := [\tilde{B}_i(t)', 0]'_{n \times r_i}, \tag{8}$$

$$\tilde{\tilde{Q}}_i(t; \epsilon) := \text{ block diag } (\tilde{Q}_i(t, \epsilon), 0); \tilde{\tilde{Q}}_{if}(\epsilon) := \text{ block diag } (\tilde{Q}_{if}(\epsilon), 0), \tag{9}$$

$$\tilde{A}^i(P^1, P^2; \epsilon) := \begin{pmatrix} A(\epsilon) & -\tilde{B}_i \tilde{B}_i' \hat{P} \\ KC & M(\epsilon) - KC \end{pmatrix}; \quad i = 1, 2, \tag{10}$$

$$M(\epsilon) := A(\epsilon) - \tilde{B}_1 \tilde{B}_1' \hat{P} - \tilde{B}_2 \tilde{B}_2' \hat{P}, \tag{11}$$

$$\hat{P} := \begin{pmatrix} P^1_{11} + P^1_{13} & P^1_{12} + P^1_{14} \\ P^2_{21} + P^2_{23} & P^2_{22} + P^2_{24} \end{pmatrix}; P^i_{jk} : jk'\text{th block of } P^i, \tag{12}$$

$$K := \Sigma C'(GG')^{-1}, \tag{13}$$

$$\dot{\Sigma} = A(\epsilon)'\Sigma + \Sigma A(\epsilon) + FF' - \Sigma C'(GG')^{-1} C\Sigma; \Sigma(t_0) = \Sigma_0. \tag{14}$$

Note that (6)-(7) is a set of coupled nonlinear matrix differential equations, the coupling being through the matrices $\tilde{A}^1$ and $\tilde{A}^2$.

**Theorem 3.1.** *For a given $\epsilon$, let the set of matrix differential equations (6)-(7) admit a symmetric solution $P^1(\epsilon) \geq 0$, $P^2(\epsilon) \geq 0$. Then, the stochastic differential game of section 2 admits a Nash equilibrium solution for this value of $\epsilon$, given by*

$$\gamma_i^*(t, y; \epsilon) = -B_i'[(P^i_{i1} + P^i_{i3})\hat{x}_1 + (P^i_{i2} + P^i_{i4})\hat{x}_2], t_0 \leq t \leq t_f; \quad i = 1, 2, \tag{15}$$

*where $\hat{x} := (\hat{x}_1', \hat{x}_2')'$ is generated by the Kalman filter:*

$$\begin{aligned} & d\hat{x} = A(\epsilon)\hat{x}dt - (\tilde{B}_1 \tilde{B}_1' + \tilde{B}_2 \tilde{B}_2')\hat{P}\hat{x}dt + K[dy_t - C\hat{x}dt]; \hat{x}_{t_0} = \bar{x}_0 \\ \Leftrightarrow & \\ & d\hat{x} = M(\epsilon)\hat{x}dt + K[dy_t - C\hat{x}_t dt]; \hat{x}_{t_0} = \bar{x}_0. \end{aligned} \tag{16}$$

*Proof.* We verify that under (15) the pair of inequalities (5) are satisfied. For Player 1, this reduces to the following stochastic control problem:

$$\min_{\gamma_1} E\{x'_{t_f} \tilde{Q}_{1f} x_{t_f} + \int_{t_0}^{t_f} (x_t' \tilde{Q}_1(t; \epsilon) x_t + u_t^{1'} u_t^1) dt\}$$

subject to

$$\begin{aligned} dx_t &= A(\epsilon)x_t dt - \tilde{B}_2 \tilde{B}_2' \hat{P} z_t dt + \tilde{B}_1 u_t^1 dt + F dw_t, \\ dz_t &= KCx_t dt + [M(\epsilon) - KC]\hat{x}_t dt + KG dv_t; z_{t_0} = \bar{x}_0, \end{aligned}$$

and with

$$u_t^1 = \gamma_{1t}(t, y), \quad \gamma_1 \in \Gamma_1.$$

Note that $\{z_t\}$ is the same stochastic process as $\{\hat{x}_t\}$, provided that $\gamma_1 = \gamma_1^*$.

Now, using the notation introduced prior to the statement of the Theorem, and introducing the $2n$-dimensional vector $\xi_t := (x_t', z_t')'$, the cost function to be minimized can be re-written as

$$E\{\xi_{t_f}' \tilde{\tilde{Q}}_{1f} \xi_{t_f} + \int_{t_0}^{t_f} (\xi_t' \tilde{\tilde{Q}}_1(t;\epsilon)\xi_t + u_t^{1'} u_t^1) dt\}.$$

The associated state equation is

$$d\xi_t = \tilde{A}^1 \xi_t dt + \tilde{\tilde{B}}_1 u_t^1 + \text{ block diag } (F, KG)(dw_t', dv_t')'; \xi_{t_0} = (x_0', \bar{x}_0')',$$

and the measurement process is

$$dy_t = (C, 0)\xi_t dt + G dv_t, y_{t_0} = 0.$$

Hence, the problem is a standard LQG stochastic control problem, for which the unique solution is

$$u_t = \mu(t, \hat{z}_t) = -\tilde{\tilde{B}}_1' P^1(t) \hat{z}_t, t \geq t_0, \tag{$**$}$$

where $\hat{z}_t := E[z_t \mid y_t, \mu]$ is generated by the Kalman filter

$$\left.\begin{aligned} d\hat{z}_t &= (\tilde{A}^1 - \tilde{\tilde{B}}_1 \tilde{\tilde{B}}'_1 P^1)\hat{z}_t dt + \tilde{K}[dy_t - (C,0)\hat{z}_t dt]; \hat{z}_{t_0} = (\bar{x}_0', \bar{x}_0')', \\ \tilde{K} &:= [\tilde{\Sigma}(C,0)' + (0, GG'K')'](GG')^{-1}, \\ \dot{\tilde{\Sigma}} &= \tilde{\Sigma}\tilde{A}^{1'} + \tilde{A}^1 \tilde{\Sigma} + \text{ block diag } (FF', KGG'K) - \tilde{K}(GG')\tilde{K}'; \\ \tilde{\Sigma}(t_0) &= \text{ block diag } (\Sigma_0, 0). \end{aligned}\right\} \tag{$*$}$$

