A Complete Characterization of Minimax
and Maximin Encoder-Decoder Policies for
Communication Channels with Incomplete
Statistical Description

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Abstract—The problem is considered of transmitting a sequence of independent and identically distributed Gaussian random variables over a channel whose statistical description is incomplete. The channel is modeled as one that is conditionally Gaussian, with the unknown part being controlled by a so-called “jammer” who may have access to the input to the encoder and operates under a given power constraint. By adopting a game-theoretic approach, a complete set of solutions is obtained (encoder and decoder mappings, and least-favorable distributions for the channel noise) for this statistical decision problem, under two different sets of conditions, depending on whether the encoder mapping is deterministic or stochastic. In the latter case, existence of a mixed saddle-point solution can be verified when a side channel of a specific nature is available between the transmitter and the receiver. In the former case, however, only minimax and maximin solutions can be derived.

I. INTRODUCTION AND PROBLEM FORMULATION

We consider complete characterizations for the saddle-point or minimax and maximin solutions to a class of communication problems that involve the transmission of a sequence of Gaussian random variables over a noisy channel under a given power constraint. By adopting a game-theoretic approach, a complete set of solutions is obtained (encoder and decoder mappings, and least-favorable distributions for the channel noise) for this statistical decision problem, under two different sets of conditions, depending on whether the encoder mapping is deterministic or stochastic. In the latter case, existence of a mixed saddle-point solution can be verified when a side channel of a specific nature is available between the transmitter and the receiver. In the former case, however, only minimax and maximin solutions can be derived.

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Figs. 1, 2, 3. Communication systems of Problems 1, 2, 3.

The power constraint \( E[|\beta(u)|^2] \leq k^2 \). Hence the channel output to the receiver is

\[
z = x + y + w = \gamma(u) + \beta(u) + w. \tag{1.1}
\]

At the receiver the decoder \( \delta(\cdot) \) is chosen out of the space \( \Gamma_r \) of random mappings from \( \mathbb{R} \) into \( \mathbb{R} \) so as to obtain an estimate of the input signal \( u \) based on the measurement \( z \), under a mean-squared error criterion. In accomplishing this, the receiver will also have access to the structure of the encoder mapping (but not its realized value) via the forward side channel depicted in the figure. Hence, with the mean-squared error denoted by

\[
J(\gamma, \delta, \beta) = E\{ |u - \delta(z)|^2 \}, \tag{1.2}
\]

where \( z \) is given by (1.1) and \( E \) denotes the expectation over statistics of \( u, w, y, \delta, \beta \), we assume that the pair \((y, \delta) \in \Gamma_y \times \Gamma_\delta \) will be chosen so as to minimize this quantity, while \( \beta \) is chosen to maximize the same quantity. What is sought, then, (if it exists) is a saddle-point solution \((y^*, \delta^*, \beta^*) \in \Gamma_y \times \Gamma_\delta \times \Gamma_\beta \) satisfying

\[
J(y^*, \delta^*, \beta^*) \leq J(\gamma, \delta, \beta^*) \leq J(\gamma^*, \delta^*, \beta^*),
\]

for all \((\gamma, \delta, \beta) \in \Gamma_\gamma \times \Gamma_\delta \times \Gamma_\beta \). \( \tag{1.3} \)

**Problem 2:** The communication system of Problem 2 is depicted in Fig. 2. The only difference between this problem and Problem 1 is that here the encoder mapping is restricted to be deterministic, in which case the side channel would be superfluous. As it is shown later (in Section III), this problem does not admit a saddle-point solution; hence, we will be interested in the derivation of minimax and maximin solutions. Towards this end, let \( \Gamma_{\text{ed}} \subset \Gamma_e \) be the space of all (deterministic) mappings \( \gamma : \mathbb{R} \rightarrow \mathbb{R} \) for which \( E[|\gamma(u)|^2] \) is a well-defined quantity and is bounded from above by \( c^2 \). Then, under the minimax approach, we evaluate the upper value of the zero-sum game with kernel \( J \),

\[
\hat{J}_H^* = J(\gamma^*, \delta^*, \beta^*) = \min_{\gamma \in \Gamma_\gamma} \max_{\delta \in \Gamma_\delta} J(\gamma, \delta, \beta^*) \tag{1.4}
\]

\[
= \min_{\gamma \in \Gamma_\gamma} J(\gamma, \delta^*, \beta^*) \tag{1.5}
\]

assuming that such a solution exists. The triple \((\gamma^*, \delta^*, \beta^*) \) as determined above is called the minimax solution for the communication system of Problem 2.

