A Complete Characterization of Minimax and Maximin Encoder-Decoder Policies for Communication Channels with Incomplete Statistical Description

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Abstract—The problem is considered of transmitting a sequence of independent and identically distributed Gaussian random variables over a channel whose statistical description is incomplete. The channel is modeled as one that is conditionally Gaussian, with the unknown part being controlled by a so-called "jammer" who may have access to the input to the encoder and operates under a given power constraint. By adopting a game-theoretic approach, a complete set of solutions is obtained (encoder and decoder mappings, and least-favorable distributions for the channel noise) for this statistical decision problem, under two different sets of conditions, depending on whether the encoder mapping is deterministic or stochastic. In the latter case, existence of a mixed saddle-point solution can be verified when a side channel of a specific nature is available between the transmitter and the receiver. In the former case, however, only minimax and maximin solutions can be derived.

I. Introduction and Problem Formulation

We consider complete characterizations for the saddle-point or minimax and maximin solutions to a class of communication problems that involve the transmission of a sequence of Gaussian random variables over a noisy channel under a given power constraint and in the presence of an intelligent jammer. In a related earlier work [1] a similar formulation was considered, but the jamming noise was allowed to be correlated with the output of the encoder. Here, however, we consider the case where the jamming noise is, instead, allowed to be correlated with the input to the encoder, and when the encoder mapping is either probabilistic (Problem 1) or deterministic (Problem 2) (see Section I-A). In the former we also allow for a side channel that allows transmission of information regarding the structure of the probabilistic encoder to the receiver, whereas in the latter such a forward channel is not necessary. In this framework we show that Problem 1 admits a mixed saddle point, with the least-favorable distribution for the jamming noise being Gaussian but uncorrelated with the message to be transmitted. This should be contrasted with the result of [1], where the saddle point was one point (deterministic) and the Gaussian least-favorable distribution for the jamming noise correlated with the output of the encoder. For Problem 2, however, there exists no saddle-point solution, and only minimax and maximin solutions can be obtained, as we will discuss.

In Section I-A we provide complete descriptions for these two types of problems, and in Section I-B we briefly discuss relevant results from [1] that will be used in the derivation of some tight bounds and also for comparison purposes. Section II provides the saddle-point solution for Problem 1, whereas Sections III-A and B present the derivation of minimax and maximin solutions for Problem 2.

Finally, Section IV presents a comparative discussion on the solutions of Problems 1 and 2 as well as the solution of [1]; these results are tabulated in Table I. Section IV also includes some concluding remarks.

A. A General Description

The general class of jamming problems to be treated in this paper admits two versions, to be called Problems 1 and 2, depending on whether the encoder mapping is probabilistic or not.

Problem 1: Consider the communication system depicted in Fig. 1. The input signal u is a Gaussian random variable[1] with mean zero and variance unity. The transmitter encodes the input signal u into a variable x, with the encoding policy γ being an element of the space Γ_e of random mappings satisfying the power constraint $E\{[\gamma(u)]^2\} \leq c^2$. The main communication channel is additive and memoryless, with the transmitted message being corrupted by jamming noise y, and Gaussian noise w, where the latter has mean zero and variance $\varphi > 0$. By adopting a worst-case analysis, we take the jamming noise to be (possibly) correlated with u, but independent of w, and we let $y = \beta(u)$, where the "jammer policy" β is chosen out of the space Γ_j of random mappings satisfying

[1] This could also be taken to be a sequence of independent and identically distributed Gaussian random variables and the channel used independently for each transmission, with the components of the sequence handled one by one.

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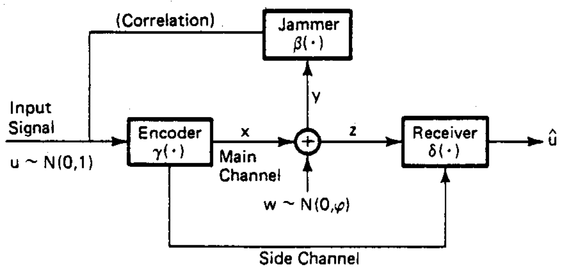

Fig. 1. Communication system of Problem 1 (γ, β and δ allowed to be mixed policies).

the power constraint $E\{[\beta(u)]^2\} \le k^2$. Hence the channel output to the receiver is

$$z = x + y + w = \gamma(u) + \beta(u) + w. \tag{1.1}$$

At the receiver the decoder $\delta(\cdot)$ is chosen out of the space Γ_r of random mappings from $\mathbb{R}$ into $\mathbb{R}$ so as to obtain an estimate of the input signal u based on the measurement z, under a mean-squared error criterion. In accomplishing this, the receiver will also have access to the structure of the encoder mapping (but not its realized value) via the forward side channel depicted in the figure. Hence, with the mean-squared error denoted by

$$J(\gamma, \delta, \beta) = E\{[u - \delta(z)]^2\}, \tag{1.2}$$

where z is given by (1.1) and E denotes the expectation over statistics of u, w, γ, δ, and β, we assume that the pair $(\gamma, \delta) \in \Gamma_e \times \Gamma_r$ will be chosen so as to minimize this quantity, while $\beta \in \Gamma_j$ is chosen to maximize the same quantity. What is sought, then, (if it exists) is a *saddle-point solution* $(\gamma^*, \delta^*, \beta^*) \in \Gamma_e \times \Gamma_r \times \Gamma_j$ satisfying

$$J(\gamma^*, \delta^*, \beta) \le J(\gamma^*, \delta^*, \beta^*) \le J(\gamma, \delta, \beta^*),$$
$$\text{for all } (\gamma, \delta, \beta) \in \Gamma_e \times \Gamma_r \times \Gamma_j. \tag{1.3}$$

