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TECHNICAL NOTE

Some Thoughts on Saddle-Point Conditions and Information Structures in Zero-Sum Differential Games

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Abstract. For a very simple two-stage, linear–quadratic, zero-sum difference game with dynamic information structure, we show that (i) there exist nonlinear saddle-point strategies which require the same existence conditions as the well-known linear, closed-loop, no-memory solution and (ii) there exist both linear and nonlinear saddle-point strategies which require more stringent conditions than the unique open-loop solution. We then discuss the implication of this result with respect to the existence of saddle points in zero-sum differential games for different information patterns.

Key Words. Game theory, linear–quadratic games, zero-sum differential games, saddle-point solutions, information patterns.

1. Introduction

It has recently been shown in Refs. 1 and 2 that deterministic nonzero-sum differential games (NZSDG) with dynamic information structures admit uncountably many Nash equilibrium solutions, with the conditions of existence and the Nash equilibrium point being different in each case. Since zero-sum differential games (ZSDG) are special types of NZSDG, the question now arises as to whether the same kind of phenomenon can be observed in the saddle-point solution of ZSDG.

There is a result by Witsenhausen (Ref. 3), who has shown that, if a ZSDG admits a saddle point for some information pattern, then it remains a

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saddle point when the information available to either or both players is increased. This might seem to be reasonable because, since one is basically solving a deterministic game, one always has the option of throwing away (not using) the extra available information; that is, the saddle-point strategy pair obtained under the old information structure can still be considered as a well-defined saddle point strategy pair under the new information pattern. It is, however, quite possible that some other strategy picked by the minimizing player under the new information pattern will be more favorable to him; that is, if the duration of the game is considered as a variable, then he can possibly extend the duration of the game beyond that dictated by the saddle-point solution obtained under the old information structure. This conjecture has, in fact, been verified in Refs. 4 and 5 within the context of linear–quadratic ZSDG, where authors have shown that, by employing pure-feedback (closed-loop, no-memory) strategies, the minimizer can actually extend the duration of the game beyond that imposed by the open-loop saddle-point solution. The question, though, has never been raised in the literature as to (i) whether there exists some other saddle-point solution under the closed-loop information structure which is even less restrictive than the pure-feedback solution and (ii) whether the open-loop saddle-point solution is at least as restrictive as every saddle-point solution obtained under the closed-loop information structure.

In the light of the new results presented in Refs. 1 and 2 within the context of NZSDG, we tend to think that the answers to the above questions are not obvious and require further investigation. In order to gain some insight into the problem, we consider, in this paper, a very simple two-stage linear–quadratic game with dynamic information pattern and show, by employing the ideas introduced in Refs. 1 and 2, the following: (i) the linear pure-feedback saddle-point solution is the least restrictive solution within the linear class; however, there exist nonlinear saddle-point strategies which are equally restrictive; and (ii) the open-loop saddle-point solution is not the most restrictive solution, i.e., there exist closed-loop saddle-point strategies which are even more restrictive (in terms of parameters of the game).

2. Zero-Sum Game

Consider the two-stage zero-sum game defined by the cost function

$$J = \tfrac{1}{2}\{x_1 + u)^2 + \alpha u^2 - \beta v^2\}, \qquad \alpha > 0, \beta > 0, \tag{1-1}$$

and the state equation

$$x_1 = x_0 + v, \tag{1-2}$$

where all variables are scalar and assume values in $\mathcal{R}^1$. u denotes the control variable of Player 1 who is the minimizer and acts at stage one. v is the control variable of Player 2 (the maximizer) who acts at stage zero. Player 2 knows only the value x_0 of the intitial state, whereas Player 1 knows x_0 as well as the outcome x_1. We denote by Γ_0 the class of all measurable maps from $\mathcal{R}^1$ into $\mathcal{R}^1$, and by Γ_1 those that map $\mathcal{R}^1 \times \mathcal{R}^1$ into $\mathcal{R}^1$. At stage zero, Player 2 picks $\gamma_0 \in \Gamma_0$; and, at stage one, Player 1 picks $\gamma_1 \in \Gamma_1$.