To obtain the maximin solution, we consider instead the lower value of the zero-sum game,

\[
J_M^* = J(\gamma^*, \delta^*, \beta^*) = \max_{\beta \in \Gamma_\beta} \min_{\gamma \in \Gamma_\gamma} J(\gamma, \delta, \beta) \tag{1.6}
\]

where the pair \((\gamma^*, \delta^*) \) is a mapping from \( \Gamma_j \) into \( \Gamma_{\text{ed}} \times \Gamma_r \), determined by

\[
(\gamma^*, \delta^*) = \arg \min_{(\gamma, \delta) \in \Gamma_\gamma \times \Gamma_\delta} J(\gamma, \delta, \beta^*). \tag{1.7}
\]

The triple \((\gamma^*, \delta^*, \beta^*) \) defined above by (1.6) and (1.7) is the maximin solution for the communication system of Problem 2. A saddle-point solution \((\gamma^*, \delta^*, \beta^*) \in \Gamma_{\text{ed}} \times \Gamma_r \times \Gamma_j \) will exist if and only if

\[
J_M^* = J_H^* = J(\gamma^*, \delta^*, \beta^*),
\]

which, however, is not the case in Problem 2, as it will be shown in Section III.

B. A Summary of Relevant Results from [1]

In the context of the general formulation of Section I-A, consider the system depicted in Fig. 3, where the jammer now has access to the output of the encoder. Hence, here (1.1) is replaced by

\[
z = x + y + w = \gamma(u) + \beta(y) + w. \tag{1.8}
\]

which constitutes the only difference between this system and that of Fig. 2. The result proven in [1] in a more general context is that a saddle point exists for this system and it has different characterizations in different regions of the parameter space, determined by the relative magnitudes of \( k, c, \) and \( \psi \). To present this result in terms of the
notation of Section I-A, let us first introduce the regions

\[ R_1: k \geq c \]
\[ R_2: k < c \]
\[ R_3: k^2 - ck + \varphi > 0 \]
\[ R_4: k^2 - ck + \varphi \leq 0. \]

Then we have the following theorem.

**Theorem 1 [I]:** The communication system of Fig. 3 (to be referred to as Problem 3) admits two saddle-point solutions \((y^*, \delta^*, \beta^*)\) and \((-y^*, -\delta^*, \beta^*)\) over the space

\[ \Gamma_e \times \Gamma_r \times \Gamma_j, \]

where

\[ y^*(u) = \begin{cases} 
\text{arbitrary,} & \text{in } R_1 \\
 cu, & \text{in } R_2
\end{cases} \] (1.10)

\[ \beta^*(x) = \begin{cases} 
-x, & \text{in } R_1 \\
 \frac{(k/c)x}{(c^2 + k^2 + \varphi)^2}, & \text{in } R_2 \cap R_3 \\
 \frac{2c(k^2 + \varphi)}{c^2 + k^2 + \varphi}, & \text{in } R_2 \cap R_4
\end{cases} \] (1.11)

where \( \eta \) is a zero-mean Gaussian random variable with variance \( k^2 \); it is independent of both \( u \) and \( w \); and the saddle-point value of \( J \) is

\[ J^*_s = \frac{k^2 + \varphi}{c^2 + k^2 + \varphi}. \] (2.3)

**Proof:** We need to prove that the solution given above satisfies the pair of saddle-point inequalities (1.3) for all \((y, \delta, \beta) \in \Gamma_e \times \Gamma_r \times \Gamma_j\) and under the stipulation that the side channel is used to carry structural information concerning probabilistic encoder mappings. The proof will be completed in two steps.

1) **Verification of the Right Side Inequality of (1.3):** Suppose that the jammer's policy is as given by (2.2). Then, the communication system of Fig. 1 (Problem 1) becomes the standard Gaussian test channel [4], for which the best encoding policy is known to be either \( y(u) = cu \) or \( y(u) = -cu \), with the corresponding decoder structures being

\[ E[u|z] = \left\{ \begin{array}{ll} 
\frac{c}{c^2 + k^2 + \varphi} & \text{for } c^2 + k^2 + \varphi \neq 0 \\
\frac{c}{c^2 + k^2 + \varphi} & \text{for } c^2 + k^2 + \varphi = 0
\end{array} \right. \]

respectively. Both these (deterministic) policies lead to the same distortion level \( J(y^*, \delta^*, \beta^*) = J^*_s \), and so does the mixed policy given in (2.1).