Problem 2: The communication system of Problem 2 is depicted in Fig. 2. The only difference between this problem and Problem 1 is that here the encoder mapping is restricted to be deterministic, in which case the side channel would be superfluous. As it will be shown later (in Section III), this problem does not admit a saddle-point solution; hence, we will be interested in the derivation of minimax and maximin solutions. Towards this end, let $\Gamma_{ed} \subset \Gamma_e$ be the space of all (deterministic) mappings γ: $\mathbb{R} \to \mathbb{R}$ for which $E\{[\gamma(u)]^2\}$ is a well-defined quantity and is bounded from above by c^2. Then, under the minimax approach, we evaluate the upper value of the zero-sum game with kernel J,

$$\bar{J}_B^* = J(\gamma^*, \delta^*, \beta^*_{(\gamma^*, \delta^*)}) = \min_{\substack{\gamma \in \Gamma_{ed} \\ \delta \in \Gamma_r}} \max_{\beta \in \Gamma_j} J(\gamma, \delta, \beta)$$
$$= \min_{\substack{\gamma \in \Gamma_{ed} \\ \delta \in \Gamma_r}} J(\gamma, \delta, \beta^*_{(\gamma, \delta)}), \tag{1.4}$$

where $\beta^*_{(\gamma, \delta)}$ is a mapping from $\Gamma_{ed} \times \Gamma_r$ into Γ_j, given by

$$\beta^*_{(\gamma, \delta)} = \arg\max_{\beta \in \Gamma_j} J(\gamma, \delta, \beta), \tag{1.5}$$

assuming that such a solution exists. The triple $(\gamma^*, \delta^*, \beta^*_{(\gamma^*, \delta^*)})$ as determined above is called the *minimax solution* for the communication system of Problem 2.

To obtain the maximin solution, we consider instead the lower value of the zero-sum game,

$$J_B^* = J(\gamma^*_{\beta^*}, \delta^*_{\beta^*}, \beta^*) = \max_{\beta \in \Gamma_j} \min_{\substack{\gamma \in \Gamma_{ed} \\ \delta \in \Gamma_r}} J(\gamma, \delta, \beta)$$
$$= \max_{\beta \in \Gamma_j} J(\gamma^*_\beta, \delta^*_\beta, \beta), \tag{1.6}$$

where the pair $(\gamma^*_\beta, \delta^*_\beta)$ is a mapping from Γ_j into $\Gamma_{ed} \times \Gamma_r$, determined by

$$(\gamma^*_\beta, \delta^*_\beta) = \arg\min_{(\gamma, \delta) \in \Gamma_{ed} \times \Gamma_r} J(\gamma, \delta, \beta). \tag{1.7}$$

The triple $(\gamma^*_\beta *, \delta^*_\beta *, \beta^*)$ defined above by (1.6) and (1.7) is the *maximin solution* for the communication system of Problem 2. A saddle-point solution $(\gamma^*, \delta^*, \beta^*) \in \Gamma_{ed} \times \Gamma_r \times \Gamma_j$ will exist if and only if

$$J_B^* = \bar{J}_B^* = J(\gamma^*, \delta^*, \beta^*),$$

which, however, is not the case in Problem 2, as it will be shown in Section III.

B. A Summary of Relevant Results from [1]

In the context of the general formulation of Section I-A, consider the system depicted in Fig. 3, where the jammer now has access to the output of the encoder. Hence, here (1.1) is replaced by

$$z = x + y + w = \gamma(u) + \beta(x) + w, \tag{1.8}$$

which constitutes the only difference between this system and that of Fig. 2. The result proven in [1] in a more general context is that a saddle point exists for this system and it has different characterizations in different regions of the parameter space, determined by the relative magnitudes of k, c, and φ. To present this result in terms of the

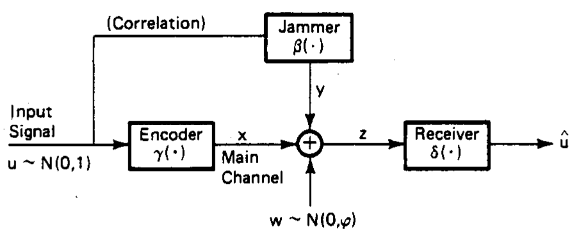

Fig. 2. Communication system of Problem 2 (γ restricted to be a deterministic mapping).

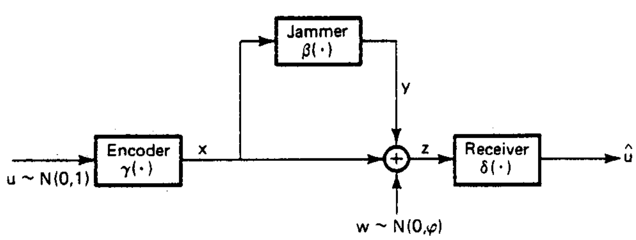

Fig. 3. Problem 3: simplified version of communication system treated in [1], which would be relevant to systems considered in this paper (i.e. Problems 1 and 2).

notation of Section I-A, let us first introduce the regions

$$\begin{aligned} &R_1: k \geq c \\ &R_2: k < c \\ &R_3: k^2 - ck + \varphi > 0 \\ &R_4: k^2 - ck + \varphi \leq 0. \end{aligned} \tag{1.9}$$

Then we have the following theorem.