We now claim that $\tilde{\Sigma} =$ block diag $(\Sigma, 0)$, where $\Sigma$ is generated by (14), solves the preceding equation. To see this, first note that with $\tilde{\Sigma}$ as given above,

$$\tilde{K} = [(\Sigma C'(GG')^{-1})', K'] \equiv [K', K']'$$

$$\tilde{\Sigma}\tilde{A}^{1'} = \begin{pmatrix} \Sigma A' & \Sigma C'K' \\ 0 & 0 \end{pmatrix}$$

from which it follows that the 11-block of $(*)$ is precisely (14). The 12-block is

$$0 = \Sigma C'K' - K(GG')K',$$

which is an identity. Finally, the 22-block is

$$0 = KGG'K - KGG'K' \equiv 0.$$

Now let $(\hat{z}_{1t}', \hat{z}_{2t}')' =: \hat{z}_t$, where $\hat{z}_{it}$ is $n$-dimensional. The first component satisfies the DE

$$d\hat{z}_1 = [A(\epsilon)\hat{z}_1 - B_1 B_1'(P_{11}^1, P_{12}^1)\hat{z}_1 - \tilde{B}_2 \tilde{B}_2' \hat{P} \hat{z}_2 - B_1 B_1'(P_{13}^1, P_{14}^1)\hat{z}_2]dt + K[dy_t - C\hat{z}_1]; z_{1t_0} = \bar{x}_0,$$

whereas the second component satisfies

$$d\hat{z}_2 = [(M(\epsilon) - KC)\hat{z}_2 + KC\hat{z}_1]dt + K[dy_t - C\hat{z}_1]; z_{2t_0} = \bar{x}_0.$$

Let $\epsilon_t := \hat{z}_{1t} - \hat{z}_{2t}$, for which a DE is

$$d\epsilon_t = [A(\epsilon) - B_1 B_1'(P_{11}^1, P_{12}^1) - KC]\epsilon_t dt, \quad \epsilon_{t_0} = 0.$$

Hence, $\hat{z}_1(y) \equiv \hat{z}_2(y)$, and using this in the equation for $\hat{z}_2$, we arrive at the conclusion that $\hat{z}_1(y) \equiv \hat{z}_2(y) \equiv \hat{x}_t(y)$, where the latter was defined by (16). Furthermore using this in (**) leads to (15) for $i = 1$. By index-symmetry, we also have an analogous result for $i = 2$. ◇

For the result of Theorem 3.1 to be of any value, we have to show that there exists a solution to the coupled equations (6)-(7). As indicated earlier, these are not standard Riccati equations because of the dependence of $\tilde{A}^1$ and $\tilde{A}^2$ on $P^1$ and $P^2$. It is at this point now that we make use of the weak coupling between the two subsystems controlled by the two players.

Towards establishing existence of a solution to (6)-(7), let us first assume that $P^i(t;\epsilon)$, $i = 1, 2$, are analytic in $\epsilon$ and

$$P^i(t;\epsilon) = \sum_{k=0}^{l} P_{(k)}^i(t)\epsilon^k + 0(\epsilon^{l+1}), \quad i = 1, 2, \tag{17}$$

uniformly in the interval $[t_0, t_f]$, and for any $l \geq 0$. Then, if we can show that the $P_{(k)}^i$'s are continuous functions of '$t$', then by an application of the *implicit function theorem* to differential equations (see, [13] Theorems 7.1 and 7.2), we have that (17) solves (6)-(7) for $\epsilon \in (-\epsilon_0, \epsilon_0)$, for some $\epsilon_0 > 0$. In order to do this, let us first obtain an expansion of $\tilde{A}^i$ in terms of $\epsilon$. Under (17), $\tilde{A}^i$ is expandable in terms of $\epsilon$ if $K$ is, and writing $K$ as $K = \sum_{n=0}^{l} K_{(n)}\epsilon^n + 0(\epsilon^{l+1})$, we have

$$K_{(n)} = \Sigma_{(n)} C'(GG')^{-1}, \tag{18}$$

where

$$\dot{\Sigma}_{(0)} = A_{(0)}\Sigma_{(0)} + \Sigma_{(0)}A_{(0)}' + FF' - \Sigma_{(0)}C'(GG')^{-1}C\Sigma_{(0)}; \quad \Sigma_{(0)}(t_0) = \Sigma_0, \tag{19}$$

$$\begin{gathered}\dot{\Sigma}_{(n)} = A_{(0)}\Sigma_{(n)} + \Sigma_{(n)}A_{(0)}' + \Sigma_{(n-1)}A_{(1)}' + A_{(1)}\Sigma_{(n-1)} - \sum_{k=0}^{n}\Sigma_{(k)}C'(GG')^{-1}C\Sigma_{(n-k)}; \\ \Sigma_{(n)}(t_0) = 0, n \geq 1,\end{gathered} \tag{20}$$

$$A_{(0)} := \text{ block diag } (A_1, A_2); A_{(1)} := \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}.$$