2) **Verification of the Left Side Inequality:** With the pair of encoding and decoding policies \((y^*, \delta^*, \beta^*)\) as given in (2.1), we first compute the mean-squared error \( J^*_s(y^*, \delta^*, \beta^*) \) on the specific structural realization of encoding-decoding policies in current use.

a) When \( \left( cu, cz/(c^2 + k^2 + \varphi) \right) \) is the realized encoder-decoder policy pair, we have

\[ J_1 = E \left\{ \left( \frac{c}{c^2 + k^2 + \varphi} - u \right)^2 \right\} \]

\[ = E \left\{ \left( \frac{c}{c^2 + k^2 + \varphi} (cu + y + w) - u \right)^2 \right\} \]

\[ = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2}{(c^2 + k^2 + \varphi)^2} E(y^2) \]

\[ + \frac{c^2}{(c^2 + k^2 + \varphi)^2} \varphi - \frac{2c(k^2 + \varphi)}{(c^2 + k^2 + \varphi)^2} E(uv). \] (2.4)
b) Now take $\gamma(u) = -cu$, and $\delta = -cz/(c^2 + k^2 + \varphi)$ to obtain

$$
\hat{J}_2 = E\left\{ \left( -\frac{cz}{c^2 + k^2 + \varphi} - u \right)^2 \right\}
$$

$$
= E\left\{ \left( -\frac{c}{c^2 + k^2 + \varphi}(-cu + y + w) - u \right)^2 \right\}
$$

$$
= \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2}{(c^2 + k^2 + \varphi)^2} E\{y^2\}
$$

$$
+ \frac{c^2}{(c^2 + k^2 + \varphi)^2}\varphi + \frac{2c(k^2 + \varphi)}{(c^2 + k^2 + \varphi)^2} E\{uy\}.
$$

(2.5)

By unconditioning the above conditional values of $J$, we obtain

$$
J = \frac{1}{2} \hat{J}_1 + \frac{1}{2} \hat{J}_2 = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2k^2}{(c^2 + k^2 + \varphi)^2} E\{y^2\} + \frac{c^2}{(c^2 + k^2 + \varphi)^2}\varphi.
$$

(2.6)

which indicates that $J$ depends only on the second moment of $y$. Hence, the maximizing solution is any random variable with second moment equal to $k^2$; the Gaussian random variable $\eta$, with mean zero and variance $k^2$, is one such random variable.

Hence

$$
J^* = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2k^2}{(c^2 + k^2 + \varphi)^2} E\{y^2\} + \frac{c^2\varphi}{(c^2 + k^2 + \varphi)^2} \frac{k^2 + \varphi}{c^2 + k^2 + \varphi} = J_A^*.
$$

This then completes the proof of Theorem 2.

**Corollary 1:** The solution $(\gamma^*, \delta^*)$ given in Theorem 2 is the almost surely (a.s.) unique optimal solution for the transmitter-receiver. For the jammer any optimal solution $\bar{\beta} \in \Gamma_j$ has the property that the random variable $y = \bar{\beta}(u)$ is uncorrelated with the message, i.e., $E\{uy\} = 0$.

**Proof.**

1) Because of the interchangeability property of saddle-point equilibria, every optimal solution for the encoder-decoder should be in equilibrium with $\beta^*$ given by (2.2). However, for $\beta = \beta^*$, we have the standard Gaussian test channel whose entire class of optimum solutions in $\Gamma_e \times \Gamma_r$ is described by

$$
(\hat{\gamma}, \hat{\delta}) = \begin{cases} 
(cu, \frac{c}{c^2 + k^2 + \varphi} z), & w.p. q \\
(-cu, -\frac{c}{c^2 + k^2 + \varphi} z), & w.p. 1 - q
\end{cases}
$$

(2.7)

with $q \in [0, 1]$. Substituting (2.7) into $J$ for a fixed $q$, we obtain

$$
J(\gamma, \delta, \beta) = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2\varphi}{(c^2 + k^2 + \varphi)^2} E\{y^2\}
$$

$$
+ \frac{c^2}{(c^2 + k^2 + \varphi)^2} \left( 1 - 2q \right) E\{uy\}.
$$

(2.8)

If $q \neq 0.5$ the jammer could choose $\beta(u) = sgn(2q - 1)ku$, which leads to

$$
J(\hat{\gamma}, \hat{\delta}, \bar{\beta}) > J^*.
$$

Hence (2.7) provides an optimal solution only if $q = 0.5$.