Theorem 1 [1]: The communication system of Fig. 3 (to be referred to as Problem 3) admits two saddle-point solutions $(\gamma^*, \delta^*, \beta^*)$ and $(-\gamma^*, -\delta^*, \beta^*)$ over the space $\Gamma_e \times \Gamma_r \times \Gamma_j$, where

$$\gamma^*(u) = \begin{cases} \text{arbitrary}, & \text{in } R_1 \\ cu, & \text{in } R_2 \end{cases} \tag{1.10}$$

$$\beta^*(x) = \begin{cases} -x, & \text{in } R_1 \\ -(k/c)x, & \text{in } R_2 \cap R_3 \\ -\left[\dfrac{k^2+\varphi}{c^2}\right]x + \eta, & \text{in } R_2 \cap R_4 \end{cases} \tag{1.11}$$

where η is a zero-mean Gaussian random variable with variance $k^2 - ((k^2+\varphi)^2/c^2)$ and is independent of x and w. Furthermore,

$$\delta^*(z) = \begin{cases} 0, & \text{in } R_1 \\ \left[\dfrac{c-k}{(c-k)^2+\varphi}\right]z, & \text{in } R_2 \cap R_3. \\ (1/c)z, & \text{in } R_2 \cap R_4 \end{cases} \tag{1.12}$$

The optimal (saddle-point) value of J is

$$J_C^*(\gamma^*, \delta^*, \beta^*) = \begin{cases} 1, & \text{in } R_1 \\ \dfrac{\varphi}{(c-k)^2+\varphi}, & \text{in } R_2 \cap R_3 \\ \dfrac{k^2+\varphi}{c^2}, & \text{in } R_2 \cap R_4 \end{cases} \tag{1.13}$$

II. The Saddle-Point Solution For Problem 1

Before presenting the solution to Problem 1, we digress to clarify the role of the side channel, which provides a forward communication link between the transmitter and the receiver. If the optimum encoder policy is deterministic (i.e., an element of Γ_{ed}), the side channel becomes superfluous and plays no role in the problem. However, if the optimum encoder policy is probabilistic, then the transmitter "mixes" between two or more elements out of Γ_{ed} according to a specific probability distribution. If this is the case, then we assume that the transmitter uses the side channel to inform the receiver of the actual outcome of the corresponding chance mechanism during each transmission of the message.[2] With this stipulation, we now present, in Theorem 2 below, the main result concerning Problem 1. In the statement of the theorem, the expression "(γ_1, δ_1) with probability (w.p.) p_1" means that both the encoding policy $\gamma_1 \in \Gamma_{ed}$ and decoding policy $\delta_1 \in \Gamma_r$ are used simultaneously with probability p_1. (This is, of course, possible because of the presence of the side channel.)

[2]As one of the referees has pointed out, since the information to be transmitted over the side channel is independent of the message, it could be viewed as information generated at a third point and fed separately to the transmitter and the receiver. Hence, it is not necessary that a side channel exist between the transmitter and the receiver, carrying information from the former to the latter. With this interpretation, the situation becomes reminiscent of that prevailing in practical anti-jamming systems (see, e.g. [3]).

Theorem 2: For the communication system of Fig. 1, there exists a saddle-point solution $(\gamma^*, \delta^*, \beta^*)$, given by (2.1)–(2.3):

$$(\gamma^*, \delta^*) = \begin{cases} \left(cu, \dfrac{c}{c^2+k^2+\varphi}z\right), & \text{w.p. } 0.5 \\ \left(-cu, -\dfrac{c}{c^2+k^2+\varphi}z\right), & \text{w.p. } 0.5 \end{cases} \tag{2.1}$$

$$\beta^*(u) = \eta, \tag{2.2}$$

where η is a zero-mean Gaussian random variable with variance k^2; it is independent of both u and w; and the saddle-point value of J is

$$J_A^* = \frac{k^2+\varphi}{c^2+k^2+\varphi}. \tag{2.3}$$

Proof: We need to prove that the solution given above satisfies the pair of saddle-point inequalities (1.3) for all $(\gamma, \delta, \beta) \in \Gamma_e \times \Gamma_r \times \Gamma_j$ and under the stipulation that the side channel is used to carry structural information concerning probabilistic encoder mappings. The proof will be completed in two steps.

1) Verification of the Right Side Inequality of (1.3): Suppose that the jammer's policy is as given by (2.2). Then, the communication system of Fig. 1 (Problem 1) becomes the standard Gaussian test channel [4], for which the best encoding policy is known to be either $\gamma(u) = cu$ or $\gamma(u) = -cu$, with the corresponding decoder structures being $E[u|z] = [c/(c^2+k^2+\varphi)]z$ or $-[c/(c^2+k^2+\varphi)]z$, respectively. Both these (deterministic) policies lead to the same distortion level $J(\gamma^*, \delta^*, \beta^*) = J_A^*$, and so does the mixed policy given in (2.1).

2) Verification of the Left Side Inequality: With the pair of encoding and decoding policies (γ^*, δ^*) as given in (2.1), we first compute the mean-squared error $\hat{J}_i(\gamma^*, \delta^*, \beta)$ conditioned on the specific structural realization of encoding-decoding policies in current use.

a) When $(cu, cz/(c^2+k^2+\varphi)$ is the realized encoder-decoder policy pair, we have

$$\begin{aligned} \hat{J}_1 &= E\left\{\left(\frac{cz}{c^2+k^2+\varphi} - u\right)^2\right\} \\ &= E\left\{\left[\frac{c}{c^2+k^2+\varphi}(cu+y+w) - u\right]^2\right\} \\ &= \frac{(k^2+\varphi)^2}{(c^2+k^2+\varphi)^2} + \frac{c^2}{(c^2+k^2+\varphi)^2}E(y^2) \\ &\quad + \frac{c^2}{(c^2+k^2+\varphi)^2}\varphi - \frac{2c(k^2+\varphi)}{(c^2+k^2+\varphi)^2}E(uy). \end{aligned} \tag{2.4}$$

b) Now take $\gamma(u) = -cu$, and $\delta^* = -cz/(c^2 + k^2 + \varphi)$ to obtain

$$\hat{J}_2 = E\left\{\left(-\frac{cz}{c^2+k^2+\varphi} - u\right)^2\right\}$$
$$= E\left\{\left[-\frac{c}{c^2+k^2+\varphi}(-cu+y+w) - u\right]^2\right\}$$
$$= \frac{(k^2+\varphi)^2}{(c^2+k^2+\varphi)^2} + \frac{c^2}{(c^2+k^2+\varphi)^2}E(y^2) + \frac{c^2}{(c^2+k^2+\varphi)^2}\varphi + \frac{2c(k^2+\varphi)}{(c^2+k^2+\varphi)^2}E(uy). \tag{2.5}$$

By unconditioning the above conditional values of J, we obtain

$$J = \frac{1}{2}\hat{J}_1 + \frac{1}{2}\hat{J}_2 = \frac{(k^2+\varphi)^2}{(c^2+k^2+\varphi)^2} + \frac{c^2}{(c^2+k^2+\varphi)^2}E(y^2) + \frac{c^2}{(c^2+k^2+\varphi)^2}\varphi, \tag{2.6}$$

which indicates that J depends only on the second moment of y. Hence, the maximizing solution is any random variable with second moment equal to k^2; the Gaussian random variable η, with mean zero and variance k^2, is one such random variable.