Note that (19) is the standard Riccati equation that arises in Kalman filtering, and hence admits a unique continuously differentiable solution. Furthermore, for each $n \geq 1$, (20) is a linear differential equation and hence admits a unique solution. Then, by the implicit function theorem, there exists a neighborhood of $\epsilon = 0$ where (18) is valid. In view of this, $\tilde{A}^i$ is expandable in $\epsilon$, $\tilde{A}^i = \sum_{n=0}^{l} \tilde{A}_{(n)}^i \epsilon^n + 0(\epsilon^{l+1})$, where

$$\tilde{A}_{(n)}^i = \begin{pmatrix} A_{(n)} & -\tilde{B}_i\tilde{B}_i'\hat{P}^{(n)} \\ K_{(n)}C & M_{(n)} - K_{(n)}C \end{pmatrix}, \quad i = 1, 2,$$

$A_{(n)} = 0$, $n \geq 2$, and $M_{(n)}$ is defined by (11) by replacing $A$ by $A_{(n)}$, $\hat{P}$ by $\hat{P}^{(n)}$, where the latter is given by (12) with the superindex $(n)$. In view of this, (6)-(7) are decomposed into

$$\dot{P}_{(0)}^i + \tilde{A}_{(0)}^{i'} P_{(0)}^i + P_{(0)}^i \tilde{A}_{(0)}^i - P_{(0)}^i \tilde{\tilde{B}}_i \tilde{\tilde{B}}_i' P_{(0)}^i + \tilde{\tilde{Q}}_i = 0; \quad P_{(0)}^i(t_f) = \tilde{\tilde{Q}}_{i_f}, \tag{21}$$

and for $n \geq 1$

$$\dot{P}^i_{(n)} + (\tilde{A}^i_{(0)} - \tilde{\tilde{B}}_i \tilde{\tilde{B}}_i' P^i_{(0)})' P^i_{(n)} + P^i_{(n)} (\tilde{A}^i_{(0)} - \tilde{\tilde{B}}_i \tilde{\tilde{B}}_i' P^i_{(0)})$$
$$+ \sum_{k=1}^{n} \tilde{A}^{i\prime}_{(k)} P^i_{(n-k)} + \sum_{k=1}^{n} P^i_{(n-k)} \tilde{A}^i_{(k)} - \sum_{k=1}^{n-1} P^i_{(n-k)} \tilde{\tilde{B}}_i \tilde{\tilde{B}}_i' P^i_{(k)} = 0; P^i_{(n)}(t_f) = 0. \tag{22}$$

Now, at the outset it may seem that (21) is a nonstandard nonlinear matrix equation; however a little scrutiny reveals that

$$P^1_{(0)} = \text{ block diag } (P^1_{11(0)}, 0, 0, 0); \quad P^2_{(0)} = \text{ block diag } (0, P^2_{22(0)}, 0, 0),$$

where $P^i_{ii(0)}$ satisfies the familiar DE

$$\dot{P}^i_{ii(0)} + A_i' P^i_{ii(0)} + P^i_{ii(0)} A_i - P^i_{ii(0)} B_i B_i' P_{ii(0)} + Q_i = 0; \quad P^i_{ii(0)}(t_f) = Q_{i_f} \tag{23}$$

which, being a Riccati equation of the type that arises in linear regulator theory, admits a unique solution $P^i_{ii(0)}(t) \geq 0$ for all $t \in [t_0, t_f]$. On the other hand, (22) also admits a unique solution for every $n \geq 1$, since it is a linear DE. Therefore, by the implicit function theorem, a solution to (6)-(7) exists and can be computed recursively up to various orders of $\epsilon$ by solving two Riccati equations and a sequence of linear equations, which is reminiscent of the sequential design of the (deterministic) linear regulator with weakly coupled subsystems [8], [10] (see also [5]), and deterministic LQ game problem [9], [12]. Hence, we arrive at the following major result.

**Theorem 3.2.** *There exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, (6)-(7) admits a unique solution which can be computed sequentially from (23) and (22). Then, the policies (15) indeed constitute a Nash equilibrium, for $\epsilon$ in the given neighborhood.* ◇

## 4. Properties of the Nash Equilibrium Solution

In the previous section we first proposed a candidate Nash equilibrium solution and then proved that the two coupled matrix DE's in terms of which this equilibrium solution is defined indeed admit solutions when the coupling parameter $\epsilon$ is small. The question we raise here is whether the proposed Nash equilibrium solution has any desirable properties to justify the choice of this particular Nash equilibrium as compared to any other Nash equilibrium the problem might admit. In continuous-time stochastic games, this seems to be a relevant question since a proof of unicity (which is a much stronger result) is quite elusive at this point. Towards answering this question, in what follows, we show that the proposed Nash equilibrium solution is admissible in the limit as $\epsilon \to 0$, and also it is well-posed in a sense to be described later. Further, we also provide a policy iteration interpretation to the approximate solutions.

Before we proceed further, we note that the series expansion of $P^i(t;\epsilon)$, $i = 1, 2$, leads to a similar series expansion for the Nash equilibrium solution as follows:

$$\gamma_i^*(t, y_0^t) = \sum_{l=0}^{n} \gamma_i^{(l)}(t, y_0^t) + 0(\epsilon^{n+1}), \tag{24}$$

where $\gamma_i^{(l)}(t, y_0^t)$ is obtained by replacing $P^i(t;\epsilon)$ by $P^i_{(l)}(t)$ in (15). Now, we are in a position to analyze the properties of the Nash equilibrium solution, and the various terms in its expansion.