2) To prove the second part of the corollary it is sufficient to observe that if $\beta$ is chosen such that $E\{uy\} \neq 0$, then one can choose an encoder-decoder policy pair as given in (2.7), with $q = 0$ if $E\{uy\} > 0$ and $q = 1$ if $E\{uy\} < 0$, leading in each case to a value of $J$ (see (2.8)) that is strictly smaller than $J^*$. This clearly shows that a $\beta \in \Gamma_j$ with $E\{uy\} \neq 0$ cannot be an optimal solution for the jammer.

### III. Derivation of Minimax and Maximin Policies for Problem 2

We now restrict the encoder policy to be a deterministic mapping, thereby eliminating the need to use a side channel between the transmitter and the receiver. The main result to be obtained below is that with this restriction the problem (Problem 2) does not admit a saddle-point solution because

$$
J_A^* - J_B^* < J_B - J_A^*.
$$

Furthermore we provide a complete characterization of a set of minimax and maximin solutions.

**A. Derivation of Minimax Solutions**

Comparing the communication systems of Figs. 2 and 3, we first observe the following property, which holds if $\gamma$ is restricted to be a deterministic mapping. For each fixed $\gamma \in \Gamma_{ed}$, to every $\beta$ that is a function of $x = \gamma(u)$ in Problem 3 (Fig. 3) corresponds a jammer policy $\bar{\beta} = \beta \circ \gamma$ in Problem 2, which represents the same random variable $v$ ($= \beta(y(\gamma)) = \bar{\beta}(u)$). The statement, however, is not necessarily true in the other direction because $\gamma$ may not be invertible. Hence, we have the inequality

$$
\sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(x)) \leq \sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(u))
$$

for every $(\gamma, \delta) \in \Gamma_{ed} \times \Gamma_r$, with $x = \gamma(u)$. Now, taking the infimum of both sides over $(\gamma, \delta)$, we obtain

$$
J^*_c = \inf_{(\gamma, \delta) \in \Gamma_{ed} \times \Gamma_r} \sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(x)) \leq J_B^*.
$$

(3.1)

where the equality on the left side follows from Theorem 1 (since Problem 3 admits a saddle point). This shows that

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3 Henceforth we will use the terminology "optimal" to refer to individual components of a pair of policies in saddle-point equilibrium.
the minimax (upper) value for Problem 2 is bounded from below by the saddle-point value of Problem 3. The following theorem now proves that the inequality in (3.1) is in fact an equality, and it also provides a set of minimax policies that achieve this value. The regions $R_1, \ldots, R_4$ used in the theorem are those introduced earlier by (1.9).

**Theorem 3:** In Problem 2 the minimax value of the mean-squared error at the receiver is

$$J^*_B = J^*_C = \begin{cases} 
1, & \text{in } R_1 \\
\frac{\varphi}{(c - k)^2 + \varphi}, & \text{in } R_2 \cap R_3 \\
\frac{k^2 + \varphi}{c^2}, & \text{in } R_2 \cap R_4 
\end{cases} \tag{3.2}$$

with a corresponding set of minimax solutions being $(\gamma^*, \delta^*, \beta^*)$ and $(-\gamma^*, -\delta^*, -\beta^*)$, where

$$\bar{\gamma}^*(u) = \begin{cases} 
\gamma(u), & \text{where } \gamma \in \Gamma_{ed} \text{ is arbitrary}; \text{ in } R_1 \\
cu, & \text{in } R_2 
\end{cases} \tag{3.3}$$

$$\bar{\delta}^*(z) = \begin{cases} 
0, & \text{in } R_1 \\
\frac{c - k}{(c - k)^2 + \varphi}z, & \text{in } R_2 \cap R_3 \\
\frac{1}{c}z, & \text{in } R_2 \cap R_4 
\end{cases} \tag{3.4}$$

$$\bar{\beta}^*(u) = \begin{cases} 
\beta(u), & \text{where } \beta \in \Gamma_j \text{ is arbitrary}; \text{ in } R_1 \\
-ku, & \text{in } R_2 \cap R_3 \\
\beta(u), & \text{where } \beta \in \Gamma_j \text{ satisfies } \|\beta(u)\| = k; \text{ in } R_2 \cap R_4 
\end{cases} \tag{3.5}$$

Here, $\|\beta(u)\| \triangleq \{E[|\beta(u)|^2]\}^{1/2}$, and $\bar{\beta}^* = \beta_{(\gamma^*, \delta^*)}$, where $\beta_{(\gamma, \delta)}$ is defined by (1.5).