Hence

$$J^* = \frac{(k^2+\varphi)^2}{(c^2+k^2+\varphi)^2} + \frac{c^2k^2}{(c^2+k^2+\varphi)^2} + \frac{c^2\varphi}{(c^2+k^2+\varphi)^2} = \frac{k^2+\varphi}{c^2+k^2+\varphi} = J_A^*.$$

This then completes the proof of Theorem 2.

Corollary 1: The solution (γ^*, δ^*) given in Theorem 2 is the almost surely (a.s.) *unique* optimal solution[3] for the transmitter-receiver. For the jammer any optimal solution $\tilde{\beta} \in \Gamma_j$ has the property that the random variable $y = \tilde{\beta}(u)$ is uncorrelated with the message, i.e., $E\{uy\} = 0$.

Proof:

1) Because of the interchangeability property of saddle-point equilibria, every optimal solution for the encoder-decoder should be in equilibrium with β^* given by (2.2). However, for $\beta = \beta^*$, we have the standard Gaussian test channel whose entire class of optimum solutions in $\Gamma_e \times \Gamma_r$ is described by

$$(\hat{\gamma}, \hat{\delta}) = \begin{cases} \left(cu, \dfrac{c}{c^2+k^2+\varphi}z\right), & \text{w.p. } q \\ \left(-cu, -\dfrac{c}{c^2+k^2+\varphi}z\right), & \text{w.p. } 1-q \end{cases}, \tag{2.7}$$

with $q \in [0,1]$. Substituting (2.7) into J for a fixed q, we obtain

$$J(\hat{\gamma}, \hat{\delta}, \beta) = \frac{(k^2+\varphi)^2}{(c^2+k^2+\varphi)^2} + \frac{c^2\varphi}{(c^2+k^2+\varphi)^2} + \frac{c^2}{(c^2+k^2+\varphi)^2}E\{y^2\} - \frac{2c(k^2+\varphi)}{(c^2+k^2+\varphi)^2}(1-2q)E\{uy\}. \tag{2.8}$$

If $q \neq 0.5$ the jammer could choose $\beta(u) = \hat{\beta}(u) = \text{sgn}(2q-1)ku$, which leads to

$$J(\hat{\gamma}, \hat{\delta}, \hat{\beta}) > J_A^*.$$

Hence (2.7) provides an optimal solution only if $q = 0.5$.

2) To prove the second part of the corollary it is sufficient to observe that if β is chosen such that $E[uy] \neq 0$, then one can choose an encoder-decoder policy pair as given in (2.7), with $q = 0$ if $E\{uy\} > 0$ and $q = 1$ if $E\{uy\} < 0$, leading in each case to a value of J (see (2.8)) that is strictly smaller than J_A^*. This clearly shows that a $\beta \in \Gamma_j$ with $E[u\beta(u)] \neq 0$ cannot be an optimal solution for the jammer.

III. Derivation of Minimax and Maximin Policies For Problem 2

We now restrict the encoder policy to be a deterministic mapping, thereby eliminating the need to use a side channel between the transmitter and the receiver. The main result to be obtained below is that with this restriction the problem (Problem 2) does not admit a saddle-point solution because

$$J_A^* = J_B^* < \bar{J}_B^* = J_C^*.$$

Furthermore we provide a complete characterization of a set of minimax and maximin solutions.

A. Derivation of Minimax Solutions

Comparing the communication systems of Figs. 2 and 3, we first observe the following property, which holds if γ is restricted to be a deterministic mapping. For each fixed $\gamma \in \Gamma_{ed}$, to every β that is a function of $x = \gamma(u)$ in Problem 3 (Fig. 3) corresponds a jammer policy $\tilde{\beta} = \beta \circ \gamma$ in Problem 2, which represents the same random variable v $(= \beta(\gamma(u)) = \tilde{\beta}(u))$. The statement, however, is not necessarily true in the other direction because γ may not be invertible. Hence, we have the inequality

$$\sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(x)) \leq \sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(u))$$

for every $(\gamma, \delta) \in \Gamma_{ed} \times \Gamma_r$, with $x = \gamma(u)$. Now, taking the infimum of both sides over (γ, δ), we obtain

$$J_C^* = \inf_{(\gamma,\delta)\in\Gamma_{ed}\times\Gamma_r} \sup_{\beta\in\Gamma_j} J(\gamma, \delta, \beta(x)) \leq \bar{J}_B^*, \tag{3.1}$$

where the equality on the left side follows from Theorem 1 (since Problem 3 admits a saddle point). This shows that

[3] Henceforth we will use the terminology "optimal" to refer to individual components of a pair of policies in saddle-point equilibrium.

the minimax (upper) value for Problem 2 is bounded from below by the saddle-point value of Problem 3. The following theorem now proves that the inequality in (3.1) is in fact an equality, and it also provides a set of minimax policies that achieve this value. The regions $R_1, \cdots, R_4$ used in the theorem are those introduced earlier by (1.9).