We first recognize that the *zero*'th order term in the expansion can be obtained by solving the original stochastic differential game after setting $\epsilon = 0$. Note that with $\epsilon = 0$ the game dynamics are completely decoupled, but the players' policy choices are still coupled through the common measurement.

To obtain the Nash equilibrium of the "zeroth order game", we first fix $u_t^2 = \gamma_2^{(0)}(t,y)$ and minimize $J_1$ over $\gamma_1 \in \Gamma_1$ subject to (1) with $\epsilon = 0$. Note that this is not a standard LQG problem because of the fact that we do not know the structure of $\gamma_2^{(0)}(t,y_0^t)$ (in general, it could even be a nonlinear policy). Despite this, we can solve the above problem, with the solution given below in the form of a lemma.

**Lemma 4.1.** *The stochastic control problem defined above admits the unique solution*

$$\gamma_1^{(0)}(t,y) = -B_1' S_1 \hat{x}_t^1, \tag{25}$$

*where* $S_1 \geq 0$ *satisfies the Riccati equation*

$$\dot{S}_1 + A_1' S_1 + S_1 A_1 - S_1 B_1 B_1' S_1 + Q_1 = 0; \quad S_1(t_f) = Q_{1f}, \tag{26}$$

*and*

$$d\hat{x}_t^1 = (A_1 - B_1 B_1' S_1)\hat{x}_t^1 dt + K_1(dy_t - (C_1\hat{x}_t^1 + C_2\hat{x}_t^2)dt); \hat{x}_{t_0}^1 = 0, \tag{27}$$

$$d\hat{x}_t^2 = A_2\hat{x}_t^2 dt + B_2\gamma_2^{(0)}(t,y_0^t)dt + K_2(dy_t - (C_1\hat{x}_t^1 + C_2\hat{x}_t^2)dt); \hat{x}_{t_0}^2 = 0, \tag{28}$$

$$K^{(0)} := \begin{pmatrix} K_1^{(0)} \\ K_2^{(0)} \end{pmatrix} = \Sigma^{(0)} C'(GG')^{-1}, \tag{29}$$

$$\dot{\Sigma}^{(0)} - A_0\Sigma^{(0)} - \Sigma^{(0)}A_0' - FF' + K^{(0)}GG'K^{(0)} = 0; \quad \Sigma^{(0)}(t_0) = \Sigma_0, \tag{30}$$

$$A_0 := \textit{ block diag } (A_1, A_2).$$

*Proof.* First, we note that $J_1(\gamma_1,\gamma_2;\epsilon=0)$ can be written as

$$\begin{aligned} J_1(\gamma_1,\gamma_2;\epsilon=0) = E\{\textstyle\int_{t_0}^{t_f} \parallel u_t^1 + B_1'S_1x_t^1 \parallel^2 dt\} + x_{t_0}^{1'}S_1(t_0)x_{t_0}^1 \\ + \textstyle\int_{t_0}^{t_f} Tr[S_1(t)F_1(t)F_1(t)']dt\}, \end{aligned} \tag{31}$$

where $u_t^1 \equiv \gamma_1(t,y_0^t)$ and Equation (31) follows from the standard "completing the squares" argument [6]. Note that (31) is independent of $\gamma_2^{(0)}(t,y_0^t)$. Now it follows, from a standard result in stochastic control, that (25) is the unique solution that minimizes (31) provided that we can show that the stochastic control problem is "neutral" i.e., $\tilde{x} := x - \hat{x}$ is independent of both $u_t^1$ and $\gamma_2^{(0)}(t,y_0^t)$. To show this, we simply note that the sigma field $\sigma_t$ with respect to which $u_t^1$ is measurable also includes the sigma field generated by $\gamma_2^{(0)}(t,y_0^t)$. Hence, one can easily modify the proof of the neutrality of the standard one-player LQG stochastic control problem [6] to apply to our problem, thus completing the proof. ◇

Now we reverse the roles of the players, fix $\gamma_1^{(0)}$ arbitrarily and minimize $J_2$ over $\Gamma_2$, with $\epsilon = 0$, to arrive again at a unique solution:

$$\gamma_2^{(0)}(t,y) = -B_2' S_2 \hat{x}_t^2, \tag{32}$$

where $S_2 \geq 0$ satisfies

$$\dot{S}_2 + A_2' S_2 + S_2 A_2 - S_2 B_2 B_2' S_2 + Q_2 = 0; \quad S_2(t_f) = Q_{2f}, \tag{33}$$

and $\hat{x}_t^2$ is given by (28) with $\gamma_2^{(0)}$ replaced by the expression in (32). Since (28) depends on $\hat{x}_t^1$ also, we will need here (27) with the term $-B_1B_1'S_1\hat{x}_t^1$ replaced by $B_1\gamma_1^{(0)}(t,y)$. Since (25) and (32) are unique responses, it readily follows that the Nash equilibrium policies are unique, and given by (19) and (32) with $\hat{x}_2^1$ and $\hat{x}_t^2$ given by (27)-(28), with $\gamma_2^{(0)}$ in (28) replaced by the expression in (32). This completes the derivation of the zeroth order solution. It is useful to note that this zeroth order solution is also the unique solution to a team problem with objective function any convex combination of $J_1^{(0)}$ and $J_2^{(0)}$, where $J_i^{(0)}$ is $J_i$ with $\epsilon = 0$. Furthermore, $S_i = P_{ii(0)}^i$, where the latter was defined by (23); (30) is identical with (19), and (29) is the same as (18) with $n = 0$. All this implies that (25) and (32) are the same as (15) with $\epsilon = 0$.