With these definitions, the pair $\{\gamma_0^* \in \Gamma_0, \gamma_1^* \in \Gamma_1\}$ is said to be a *saddle-point solution* for the zero-sum game, if it satisfies

$$J(\gamma_1^*, \gamma_0) \leq J(\gamma_1^*, \gamma_0^*) \leq J(\gamma_1, \gamma_0^*), \tag{2}$$

for all $\gamma_0 \in \Gamma_0$, $\gamma_1 \in \Gamma_1$.

Now, by applying the ideas developed in Refs. 1 and 2 to the r.h.s. of (2), we can establish the following property:

Property 2.1. For any saddle-point solution $\{\gamma_0 \in \Gamma_0, \gamma_1 \in \Gamma_1\}$, we have the relation

$$\gamma_1(x_1, x_0) = -[1/(1+\alpha)]x_1 + \psi(x_1, \bar{x}_1), \tag{3-1}$$

with

$$\bar{x}_1 = x_0 + \gamma_0(x_0), \tag{3-2}$$

for some measurable function $\psi(\,.\,,\,.\,)$ of two variables, with the additional property that $\psi(y, y) = 0$, $\forall y \in \mathcal{R}^1$.

In order to complete the characterization of all saddle-point solutions to the problem, we still have to find the strategies of Player 2 that are in equilibrium with (3-1) for different choices of ψ. We now restrict the permissible strategies of Player 1 to a proper subset of Γ_1 by assuming ψ to be twice continuously differentiable on $\mathcal{R}^1 \times \mathcal{R}^1$. The reason for this is mainly to avoid some unnecessary technical difficulties in the analysis to follow.

Substitution of (3-1) into (1-1) and maximization with respect to v yields the following first-order and second-order conditions (assuming that a maximum exists):

$$[\alpha/(1+\alpha)]x_1 + (1+\alpha)\psi(x_1, \bar{x}_1)[\partial\psi(x_1, \bar{x}_1)/\partial x_1] - \beta v = 0, \tag{4-1}$$

$$[\alpha/(1+\alpha) + \partial\psi/\partial x_1]^2 + (1+\alpha)\psi(\partial^2\psi/\partial x_1^2) + \alpha[1/(1+\alpha) - \partial\psi/\partial x_1]^2 - \beta < 0, \tag{4-2}$$

where x_1 depends on v through (1-1).

Now, at equilibrium $x_1 = \bar{x}_1$, and hence $\psi(x_1, \bar{x}_1) = 0$. This fact, used in (4-1), yields a unique solution for v, namely,

$$v^* = \gamma_0^*(x_0) = \{\alpha/[\beta + \alpha(\beta - 1)]\}x_0; \tag{5-1}$$

and this is a well-defined maximizing solution under the sufficiency condition (4-2) at equilibrium, which is

$$\beta > \alpha/(1+\alpha) + (1+\alpha)(\partial\psi/\partial x_1)^2_{x_1=\bar{x}_1}, \tag{5-2}$$

$$\bar{x}_1 = \{\beta(1+\alpha)/[\beta+\alpha(\beta-1)]\}x_0. \tag{5-3}$$

Hence, we have the following theorem.

Theorem 2.1. For the zero-sum game with dynamic information structure considered in this paper, the pair

$$\gamma_1^*(x_1, x_0) = -[1/(1+\alpha)]x_1 + \psi(x_1, \bar{x}_1), \tag{6-1}$$

$$\gamma_0^*(x_0) = \{\alpha/[\beta+\alpha(\beta-1)]\}x_0, \tag{6-2}$$

with $\bar{x}_1$ given by (5-3) constitutes a *saddle-point solution* for any measurable $\psi(\,.\,,\,.\,)$, subject to the restrictions given above. The condition of existence of such a solution is given by (5-2) and depends on the specific choice of ψ.