**Proof:** Since

$$J_B^* \leq \sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(u)), \quad \text{for all } \gamma \in \Gamma_{ed}, \gamma \in \Gamma_j, \beta \in \Gamma_j,$$

the proof is completed by showing that with $\bar{\gamma}^*, \bar{\delta}^*$ as given above,

$$J_B^* \leq \sup_{\beta \in \Gamma_j} J(\bar{\gamma}^*, \bar{\delta}^*, \beta(u)) = J(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*(u)) = J_C^* \tag{3.6}$$

which, in view of inequality (3.1), leads to $J_B^* = J_C^*$, and that $(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*)$ is a minimax solution.

To show the validity of the right side equality in (3.6), first consider region $R_1$. Here, since

$$J(\gamma^*, \delta^*, \beta) = E\{u^2\} - 1 = J_C^*, \quad \text{for all } \beta \in \Gamma_j,$$

the result is clearly valid.

In region $R_2 \cap R_3$, we have

$$J(\gamma^*, \delta^*, \beta) = E\left\{\left[u - \frac{c - k}{(c - k)^2 + \varphi}(cu + \beta(u) + w)\right]^2\right\}$$

$$= \frac{\varphi + k(k-c)}{c^2} \left[\frac{\varphi + k(k-c)}{(c - k)^2 + \varphi}\right]^2$$

$$+ 2\frac{\varphi + k(k-c)}{(c - k)^2 + \varphi}E[u\beta(u)](k-c)$$

$$+ \frac{(c - k)^2}{(c - k)^2 + \varphi}E[\beta(u)]^2 + \frac{(c - k)^2\varphi}{[(c - k)^2 + \varphi]^2}.$$

(3.7)

Since the coefficient of $E[u\beta(u)]$ is negative in $R_2 \cap R_3$, the above expression is maximized uniquely by choosing $\beta^*(u) = -ku$, which, when substituted into (3.7), leads to

$$J(\gamma^*, \delta^*, -ku) = \frac{\varphi}{(c - k)^2 + \varphi} = J_C^*.$$

Finally, in $R_2 \cap R_4$, we have

$$J(\gamma^*, \delta^*, \beta) = E\{\left[u - \frac{1}{c}(cu + \beta(u) + w)\right]^2\}$$

$$= \frac{1}{c^2}E\{[\beta(u)]^2\} + \frac{\varphi}{c^2},$$

which is maximized by choosing $\beta^*(u)$ any random variable with second moment equal to $k^2$. The corresponding value of $J$ is then

$$\frac{k^2 + \varphi}{c^2} = J_C^*.$$

The following theorem now extends the result of Theorem 3 and provides a complete characterization for the minimax encoder-decoder mappings.

**Theorem 4:** In Problem 2 let the decoder be restricted to linear mappings, $\delta(z) = \Delta z$, $\Delta \in \mathbb{R}$. Then: 1) the class of optimum jammer policies $\beta_{(\gamma, \delta)}$ defined by (1.5) is completely described by

$$\beta_{(\gamma, \delta)}(u) = \begin{cases} 
\beta(u), & \text{where } \beta \in \Gamma_j \text{ is arbitrary}; \text{ in } R_1 \\
-ku, & \text{in } R_2 \cap R_3 \\
\beta(u), & \text{where } \beta \in \Gamma_j \text{ satisfies } \|\beta(u)\| = k; \text{ in } R_2 \cap R_4 
\end{cases} \tag{3.8}$$

and 2) the set of all encoder-decoder policies that minimize $J(\gamma, \delta, \beta_{(\gamma, \delta)})$ is precisely the set given in Theorem 3; that is, the minimax solutions $(\pm \gamma^*, \pm \delta^*)$ constitute a complete characterization.
Proof:

1) Note that with \( \delta(z) = \Delta z \), \( J(\gamma, \delta, \beta) = E[\{u - \Delta[y(u) + \beta(u) + w]\}^2] \), and when \( \Delta = 0 \), \( J \) becomes independent of \( \beta \), thus making every \( \beta \in \Gamma_j \) a maximizing solution. If \( \Delta \neq 0 \)

\[
J(\gamma, \delta, \beta) = \Delta^2 E\left[ \left( \frac{1}{\Delta} |u - \Delta y(u)| - \beta(u) \right)^2 \right] + \Delta^2 \phi,
\]

\[
(*)
\]

and maximizing this over \( \beta \in \Gamma_j \), under the condition \( \|u - \Delta y(u)\| \neq 0 \), we obtain, uniquely,

\[
\beta^*_\gamma(u) = -\frac{1}{\Delta} \frac{1}{\Delta} |u - \Delta y(u)| k, \quad \Delta \neq 0
\]

where \( \gamma \in \Gamma_j \) is arbitrary, if \( \Delta = 0 \),

\[
\beta^*_\gamma(u) = 0
\]