Next we show that the Nash equilibrium pair obtained in the previous section is an *admissible* solution of the original problem as $\epsilon \to 0$. First, we recall the notion of an admissible solution [1]:

**Definition 4.1.** *A Nash equilibrium pair* $\{\gamma_1^{n1}, \gamma_2^{n1}\}$ *is said to be dominated by another Nash equilibrium pair* $\{\gamma_1^{n2}, \gamma_2^{n2}\}$ *if the following pair of inequalities are satisfied:*

$$J_i(\gamma_1^{n2}, \gamma_2^{n2}) \leq J_i(\gamma_1^{n1}, \gamma_2^{n1}), \qquad i = 1, 2,$$

*with strict inequality holding for at least one player. A Nash equilibrium pair is said to be admissible if it is not dominated by any other Nash equilibrium pair.* ◇

It is well-acknowledged in literature that, in the absence of a proof of uniqueness, one criterion for choosing a particular pair of Nash equilibrium policies is its admissibility. In what follows, we will show that the Nash equilibrium pair $\{\gamma_1^*, \gamma_2^*\}$, obtained in the previous section, is admissible in the limit as $\epsilon \to 0$.

First consider the following team problem:

$$\min_{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2} \{J_1(\gamma_1, \gamma_2) + J_2(\gamma_1, \gamma_2)\} \tag{34}$$

subject to the state equation (1). Let the optimal team cost be denoted by $J_\epsilon^t$. From standard LQG stochastic control theory [6], $J_\epsilon^t$ is given by

$$\begin{aligned} J_\epsilon^t = E\{\int_{t_0}^{t_f} Tr[\Sigma^t(t;\epsilon) P^t(t;\epsilon) B^t B^{t'} P^t(t;\epsilon)]dt + x_{t_0}' P^t(t_0) x_{t_0}\} \\ + \int_{t_0}^{t_f} Tr[P^t(t;\epsilon) F(t) F(t)']dt, \end{aligned} \tag{35}$$

where $P^t(t;\epsilon)$ and $\Sigma^t(t;\epsilon)$ satisfy the following Riccati differential equations:

$$\dot{P}^t + A'P^t + P^t A - P^t B^t B^{t'} P^t + Q^t = 0; \; P^t(t_f) = Q_f^t, \tag{36}$$

$$\dot{\Sigma}^t = A\Sigma^t + \Sigma^t A' - \Sigma^t C'(GG')^{-1} C \Sigma^t + FF' = 0; \; \Sigma^t(t_0) = \Sigma_0, \tag{37}$$

and

$$B^t := (\tilde{B}_1, \tilde{B}_2); \; Q^t := block\ diag\{Q_1 + \epsilon Q_{21}, Q_2 + \epsilon Q_{12}\}$$

$$Q_f^t := block\ diag\{Q_{1f} + \epsilon Q_{21f}, Q_{2f} + \epsilon Q_{12f}\}.$$

Applying the implicit function theorem ([13] Theorems 7.1 and 7.2) to (36) and (37), we can show that

$$P^t(t;\epsilon) = block\ diag\{P_{11(0)}(t), P_{22(0)}(t)\} + 0(\epsilon)$$

and

$$\Sigma^t(t;\epsilon) = \Sigma_{(0)}(t) + 0(\epsilon).$$

Now applying the dominated convergence theorem [11] to (35), we have that

$$\begin{aligned} J^{(0)} := \lim_{\epsilon \to 0} J_\epsilon^t = \sum_{i=1,2} E\{\int_{t_0}^{t_f} Tr[\Sigma_{(0)}(t) P_{ii(0)}(t) B^i B^{i'} P_{ii(0)}(t)]dt + x_{t_0}^{i'} P_{ii(0)}(t_0) x_{t_0}^i\} \\ + \int_{t_0}^{t_f} Tr[P_{ii(0)}(t) F_i(t) F_i(t)']dt. \end{aligned} \tag{38}$$

In a similar manner, we can also show that $\lim_{\epsilon \to 0} J_\epsilon(\gamma_1^*, \gamma_2^*) = J^{(0)}$, where

$$J_\epsilon(\gamma_1, \gamma_2) := J_1(\gamma_1, \gamma_2) + J_2(\gamma_1, \gamma_2).$$

Now we observe the following inequality:

$$J_\epsilon^t \leq J_\epsilon(\gamma_1^*, \gamma_2^*). \tag{39}$$

From the discussion above, (39) becomes an equality in the limit as $\epsilon \to 0$. Hence, asymptotically by using another Nash equilibrium pair, the cost for at least one of the players will increase, implying that $\{\gamma_1^*, \gamma_2^*\}$ is an admissible Nash equilibrium pair in the limit as $\epsilon \to 0$. This leads to the following theorem.

**Theorem 4.1.** *The pair of Nash equilibrium strategies* $\{\gamma_1^*(.;\epsilon), \gamma_2^*(.;\epsilon)\}$ *given by (15) satisfies the following two properties:*

1. *The zeroth order solution* $\{\gamma_1^{(0)}(t, y_0^t), \gamma_2^{(0)}(t, y_0^t)\} := \{\gamma_1^*(t, y_0^t; 0), \gamma_2^*(t, y_0^t; 0)\}$ *is the same as the unique Nash equilibrium solution of the problem with* $\epsilon = 0$ *i.e., the solution is continuous at* $\epsilon = 0$.
2. $\{\gamma_1^*(.;\epsilon), \gamma_2^*(.;\epsilon)\}$ *is an admissible Nash equilibrium pair in the limit as* $\epsilon \to 0$.