3. Some Comment and Discussion

It should, first of all, be noted that, unlike the nonzero-sum game discussed in Ref. 1, the saddle-point strategy of Player 2 is independent of the specific choice of ψ; consequently, the saddle-point cost $J(\gamma_1^*, \gamma_0^*)$ is independent of ψ, whenever a saddle point exists. However, existence of a saddle point depends very much on what representation Player 1 employs. This is already partly known in the literature through papers like Refs. 4 and 5 where authors have mentioned, within the context of linear–quadratic, zero-sum differential games, that the conjugate-point condition of the Riccati equations involved will be different when both players play open-loop strategies than in the case when both players play pure-feedback strategies. As we mentioned in Section 1, it has even been shown that the duration of the game will be longer for the closed-loop feedback case than for the open-loop case. What this really corresponds to, for the problem treated here, is that (assuming that α is fixed *a priori*) the existence condition on β should be less restrictive in the feedback (no-memory, closed-loop) case than in the open-loop case. Since $\psi(x_1, \bar{x}_1) \equiv 0$ for the former and $\psi(x_1, \bar{x}_1) \equiv [1/(1+\alpha)](x_1 - \bar{x}_1)$ for the latter, this can easily be seen to be the case.

An interesting result of our analysis, though, is that, for any fixed α, the minimum of the r.h.s. of (5-2) is $\alpha/(1+\alpha)$, which corresponds to the constraint imposed on β by the closed-loop, no-memory strategy. However, this least bound is attained *not only* for that particular representation

employed by Player 1 (probably contrary to what intuition might say). For any saddle-point solution (6) with

$$\left(\frac{\partial \psi}{\partial x_1}\right)_{x_1=\bar{x}_1} = 0,$$

the existence condition will still be the one imposed by the well-known linear solution. In particular, the nonlinear strategy

$$\gamma_1^*(x_1, x_0) = -[1/(1+\alpha)]x_1 + \sum_{i=2}^{N} a_i[\sin(x_1 - \bar{x}_1)]^i,$$

together with

$$\gamma_0^*(x_0) = \{\alpha/[\beta + \alpha(\beta - 1)]\}x_0,$$

will constitute a saddle-point solution for any a_i and any integer N, under the existence condition imposed by the well-known linear solution.

This observation, in our opinion, will have a great impact on the research activities involving existence of saddle-point solutions in deterministic differential games. In obtaining existence conditions under the closed-loop information structure, one should be careful to take into consideration all possible *representations* of the same control value and the *nonuniqueness* arising from these different representations. To give one example, for the LQ differential game it is well known that, under both the open-loop (OL) and the closed-loop no-memory (CLNM) information structures, the saddle-point cost of the game will be unique (even though it is attained by different linear strategies in each case), provided that both conjugate-point conditions are satisfied. It is also partly known that these existence conditions are less restrictive for the CLNM information structure than for the OL information structure. Hence, if the minimizing player is given the option of picking one or the other, he would definitely play a pure-feedback strategy. However, it is not known yet (and the question has never been raised before) whether, for the general, closed-loop information structure, some other kind of strategy would require an even less restrictive existence condition. It is within the realm of possibility that, by playing an appropriate nonlinear strategy, which takes into account the present as well as the past values of the state vector, the minimizing player can extend the duration of the game beyond that imposed by the linear CLNM strategy. Investigation of what seems to be a very interesting feature of saddle points in differential games still remains a challenge for future research. What we know today, though, is that, at least within the context of the ZS game discussed in this paper, there will exist nonlinear saddle-point strategies which require the same existence conditions as the least restrictive linear solution.

One final remark is that, as an answer to the second question raised in Section 1, it should be clear, from the existence conditions (5-2) and the fact that the OL solution is characterized by $\psi(x_1, \bar{x}_1) \equiv [1/(1+\alpha)](x_1 - \bar{x}_1)$, that there exist other saddle-point strategies which are even more restrictive than the OL solution.

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