Theorem 3: In Problem 2 the minimax value of the mean-squared error at the receiver is

$$\bar{J}_B^* = J_C^* = \begin{cases} 1, & \text{in } R_1 \\ \dfrac{\varphi}{(c-k)^2+\varphi}, & \text{in } R_2 \cap R_3 \\ \dfrac{k^2+\varphi}{c^2}, & \text{in } R_2 \cap R_4 \end{cases} \tag{3.2}$$

with a corresponding set of minimax solutions being $(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*)$ and $(-\bar{\gamma}^*, -\bar{\delta}^*, -\bar{\beta}^*)$, where

$$\bar{\gamma}^*(u) = \begin{cases} \gamma(u), & \text{where } \gamma \in \Gamma_{ed} \text{ is arbitrary; in } R_1 \\ cu, & \text{in } R_2 \end{cases} \tag{3.3}$$

$$\bar{\delta}^*(z) = \begin{cases} 0, & \text{in } R_1 \\ \dfrac{c-k}{(c-k)^2+\varphi} z, & \text{in } R_2 \cap R_3 \\ (1/c)z, & \text{in } R_2 \cap R_4 \end{cases} \tag{3.4}$$

$$\bar{\beta}^*(u) = \begin{cases} \beta(u), & \text{where } \beta \in \Gamma_j \text{ is arbitrary; in } R_1 \\ -ku, & \text{in } R_2 \cap R_3 \\ \beta(u), & \text{where } \beta \in \Gamma_j \text{ satisfies } \|\beta(u)\| = k; \\ & \text{in } R_2 \cap R_4 \end{cases} \tag{3.5}$$

Here, $\|\beta(u)\| \triangleq \{E\{|\beta(u)|^2\}\}^{1/2}$, and $\bar{\beta}^* = \beta^*_{(\bar{\gamma}^*, \bar{\delta}^*)}$, where $\beta^*_{(\gamma,\delta)}$ is defined by (1.5).

Proof: Since

$$\bar{J}_B^* \le \sup_{\beta \in \Gamma_j} J(\gamma, \delta, \beta(u)), \qquad \text{for all } \gamma \in \Gamma_{ed}, \gamma \in \Gamma_r,$$

the proof is completed by showing that with $\bar{\gamma}^*, \bar{\delta}^*$ as given above,

$$\bar{J}_B^* \le \sup_{\beta \in \Gamma_j} J(\bar{\gamma}^*, \bar{\delta}^*, \beta(u)) = J(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*(u)) = J_C^* \tag{3.6}$$

which, in view of inequality (3.1), leads to $\bar{J}_B^* = J_C^*$, and that $(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*)$ is a minimax solution.

To show the validity of the right side equality in (3.6), first consider region $R1$. Here, since

$$J(\gamma^*, \delta^*, \beta) = E\{(u-0)^2\} = 1 = J_C^*, \qquad \text{for all } \beta \in \Gamma_j,$$

the result is clearly valid.

In region $R_2 \cap R_3$, we have

$$\begin{aligned} J(\gamma^*, \delta^*, \beta) &= E\left\{\left[u - \frac{c-k}{(c-k)^2+\varphi}(cu + \beta(u) + w)\right]^2\right\} \\ &= \left[\frac{\varphi + k(k-c)}{\varphi + (c-k)^2}\right]^2 \\ &\quad + 2\frac{\varphi + k(k-c)}{[\varphi + (c-k)^2]^2} E[u\beta(u)](k-c) \\ &\quad + \frac{(c-k)^2}{[(c-k)^2+\varphi]^2} E[\beta(u)]^2 + \frac{(c-k)^2\varphi}{[(c-k)^2+\varphi]^2}. \end{aligned} \tag{3.7}$$

Since the coefficient of $E[u\beta(u)]$ is negative in $R_2 \cap R_3$, the above expression is maximized uniquely by choosing $\beta^*(u) = -ku$, which, when substituted into (3.7), leads to

$$J(\gamma^*, \delta^*, -ku) = \frac{\varphi}{(c-k)^2+\varphi} = J_C^*.$$

Finally, in $R_2 \cap R_4$, we have

$$\begin{aligned} J(\gamma^*, \delta^*, \beta) &= E\left\{\left[u - \frac{1}{c}(cu + \beta(u) + w)\right]^2\right\} \\ &= \frac{1}{c^2} E\{[\beta(u)]^2\} + \frac{\varphi}{c^2}, \end{aligned}$$

which is maximized by choosing $\beta^*(u)$ any random variable with second moment equal to k^2. The corresponding value of J is then

$$\frac{k^2+\varphi}{c^2} = J_C^*.$$

The following theorem now extends the result of Theorem 3 and provides a complete characterization for the minimax encoder-decoder mappings.

Theorem 4: In Problem 2 let the decoder be restricted to linear mappings, $\delta(z) = \Delta z$, $\Delta \in \mathbb{R}$. Then: 1) the class of optimum jammer policies $\beta^*_{(\gamma,\delta)}$ defined by (1.5) is completely described by

$$\beta^*_{(\gamma,\delta)}(u) = \begin{cases} \beta(u), & \text{where } \beta \in \Gamma_j \text{ is arbitrary; if } \Delta = 0 \\ -\dfrac{k}{\|u - \Delta\gamma(u)\|}[u - \Delta\gamma(u)], & \\ & \text{if } \Delta \ne 0 \text{ and } \|u - \Delta\gamma(u)\| \ne 0 \\ \beta(u), & \text{where } \beta \in \Gamma_j \text{ satisfies } \|\beta(u)\| = k, \\ & \text{if } \|u - \Delta\gamma(u)\| = 0. \end{cases} \tag{3.8}$$

and 2) the set of all encoder-decoder policies that minimize $J(\gamma, \delta, \beta^*_{(\gamma,\delta)})$ is precisely the set given in Theorem 3; that is, the minimax solutions $(\pm\gamma^*, \pm\delta^*)$ constitute a complete characterization.