*Proof*: See the discussion preceding the statement of the theorem. ◇

Next, we study some properties of the approximate solution. First, we can show (see, Appendix A) that, for any pair of permissible policies $\{\gamma_1, \gamma_2\}$, the following pair of inequalities hold:

$$J_1(\gamma_1^{(0)}, \gamma_2^{(0)}) \leq J_1(\gamma_1, \gamma_2^{(0)}) + 0(\epsilon), \tag{40a}$$

$$J_2(\gamma_1^{(0)}, \gamma_2^{(0)}) \leq J_2(\gamma_1^{(0)}, \gamma_2) + 0(\epsilon). \tag{40b}$$

In other words, the *zeroth* order solution is an $0(\epsilon)$ Nash equilibrium solution. Further, by noting that $lim_{\epsilon\to 0} J_\epsilon(\gamma_1^{(0)}, \gamma_2^{(0)}) = J^{(0)}$, we can conclude that it is an admissible $0(\epsilon)$ Nash equilibrium pair.

Noting the fact that the *zeroth* order solution provides an $0(\epsilon)$ equilibrium, the natural next step is to seek a pair of policies $\gamma_i^{(0)} + \epsilon\gamma_i^{(1)}$, $i = 1, 2$, which provide an $0(\epsilon^2)$ equilibrium i.e., for any pair of permissible policies $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ the following pair of inequalities are satisfied:

$$J_1(\gamma_1^{(0)} + \epsilon\gamma^{(1)}, \gamma_2^{(0)} + \epsilon\gamma_2^{(1)}) \leq J_1(\gamma_1, \gamma_2^{(0)} + \epsilon\gamma_2^{(1)}) + 0(\epsilon^2), \tag{41a}$$

$$J_2(\gamma_1^{(0)} + \epsilon\gamma_1^{(1)}, \gamma_2^{(0)} + \epsilon\gamma_2^{(1)}) \leq J_2(\gamma_1^{(0)} + \epsilon\gamma_1^{(1)}, \gamma_2) + 0(\epsilon^2). \tag{41b}$$

Further, to ensure that the resulting $0(\epsilon^2)$ Nash equilibrium provides the least possible cost to both players (up to $0(\epsilon^2)$), we also require that

$$J_1(\gamma_1^{(0)} + \epsilon\gamma_1^{(1)}, \gamma_2^{(0)} + \epsilon\gamma_2^{(1)}) = \min_{\gamma_1\in\Gamma_1} J_1(\gamma_1, \gamma_2^{(0)}) + 0(\epsilon^2), \tag{42a}$$

$$J_2(\gamma_1^{(0)} + \epsilon\gamma_1^{(1)}, \gamma_2^{(0)} + \epsilon\gamma_2^{(1)}) = \min_{\gamma_2\in\Gamma_2} J_2(\gamma_1^{(0)}, \gamma_2) + 0(\epsilon^2). \tag{42b}$$

We have shown in Appendix B that $\gamma_i^{(1)}$, $i = 1, 2$, obtained by replacing $P^i$ by $P^{i(1)}$ in (15), satisfies (42a)-(42b), and (41a)-(41b) can be verified in a similar fashion as in Appendix A.

Note that the equations (40a), (40b), (41a), (41b) describe the first two steps of a modified policy iteration scheme as follows: In a normal policy iteration, at each step, one would obtain the best response of a player given the policy of the other player. But, in our modified policy iteration, at the $n$'th step, we seek a response that is $0(\epsilon^{n+1})$ close to the optimal response. The reason is that, a normal policy iteration would result in an increasing estimator size at each step whereas in our iteration, the estimator size is a constant. The justification for this modification is provided by *Theorem 3.2* of the previous section which states the fact that there does indeed exist a finite-dimensional equilibrium for the weakly coupled stochastic Nash game for small enough $\epsilon$.

It can be shown in a manner analogous to Appendices A and B that, in general, the pair of strategies $\{\sum_{l=0}^{n} \gamma_i^{(l)}(t, y_0^t)\}_{i=1,2}$ satisfy the following pair of inequalities:

$$J_1(\sum_{l=0}^{n} \gamma_1^{(l)}, \sum_{l=0}^{n} \gamma_2^{(l)}) \leq J_1(\gamma_1, \sum_{l=0}^{n} \gamma_2^{(l)}(t, y_0^t)) + 0(\epsilon^{n+1}), \tag{43a}$$

$$J_2(\sum_{l=0}^{n} \gamma_1^{(l)}, \sum_{l=0}^{n} \gamma_2^{(l)}) \leq J_2(\sum_{l=0}^{n} \gamma_1^{(l)}, \gamma_2) + 0(\epsilon^{n+1}), \tag{43b}$$

Furthermore, the following order relationships are also satisfied:

$$J_1(\sum_{l=0}^{n}\gamma_1^{(l)}, \sum_{l=0}^{n}\gamma_2^{(l)}) = \min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \sum_{l=0}^{n-1}\gamma_2^{(l)})) + 0(\epsilon^{n+1}), \tag{44a}$$

$$J_2(\sum_{l=0}^{n}\gamma_1^{(l)}, \sum_{l=0}^{n}\gamma_2^{(l)}) = \min_{\gamma_2 \in \Gamma_2} J_2(\sum_{l=0}^{n-1}\gamma_1^{(l)}, \gamma_2) + 0(\epsilon^{n+1}). \tag{44b}$$

It should be noted that, while (43a)-(43b) ensure that the resulting solution at each step of the iteration is an $0(\epsilon^{n+1})$ Nash equilibrium, equations (44a)-(44b) ensure that among all strategies satisfying (43a)-(43b), the pair of strategies $\{\sum_{l=0}^{n}\gamma_i^{(l)}(t, y_0^t)\}_{i=1,2}$ provides the least possible cost to both players (up to $0(\epsilon^{n+1})$).