Substituting this solution into \( J \), we obtain for \( \Delta \geq 0 \)

\[
\gamma^*_\Delta(u) = \begin{cases} 
    cu, & \text{if } \Delta < 1/c, \\
    (1/\Delta)u, & \text{if } \Delta \geq 1/c,
\end{cases}
\]

for \( \Delta < 0 \) we have

\[
\gamma^*_\Delta(u) = -\gamma^*_\Delta(u).
\]

and for \( \Delta \leq 0 \)

\[
J(\gamma^*_\Delta, \delta, \beta^*_{\gamma^*_\Delta, 8}) = J^-(\Delta) = J^+(\Delta) = J^+(\Delta).
\]

It can easily be shown that

\[
\min_{0 \leq \Delta \leq 1/c} \left[ \left( 1 + (k - c) \right)^2 + \Delta^2 \phi \right] \leq \frac{\phi}{(k - c)^2 + \phi}, \quad \text{in } R_1 \cap R_2
\]

\[
= \frac{k^2 + \phi}{c^2}, \quad \text{in } R_1 \cap R_4
\]

\[
\beta^*(u) = \eta, \quad \text{in } R_1
\]

and that

\[
\min_{\Delta \geq 1/c} \left[ \Delta^2 (\phi + k^2) \right] = \frac{\phi + k^2}{c^2}
\]

with the minimizing arguments being, respectively,

\[
\Delta = \begin{cases} 
   \frac{c - k}{(k - c)^2 + \phi}, & \text{in } R_2 \cap R_3, \\
   \frac{1}{c}, & \text{in } R_2 \cap R_4, \\
   0, & \text{in } R_1
\end{cases}
\]

\[\Delta = 1/c.\]

Hence

\[
\min J^+(\Delta) = \begin{cases} 
   1, & \text{in } R_1, \\
   \frac{\phi}{(k - c)^2 + \phi}, & \text{in } R_2 \cap R_3, \\
   \frac{k^2 + \phi}{c^2}, & \text{in } R_2 \cap R_4
\end{cases}
\]

Furthermore since \( J^-(\Delta) = J^+(-\Delta) \)

\[
\min J^-(\Delta) = \min J^+(\Delta), \quad \Delta \geq 0
\]

and this completes the proof of part 2).

B. Derivation of Maximin Solutions

Now, comparing the communication systems of Figs. 1 and 3, we observe that since \( \Gamma_j \supseteq \Gamma \), taking the supremum of both sides over \( \beta \in \Gamma_j \), we obtain the bound

\[
J^*_\beta = \sup_{\beta \in \Gamma_j, \gamma \in \Gamma_{\gamma \beta}, \delta \in \Gamma} J(\gamma, \delta, \beta) \leq \max_{\beta \in \Gamma_j, \gamma \in \Gamma_{\gamma \beta}, \delta \in \Gamma} J(\gamma, \delta, \beta)
\]

where the equality follows because Problem 1 admits a saddle-point solution. This then says that \( J^*_\beta \), the maximin value for Problem 2, is bounded from below by \( J^*_\beta \). In the following we show that this is in fact an equality and the maximin value is precisely \( J^*_\beta \).

Theorem 5: In Problem 2 the maximin value of the

\[
J^*_\beta = \left( k^2 + \phi \right)/(c^2 + k^2 + \phi).
\]

with a corresponding set of maximin solutions being \((\gamma^*, \delta^*, \beta^*)\) and \((-\gamma^*, -\delta^*, -\beta^*)\), where

\[
\gamma^*(u) = cu, \quad \delta^*(z) = [c/(c + k^2 + \phi)] z, \quad \beta^*(\gamma) = \eta
\]

where \( \eta \) is a Gaussian variable with \( E[\eta^2] = 0 \) and \( E[\eta] = k^2 \). (Here, \( \gamma^* \triangleq \gamma^*_\beta, \delta^* \triangleq \delta^*_\beta, \) where \( \gamma^*_\beta, \delta^*_\beta \) are as defined in (1.7).)