Proof:

1) Note that with $\delta(z)=\Delta z$, $J(\gamma,\delta,\beta)=E\{[u-\Delta[\gamma(u)+\beta(u)+w]]^2\}$, and when $\Delta=0$, J becomes independent of β, thus making every $\beta\in\Gamma_j$ a maximizing solution. If $\Delta\neq 0$

$$J(\gamma,\delta,\beta)=\Delta^2E\left\{\left[\frac{1}{\Delta}[u-\Delta\gamma(u)]-\beta(u)\right]^2\right\}+\Delta^2\varphi, \tag{$*$}$$

and maximizing this over $\beta\in\Gamma_j$, under the condition $\|u-\Delta\gamma(u)\|\neq 0$, we obtain, uniquely,

$$\beta^*_{(\gamma,\delta)}(u)=-\frac{1}{\Delta}[u-\Delta\gamma(u)]k\Big/\left\|\frac{1}{\Delta}[u-\Delta\gamma(u)]\right\|,$$

which verifies (3.7) in the second region. Finally, if $\Delta\neq 0$, but $\|u-\Delta\gamma(u)\|=0$, $(*)$ becomes

$$J(\gamma,\delta,\beta)=\Delta^2\|\beta(u)\|^2+\Delta^2\varphi,$$

with maximum achieved by any $\beta\in\Gamma_j$ using maximum power, i.e., $\|\beta(u)\|^2=k^2$. This completes the proof of part 1).

2) Substituting $\beta^*_{(\gamma,\delta)}$ given by (3.7) in J and minimizing the resulting expression over $\gamma\in\Gamma_{ed}$ for fixed $\Delta\in\mathbb{R}$, we obtain for $\Delta\geq 0$

$$\gamma^+_{(\Delta)}(u)=\begin{cases} cu, & \text{if } \Delta<1/c, \quad \Delta\neq 0\\ (1/\Delta)u, & \text{if } \Delta\geq 1/c, \quad \Delta\neq 0\\ \gamma(u), & \text{where } \gamma\in\Gamma_j \text{ is arbitrary,}\\ & \text{if } \Delta=0\end{cases}$$

and for $\Delta<0$ we have

$$\gamma^-_{(\Delta)}(u)=-\gamma^+_{(\Delta)}(u).$$

Substituting this solution into J, we obtain for $\Delta\geq 0$

$$J\left(\gamma^+_{(\Delta)},\delta,\beta_{(\gamma^+,\delta)}\right)$$
$$\equiv J^+(\Delta)=\begin{cases}[1+(k-c)\Delta]^2+\Delta^2\varphi, & 0\leq\Delta<1/c\\ \Delta^2(\varphi+k^2), & \Delta\geq 1/c\end{cases}$$

and for $\Delta\leq 0$

$$J\left(\gamma^-_{(\Delta)},\delta,\beta_{(\gamma^-,\delta)}\right)\equiv J^-(\Delta)=J^+(-\Delta).$$

It can easily be shown that

$$\min_{0\leq\Delta\leq 1/c}\left[[1+(k-c)\Delta]^2+\Delta^2\varphi\right]$$
$$=\begin{cases}\dfrac{\varphi}{(k-c)^2+\varphi}, & \text{in } R_2\cap R_3\\ \dfrac{k^2+\varphi}{c^2}, & \text{in } R_2\cap R_4\\ 1, & \text{in } R_1\end{cases}$$

and that

$$\min_{\Delta\geq 1/c}\left[\Delta^2(\varphi+k^2)\right]=\frac{\varphi+k^2}{c^2}$$

with the minimizing arguments being, respectively,

$$\Delta=\begin{cases}\dfrac{c-k}{(k-c)^2+\varphi}, & \text{in } R_2\cap R_3\\ 1/c, & \text{in } R_2\cap R_4\\ 0, & \text{in } R_1\end{cases}$$

$$\Delta=1/c.$$

Hence

$$\min_{\Delta\geq 0}J^+(\Delta)=\begin{cases}1, & \text{in } R_1\\ \dfrac{\varphi}{(k-c)^2+\varphi}, & \text{in } R_2\cap R_3\\ \dfrac{k^2+\varphi}{c^2}, & \text{in } R_2\cap R_4\end{cases}.$$

Furthermore since $J^-(\Delta)=J^+(-\Delta)$

$$\min_{\Delta\leq 0}J^-(\Delta)=\min_{\Delta\geq 0}J^+(\Delta),$$

and this completes the proof of part 2).

B. Derivation of Maximin Solutions

Now, comparing the communication systems of Figs. 1 and 3, we observe that since $\Gamma_e\supset\Gamma_{ed}$

$$\inf_{\gamma\in\Gamma_e,\delta\in\Gamma_r}J(\gamma,\delta,\beta)\leq\inf_{\gamma\in\Gamma_{ed},\delta\in\Gamma_r}J(\gamma,\delta,\beta)$$

for all $\beta\in\Gamma_j$. Taking the supremum of both sides over $\beta\in\Gamma_j$, we obtain the bound

$$J_A^*=\sup_{\beta\in\Gamma_j}\inf_{\gamma\in\Gamma_e,\delta\in\Gamma_r}J(\gamma,\delta,\beta)\leq J_B^*, \tag{3.9}$$

where the equality follows because Problem 1 admits a saddle-point solution. This then says that J_B^*, the maximin value for Problem 2, is bounded from below by J_A^*. In the following we show that this is in fact an equality and the maximin value is precisely J_A^*.

Theorem 5: In Problem 2 the maximin value of the mean-squared error at the receiver is

$$J_B^*=J_A^*=(k^2+\varphi)/(c^2+k^2+\varphi), \tag{3.10}$$

with a corresponding set of maximin solutions being $(\boldsymbol{\gamma}^*,\boldsymbol{\delta}^*,\boldsymbol{\beta}^*)$ and $(-\boldsymbol{\gamma}^*,-\boldsymbol{\delta}^*,-\boldsymbol{\beta}^*)$, where

$$\boldsymbol{\gamma}^*(u)=cu \tag{3.11}$$

$$\boldsymbol{\delta}^*(z)=\left[c/(c^2+k^2+\varphi)\right]z \tag{3.12}$$

$$\boldsymbol{\beta}^*(u)=\eta, \tag{3.13}$$

where η is a Gaussian variable with $E[u\eta]=0$ and $E[\eta^2]=k^2$. (Here, $\boldsymbol{\gamma}^*\triangleq\gamma^*_{\boldsymbol{\beta}^*}$, $\boldsymbol{\delta}^*\triangleq\delta^*_{\boldsymbol{\beta}^*}$, where $\gamma^*_\beta,\delta^*_\beta$ are as defined in (1.7).)