## 5. Conclusions

It is well known that discrete-time LQG stochastic Nash games [4] and continuous-time zero-sum LQG stochastic games [3] admit unique linear, finite-dimensional equilibria. Towards answering the question whether this fact is true also for continuous-time LQG stochastic games, here we have studied systems which are weakly coupled spatially and strongly coupled informationally. Using perturbation techniques, we have shown the existence of a finite-dimensional Nash equilibrium for the stochastic game for small enough $\epsilon$.

We have also shown that the Nash equilibrium is $0(\epsilon)$ *admissible*, and furthermore it exhibits a well-posedness property i.e., the Nash equilibrium solution with $\epsilon = 0$ is the same as the unique Nash equilibrium of the *zeroth* order stochastic game. Also as a means of reducing computational complexity, we have presented an iterative technique to compute the approximate Nash equilibrium, which involves the solutions of two (decoupled) Riccati differential equations, and a sequence of linear differential equations, as compared to the complicated coupled nonlinear differential equations which have to be solved to obtain the exact solution. Further, we have interpreted the Nash equilibrium solution as the output of a modified policy iteration, which may serve as a useful tool in proving the unicity of the Nash equilibrium. We anticipate the results presented here to extend to the multiple (more than two) player situations, but the expressions there are very complicated, and hence are not provided here.

## References

[1] T. Başar. Informationally Nonunique Equilibrium solutions in Differential Games. *SIAM Journal of Control and Optimization*, Vol.15, pp. 636-660, 1977.

[2] T. Başar. Two-criteria LQG decision problems with one-step delay observation sharing pattern. *Information and Control*, 38(1):21–50, July 1978.

[3] T. Başar. On the saddle-point solution of a class of stochastic differential games. *Journal on Optimization Theory and Applications*, 33(4):539–556, April 1981.

[4] T. Başar. *Theory of Stochastic Dynamic Noncooperative Games.* Basil Blackwell, Oxford, England, 1991. Under Preparation.

[5] A. Bensoussan. *Perturbation Methods in Optimal Control.* John Wiley, Gauthier-Villars, Chichester, England, 1988.

[6] M.H.A. Davis. *Linear Estimation and Stochastic Control.* Chapman and Hall, London, 1977.

[7] W. H. Fleming. Optimal continuous-parameter stochastic control. *SIAM Review*, 11:470–509, 1969.

[8] P.V. Kokotovic, W.R. Perkins, J.B. Cruz, and G. D'Ans. $\epsilon$-Coupling Method for Near-Optimum Design of Large-Scale Linear Systems. *Proceedings IEE*, 116:889-892, 1969.

[9] Ü. Özgüner and W.R. Perkins. A Series Solution to the Nash Strategy for Large Scale Interconnected Systems. *Automatica*, 13:313-315, 1977.

[10] A. A. Pervozvanskii and V. G. Gaitsgori. *Theory of Suboptimal Decisions.* Kluwer, Amsterdam, The Netherlands, 1988.

[11] H.L. Royden, *Real Analysis*, The Macmillan Company, New York, 1968.

[12] R. Srikant and T. Başar. Iterative Computation of Noncooperative Equilibria in Nonzero-sum Differential Games with Weakly Coupled Players. To be published in the *Journal of Optimization Theory and Applications*, 1991.

[13] A.N. Tikhonov, A.B. Vasil'eva, and A.G. Sveshnikov. *Differential Equations.* Springer-Verlag, New York, 1985.

# Appendix A

Verification of (40a)-(40b):

Consider the following standard LQG stochastic control problem:

$$\min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \gamma_2^{(0)}).$$

The minimizing solution to the above problem is given by [6]

$$\tilde{\gamma}_1^*(t, y_0^t) = -B_1^{f}(t) P^f(t) \hat{x}^f(t), \tag{45}$$

where

$$d\hat{x}^f = (A^f - B_1^f B_1^{f'} P^f)\hat{x}^f dt + K^f(dy - C^f \hat{x}^f dt);\ \hat{x}^f(t_0) = \bar{x}_0^f, \tag{46}$$

$$\dot{P}^f + A^{f'} P^f + P^f A^f - P^f B^f B^{f'} P^f + Q^f = 0;\ P^f(t_f) = Q_f^f, \tag{47}$$

$$\dot{\Sigma}^f = A^f \Sigma^f + \Sigma^f A^{f'} - K^f GG' K^f + F^f F^{f'};\ \Sigma^f(t_0) = \Sigma_0^f, \tag{48}$$

and

$$A^f := \begin{pmatrix} A_1 & \epsilon A_{21} & 0 & 0 \\ \epsilon A_{12} & A_2 & 0 & -B_2 B_2' P_{22(0)}^2 \\ K_1^{(0)} C_1 & K_1^{(0)} C_2 & A_1 - B_1 B_1' P_{11(0)}^1 - K_1^{(0)} C_1 & -K_1^{(0)} C_2 \\ K_2^{(0)} C_1 & K_2^{(0)} C_2 & -K_2^{(0)} C_2 & A_2 - B_2 B_2' P_{22(0)}^2 - K_2^{(0)} C_2 \end{pmatrix},$$