Proof: First note the inequality

\[
J^*_\beta = \sup_{\beta \in \Gamma_j, \gamma \in \Gamma_{\gamma \beta}, \delta \in \Gamma} J(\gamma, \delta, \beta) \leq \sup_{\beta \in \Gamma_j, \gamma \in \Gamma_{\gamma \beta}, \delta \in \Gamma} J(\gamma, \delta, \beta),
\]

and that

\[
\min_{\Delta \geq 1/c} \left[ \Delta^2 (\phi + k^2) \right] = \frac{\phi + k^2}{c^2}
\]

(3.14)
which follows because the infimum on the right side is taken over a smaller set. Now, for fixed \( \gamma(u) = au, \delta(z) = \Delta z \), note that

\[
J(\gamma, \delta, \beta) = E\left[ (u - \Delta(au + \beta(u) + w))^2 \right]
\]

\[
= (1 - \Delta^2(\alpha + \beta^2(u)) + 2\Delta E[\beta^2(u)]
\]

and consider minimization of this functional over \((\alpha, \Delta) \in \mathbb{R} \times \mathbb{R}, |a| \leq c\). Let \( p^* \triangleq E[\beta^2(u)], \nu \triangleq E[\mu \beta(u)] \). Then, differentiating \( J \) with respect to \( \Delta \) twice

\[
\frac{\partial}{\partial \Delta} J = 2\Delta(\Delta - 1) + 2\Delta(p + p^2) + 4\nu \Delta > 0
\]

and setting \((\partial/\partial \Delta)J_{\Delta*} = 0\)

\[
\Rightarrow \alpha(\Delta^2(\alpha - 1) + \Delta^2(p + p^2) + (2\Delta^2 - 1)v = 0, \]

we obtain

\[
\Delta^* = \frac{\alpha - v}{\alpha^2 + p^2 + \nu + 2\nu \alpha}
\]  

(3.15)

as the unique minimizing \( \Delta^* \), for fixed \( \alpha \), since the second derivative is positive.

Substituting \( \Delta^* \) back into \( J \), we obtain

\[
J_{\Delta^*} = \frac{p^2 + \nu - v^2}{\alpha^2 + p^2 + \nu + 2\nu \alpha},
\]

and minimization of \( J_{\Delta^*} \) with respect to \( \alpha \) yields

\[
\alpha^* = \begin{cases} 
  c, & \text{if } v > 0 \\
  -c, & \text{if } v < 0 \\
  c \text{ or } -c, & \text{if } v = 0
\end{cases}
\]  

(3.16)

Substituting this solution back into \( J_{\Delta^*, \alpha^*} \), we obtain \( J_{\Delta^*, \alpha^*} \) as follows:

\[
J_{\Delta^*, \alpha^*} = \begin{cases} 
  \frac{p^2 + \nu - v^2}{c^2 + p^2 + \nu - 2cv}, & \text{if } v \leq 0 \\
  \frac{p^2 + \nu - v^2}{c^2 + p^2 + \nu + 2cv}, & \text{if } v \geq 0
\end{cases}
\]  

(3.17)

Note that \( J_{\Delta^*, \alpha^*} \) is an increasing function of \( p^2 \) and hence its maximum over \( p^2 \) subject to \( p^2 \leq k^2 \) is attained at \( p^2 = k^2 \). To further maximize it over \( v \), we first differentiate \( J_{\Delta^*, \alpha^*} \) with respect to \( v \), and we obtain (after substitution of \( p^2 = k^2 \))

\[
\frac{dJ_{\Delta^*, \alpha^*}}{dv} = \begin{cases} 
  \frac{2(v - c)(k^2 + \nu - cv)}{(k^2 + \nu + c - 2cv)^2}, & \text{if } v < 0 \\
  \frac{2(v + c)(k^2 + \nu + cv)}{(k^2 + \nu + c + 2cv)^2}, & \text{if } v > 0
\end{cases}
\]

which shows that \( J_{\Delta^*, \alpha^*}(v) \), which is continuous in \( v \), is decreasing for \( v > 0 \) and increasing for \( v < 0 \), thereby admitting a unique maximum at \( v = 0 \). Hence, \( \arg \max J_{\Delta^*, \alpha^*} = 0 \). Substituting \( v = 0 \) back into (3.15) and (3.16), we obtain

\[
\alpha^* = c
\]

\[
\Delta^* = \frac{c}{c^2 + k^2 + \nu}
\]  

(3.18)

and as above \( \beta^*(u) \) is any second-order random variable \( \eta \) such that

\[
E[\eta u] = 0, \quad E[\eta^2] = k^2.
\]  

Finally, using \( v = 0 \) in (3.16), we arrive at the following expression for the right side of (3.13):

\[
J_{\beta^*} \leq \frac{(c^2 + k^2 + \nu)}{(c^2 + k^2 + \nu)^2} J_{\Delta^*}.
\]

This, together with inequality (3.8), validates the equality

\[
J_{\beta^*} = J_{\Delta^*}.
\]

The remaining statements of the theorem follow from this equality and from expressions (3.18) and (3.19). The fact that \( \eta \) has to be Gaussian follows from the linear structure of \( \gamma^* \) and \( \delta^* \).