Proof: First note the inequality

$$J_B^*\triangleq\sup_{\beta\in\Gamma_j}\inf_{\substack{\gamma\in\Gamma_{ed}\\ \delta\in\Gamma_r}}J(\gamma,\delta,\beta)\leq\sup_{\beta\in\Gamma_j}\inf_{\substack{\alpha,\Delta\in\mathbb{R}\\ \gamma(u)=\alpha u\\ \delta(z)=\Delta z\\ |\alpha|\leq c}}J(\gamma,\delta,\beta), \tag{3.14}$$

which follows because the infimum on the right side is taken over a smaller set. Now, for fixed $\gamma(u) = \alpha u, \delta(z) = \Delta z$, note that

$$\begin{aligned} J(\gamma, \delta, \beta) &= E\{[u - \Delta(\alpha u + \beta(u) + w)]^2\} \\ &= (1 - \Delta\alpha)^2 + \Delta^2\varphi + \Delta^2 E[\beta^2(u)] \\ &\quad + 2\Delta(\Delta\alpha - 1)E[u\beta(u)] \end{aligned}$$

and consider minimization of this functional over $(\alpha, \Delta) \in \mathbb{R} \times \mathbb{R}$, $|\alpha| \leq c$. Let $p^2 \triangleq E[\beta^2(u)], v \triangleq E[u\beta(u)]$. Then, differentiating J with respect to Δ twice

$$\frac{\partial}{\partial\Delta}J = 2\alpha(\Delta\alpha - 1) + 2\Delta(\varphi + p^2) + 2(2\alpha\Delta - 1)v$$

$$\frac{\partial^2}{\partial\Delta^2}J = 2\alpha^2 + 2\varphi + 2p^2 + 4\alpha v > 0$$

and setting $(\partial/\partial\Delta)J|_{\Delta^*} = 0$

$$\Rightarrow \alpha(\Delta^*\alpha - 1) + \Delta^*(\varphi + p^2) + (2\alpha\Delta^* - 1)v = 0,$$

we obtain

$$\Delta^* = \frac{\alpha + v}{\alpha^2 + p^2 + \varphi + 2\alpha v} \tag{3.15}$$

as the unique minimizing Δ^*, for fixed α, since the second derivative is positive.

Substituting Δ^* back into J, we obtain

$$J|_{\Delta^*} = \frac{p^2 + \varphi - v^2}{\alpha^2 + p^2 + \varphi + 2\alpha v},$$

and minimization of $J|_{\Delta^*}$ with respect to α yields

$$\alpha^* = \begin{cases} c, & \text{if } v > 0 \\ -c, & \text{if } v < 0. \\ c \text{ or } -c, & \text{if } v = 0 \end{cases} \tag{3.16}$$

Substituting this solution back into $J|_{\Delta^*}$, we obtain $J|_{\Delta^*, \alpha^*}$ as follows:

$$J|_{\Delta^*,\alpha^*} = \begin{cases} \dfrac{p^2 + \varphi - v^2}{c^2 + p^2 + \varphi - 2cv}, & \text{if } v \leq 0 \\ \dfrac{p^2 + \varphi - v^2}{c^2 + p^2 + \varphi + 2cv}, & \text{if } v \geq 0 \end{cases} \tag{3.17}$$

Note that $J|_{\Delta^*,\alpha^*}$ is an increasing function of p^2 and hence its maximum over p^2 subject to $p^2 \leq k^2$ is attained at $p^2 = k^2$. To further maximize it over v, we first differentiate $J|_{\Delta^*,\alpha^*}$ with respect to v, and we obtain (after substitution of $p^2 = k^2$)

$$\frac{dJ|_{\Delta^*,\alpha^*}}{dv} = \begin{cases} -\dfrac{2(v - c)(k^2 + \varphi - cv)}{(k^2 + \varphi + c^2 - 2cv)^2} > 0, & \text{if } v < 0 \\ -\dfrac{2(v + c)(k^2 + \varphi + cv)}{(k^2 + \varphi + c^2 + 2cv)} < 0, & \text{if } v > 0 \end{cases}$$

which shows that $J|_{\Delta^*,\alpha^*}(v)$, which is continuous in v, is decreasing for $v > 0$ and increasing for $v < 0$, thereby admitting a unique maximum at $v = 0$. Hence, $\arg(\max J|_{\Delta^*,\alpha^*}) = 0$. Substituting $v = 0$ back into (3.15) and (3.16), we obtain

$$\alpha^* = c$$
$$\Delta^* = \frac{c}{c^2 + k^2 + \varphi} \tag{3.18}$$

and as shown above $\beta^*(u)$ is any second-order random variable η such that

$$E(u\eta) = 0 \qquad E(\eta^2) = k^2. \tag{3.19}$$

Finally, using $v = 0$ in (3.16), we arrive at the following expression for the right side of (3.13):

$$\boldsymbol{J}_B^* \leq (k^2 + \varphi)/(c^2 + k^2 + \varphi) \equiv J_A^*.$$

This, together with inequality (3.8), validates the equality

$$\boldsymbol{J}_B^* = J_A^*.$$

The remaining statements of the theorem follow from this equality and from expressions (3.18) and (3.19). The fact that η has to be Gaussian follows from the linear structure of $\boldsymbol{\gamma}^*$ and $\boldsymbol{\delta}^*$.

We conclude this section with the following counterpart of Theorem 4, which provides a complete characterization of maximin solutions for Problem 2.