$B^f := (B_1', 0, 0, 0)'$; $F^f := block\ diag\{F_1, F_2, K^{(0)} G\}$; $\bar{x}_0^f := (\bar{x}_0', 0, 0)'$,

$M^f := (0, 0, I)'$; $Q^f := block\ diag\{Q_1, \epsilon Q_{12}, 0, P_{22(0)}^2 B_2 B_2' P_{22(0)}^2\}$,

$Q_f^f := block\ diag\{Q_{1f}, \epsilon Q_{12f}, 0, 0\}$; $C^f := (C_1, C_2, 0, 0)$

$K^f := (\Sigma^f C^{f'} + F^f M^f G')(GG')^{-1}$; $\Sigma_0^f := block\ diag\{\Sigma_0, 0, 0\}$.

Note that with $\epsilon = 0$,

$$P^f(t; \epsilon = 0) \equiv (P^1_{11(0)}, 0, 0, 0).$$

Also, by applying the implicit function theorem ([13] Theorems 7.1 and 7.2]) to (47), we have that $P(t; \epsilon) = P(t; \epsilon = 0) + 0(\epsilon)$, which implies that

$$\tilde{\gamma}_1^*(t, y_0^t) = \gamma_1^{(0)}(t, y_0^t) + 0(\epsilon).$$

Hence, from the quadratic nature of the performance index it follows that

$$J_1(\gamma_1^{(0)}, \gamma_2^{(0)}) = J_1(\tilde{\gamma}_1^*, \gamma_2^{(0)}) + 0(\epsilon).$$

Equivalently,

$$min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \gamma_2^{(0)}) = J_1(\tilde{\gamma}_1^{(0)}, \gamma_2^{(0)}) + 0(\epsilon),$$

which implies (40a). Equation (40b) can be verified in a similar fashion.

# Appendix B

Verification of (42a)-(42b):

From standard LQG stochastic control theory [6], $J_1^f := \min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \gamma_2^{(0)})$ is given by

$$\begin{aligned} J_1^f = E\{\int_{t_0}^{t_f} Tr[\Sigma^f(t;\epsilon) P^f(t;\epsilon) B^f B^{f'} P^f(t;\epsilon)] dt\} + x_0^{f'} P^f(t_0) x_0^f \\ + \int_{t_0}^{t_f} Tr[P^f(t;\epsilon) F^f(t) F^f(t)'] dt + Tr[\Sigma_0^f P^f(t_0)]. \end{aligned} \tag{49}$$

Before we proceed further, we make the observation that $A^f$ is equivalent to $\tilde{A}^1$ given by (10), with $P^i$ replaced by $P^i_{(0)}$, $i = 1, 2$. Hence, if $P^f(t;\epsilon)$ is expanded as

$$P^f(t;\epsilon) = P^f_{(0)}(t) + \epsilon P^f_{(1)}(t) + 0(\epsilon^2),$$

then $P^f_{(1)}(t) \equiv P^1_{(1)}(t)$, since $P^1_{(1)}(t)$ depends only on $P^1_{(0)}(t)$, and $P^1_{(0)}(t) \equiv P^f_{(0)}(t)$ as observed in Appendix A.

Next we consider the expression for

$$J_1^{f1} := \min_{\gamma_1 \in \Gamma_1} J_1(\gamma_1, \gamma_2^{(0)} + \epsilon \gamma_2^{(1)}),$$

which is given by

$$\begin{aligned} J_1^{f1} = E\{\int_{t_0}^{t_f} Tr[\Sigma^{f1}(t;\epsilon) P^{f1}(t;\epsilon) B^f B^{f'} P^{f1}(t;\epsilon)] dt\} + \bar{x}_0^{f'} P^{f1}(t_0) \bar{x}_0^f \\ + \int_{t_0}^{t_f} Tr[P^{f1}(t;\epsilon) F^f(t) F^f(t)'] dt + Tr[\Sigma_0^f P^{f1}(t_0)], \end{aligned} \tag{50}$$

where

$$\dot{P}^{f1} + A^{f1'} P^{f1} + P^{f1} A^{f1} - P^{f1} B^f B^{f'} P^{f1} + Q^{f1} = 0; \; P^f(t_f) = Q^f_f, \tag{51}$$

$$\dot{\Sigma}^{f1} = A^{f1} \Sigma^{f1} + \Sigma^{f1} A^{f1'} - K^{f1} G G' K^{f1}; \; \Sigma^{f1}(t_0) = \Sigma_0^f, \tag{52}$$

$A^{f1}$ is given by (10), with $P^i$ replaced by $P^i_{(0)} + \epsilon P^i_{(1)}$, $i = 1, 2$, and $Q^{f1}$ and $K^{f1}$ are defined analogous to $Q^f$ and $K^f$, respectively. Hence, for the same reasons as before, $P^1_{(0)}(t) \equiv P^{f1}_{(0)}(t)$ and $P^1_{(1)}(t) \equiv P^{f1}_{(1)}(t)$. We can also show that both $\Sigma_f(t)$ and $\Sigma^{f1}(t)$ are of the form *block diag*$\{\Sigma, 0\}$. (To show this, we can mimic the proof of the fact that $\tilde{\Sigma}$ is in this form, which formed part of the proof of *Theorem 3.1*.) All the discussion above leads to the conclusion that $J_1^{f1} = J_1^f + 0(\epsilon^2)$. Hence, from (41a), it follows that (42a) is satisfied. Equation (42b) can be verified in a similar fashion.