We conclude this section with the following counterpart of Theorem 4, which provides a complete characterization of maximin solutions for Problem 2.

**Theorem 6:** In Problem 2 let the encoder and decoder be restricted to linear mappings, \( \gamma(u) = au, \delta(z) = \Delta z, (\alpha, \Delta) \in \mathbb{R} \times \mathbb{R}, |a| \leq c \). Then 1) the class of optimum linear encoder-decoder policies \( \gamma^*, \delta^* \) defined by (1.7) is completely described by

\[
\gamma^*(u) = \begin{cases} 
  cu, & \text{if } E[\mu \beta(u)] > 0 \\
  -cu, & \text{if } E[\mu \beta(u)] < 0 \\
  cu \text{ or } -cu, & \text{if } E[\mu \beta(u)] = 0
\end{cases}
\]  

(3.20)

and 2) the set of all jammer policies that maximize \( J(\gamma^*, \delta^*, \beta) \) over \( \beta \in \Gamma \) is precisely the set of random variables given in Theorem 5.

**Proof:** This result follows basically from the line of argument used in the proof of Theorem 5 and, in particular, from what led to expressions (3.15), (3.16), and (3.19).

**IV. A COMPARATIVE STUDY AND CONCLUDING REMARKS**

The main conclusion to be drawn from the analyses of this paper is that the communication system of Fig. 1 with partially unknown channel statistics admits a saddle-point solution if the encoder structure is allowed to be probabilistic, whereas a saddle-point solution does not exist if the encoder mapping is restricted to be deterministic. This latter property follows from the results of Theorems 3 and 5, which indicate that for Problem 2 the minimax value \( J_{\beta^*} \) is strictly greater than the maximin value \( J_{\alpha^*} \), thus ruling out the possibility of existence of a saddle-point solution for Problem 2. Another important property is that the minimax value for Problem 2 coincides with the saddle-point value of Problem 3, and the maximin value of...
TABLE I

SUMMARY OF SOLUTIONS TO PROBLEMS 1, 2, AND 3

<table>
<thead>
<tr>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Problem 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saddle-Point Solution</td>
<td>Minimax Solution</td>
<td>Saddle-Point Solution</td>
</tr>
<tr>
<td>$(\gamma^<em>, \delta^</em>, \beta^*)$</td>
<td>$(\gamma^<em>, \delta^</em>, \beta^<em>) &amp; (-\gamma^</em>, -\delta^<em>, -\beta^</em>)$</td>
<td>$(\gamma^<em>, \delta^</em>, \beta^<em>) &amp; (-\gamma^</em>, -\delta^<em>, -\beta^</em>)$</td>
</tr>
</tbody>
</table>
| $(\gamma^*, \delta^*) = \left\{ \begin{array}{l} \gamma^* = \frac{cu}{c^2 + k^2 + \psi} \\
\delta^* = \frac{cz}{c^2 + k^2 + \psi} \end{array} \right.$ | $\gamma^*(u) = cu$ | $\gamma^*(u) = cu$ |
| $R_1: (k \geq c)$ | $\delta^*(x) = \frac{c}{c^2 + k^2 + \psi}$ | $\delta^*(x) = \frac{c}{c^2 + k^2 + \psi}$ |
| $\gamma^*(u) = \text{arbitrary}$ | $\delta^*(z) = 0$ | $\delta^*(z) = 0$ |
| $\beta^*(u) = \eta - N(0, k^2)$ | $\beta^*(x) = -x$ | $\beta^*(x) = -x$ |
| $J_\beta^* = 1$ | independent of $u$ and $w$ | independent of $u$ and $w$ |

Problem 2 coincides with the saddle-point value of Problem 1. All these results have been collected together and displayed in Table I. Another important observation that can be made at this point is that the least-favorable distribution for the jamming noise is Gaussian, which is correlated with the message in Problem 3, whereas it is uncorrelated with $u$ in Problem 1.

Extensions of the results of this paper to vector channels can be found in [2]. A counterpart of these results could also be obtained (though not immediately) for continuous-time channels with incomplete statistical description. This is a topic currently under study.

REFERENCES