Theorem 6: In Problem 2 let the encoder and decoder be restricted to linear mappings, $\gamma(u) = \alpha u, \delta(z) = \Delta z, \alpha, \Delta \in \mathbb{R}, |\alpha| \leq c$. Then 1) the class of optimum linear encoder-decoder policies $\gamma_\beta^*, \delta_\beta^*$ defined by (1.7) is completely described by

$$\gamma_\beta^*(u) = \begin{cases} cu, & \text{if } E[u\beta(u)] > 0 \\ -cu, & \text{if } E[u\beta(u)] < 0 \\ cu \text{ or } -cu, & \text{if } E[u\beta(u)] = 0 \end{cases} \tag{3.20}$$

$$\delta_\beta^*(z) = \frac{E[u\gamma_\beta^*(u)] + E[u\beta(u)]}{\varphi + \|\gamma_\beta^*(u) + \beta(u)\|^2} z, \tag{3.21}$$

and 2) the set of all jammer policies that maximize $J(\gamma_\beta^*, \delta_\beta^*, \beta)$ over $\beta \in \Gamma_j$ is precisely the set of random variables given in Theorem 5.

Proof: This result follows basically from the line of argument used in the proof of Theorem 5 and, in particular, from what led to expressions (3.15), (3.16), and (3.19).

IV. A Comparative Study and Concluding Remarks

The main conclusion to be drawn from the analyses of this paper is that the communication system of Fig. 1 with partially unknown channel statistics admits a saddle-point solution if the encoder structure is allowed to be probabilistic, whereas a saddle-point solution does not exist if the encoder mapping is restricted to be deterministic. This latter property follows from the results of Theorems 3 and 5, which indicate that for Problem 2 the minimax value $\bar{J}_B^*$ is strictly greater than the maximin value $\boldsymbol{J}_B^*$, thus ruling out the possibility of existence of a saddle-point solution for Problem 2. Another important property is that the minimax value for Problem 2 coincides with the saddle-point value of Problem 3, and the maximin value of

TABLE I
SUMMARY OF SOLUTIONS TO PROBLEMS 1, 2, AND 3

Problem 1	Problem 2		Problem 3
Saddle-Point Solution	Minimax Solution	Maximin Solution	Saddle-Point Solution
$(\gamma^*, \delta^*, \beta^*)$	$(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*) \& (-\bar{\gamma}^*, -\bar{\delta}^*, -\bar{\beta}^*)$	$(\gamma^*, \delta^*, \beta^*) \& (-\gamma^*, -\delta^*, -\beta^*)$	$(\gamma^*, \delta^*, \beta^*) \& (-\gamma^*, -\delta^*, \beta^*)$
$(\gamma^*, \delta^*) = \begin{cases} \left(cu, \dfrac{cz}{c^2 + k^2 + \varphi}\right) & \text{w.p. } 0.5 \\ \left(-cu, -\dfrac{cz}{c^2 + k^2 + \varphi}\right) & \text{w.p. } 0.5 \end{cases}$	$R_1 : (k \geq c)$ $\bar{\gamma}^* =$ arbitrary $\bar{\beta}^* =$ arbitrary $\bar{\delta}^*(z) = 0$ $\bar{J}_B^* = 1$	$\gamma^*(u) = cu$ $\delta^*(z) = \dfrac{c}{c^2 + k^2 + \varphi} z$ $\beta^*(u) = \eta \sim N(0, k^2)$ independent of u and w	R_1: $\gamma^* =$ arbitrary $\delta^*(z) = 0$ $\beta^*(x) = -x$ $J_c^* = 1$
$\beta^*(u) = \eta \sim N(0, k^2)$ independent of u and w $J_A^* = \dfrac{k^2 + \varphi}{k^2 + \varphi + c^2}$	$R_2 \cap R_3 : (k < c \,\&\, k^2 - ck + \varphi > 0)$ $\bar{\gamma}^*(u) = cu$ $\bar{\delta}^*(z) = \dfrac{c - k}{(c - k)^2 + \varphi} z$ $\bar{\beta}^*(u) = -ku$ $\bar{J}_B^* = \dfrac{\varphi}{(c - k)^2 + \varphi}$	$J_B^* = \dfrac{k^2 + \varphi}{k^2 + \varphi + c^2}$	$R_2 \cap R_3$: $\gamma^*(u) = cu$ $\delta^*(z) = \dfrac{c - k}{(c - k)^2 + \varphi} z$ $\beta^*(x) = -\dfrac{k}{c} x$ $J_c^* = \dfrac{\varphi}{(c - k)^2 + \varphi}$
	$R_2 \cap R_4 : (k < c \,\&\, k^2 - ck + \varphi \leq 0)$ $\bar{\gamma}^*(u) = cu$ $\bar{\delta}^*(z) = \dfrac{z}{c}$ $\bar{\beta}^*(u) = \eta$ s.t. $E(\eta^2) = k^2$ $\bar{J}_B^* = \dfrac{k^2 + \varphi}{c^2}$		$R_2 \cap R_4$: $\gamma^*(u) = cu$ $\delta^*(z) = \dfrac{z}{c}$ $\beta^*(x) - \dfrac{k^2 + \varphi}{c^2} x + \eta$ $\eta \sim N\left(0, k^2 - \dfrac{(k^2 + \varphi)^2}{c^2}\right)$ $J_c^* = \dfrac{k^2 + \varphi}{c^2}$

Problem 2 coincides with the saddle-point value of Problem 1. All these results have been collected together and displayed in Table I. Another important observation that can be made at this point is that the least-favorable distribution for the jamming noise is Gaussian, which is correlated with the message in Problem 3, whereas it is uncorrelated with u in Problem 1.

Extensions of the results of this paper to vector channels can be found in [2]. A counterpart of these results could also be obtained (though not immediately) for continuous-time channels with incomplete statistical description. This is a topic currently under study.

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