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Performance Bounds for Hierarchical Systems under Partial Dynamic Information[1]

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Abstract. This paper discusses a general approach to obtain optimum performance bounds for $(N+1)$-person deterministic decision problems, $N+1>2$, with several levels of hierarchy and under partial dynamic information. Both cooperative and noncooperative modes of decision making are considered at the lower levels of hierarchy; in each case, it is shown that the optimum performance of the decision maker at the top of the hierarchy can be obtained by solving a sequence of open-loop (static) optimization problems. A numerical example included in the paper illustrates the general approach.

Key Words. Hierarchical decision problems, dynamic games, partial dynamic information, Stackelberg solutions, Nash solutions, Pareto-optimal solutions.

1. Introduction

This paper addresses a general class of deterministic multiperson dynamic decision problems that involve hierarchy in command and control, decentralization in information, and possibly different objective functionals for different decision units, with the objective functional of the decision maker at the top of the hierarchy (known as the coordinator or leader) representing that of the entire system. The prime objective in such problems is to obtain a set of decision laws for the coordinator, which would lead to an optimum performance for the overall system, by also taking into account the hierarchies in decision and control, and possible rational responses of the other decision units.

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In general, this is an extremely challenging class of problems to solve, the difficulty being due mainly to the dynamic nature of the information available to the decision units. This difficulty can be circumvented partially by assuming certain structural forms for the decision rules and by seeking the solution in that restricted class—that is, by confining attention to suboptimal policies. However, if the achievable optimum performance in the general class is not known, it is not possible to assess the degree of suboptimality of such designs and to judge whether the resulting performance is satisfactory or not. Hence, it is highly desirable to develop techniques that would yield (tight) bounds for the attainable optimum performance in the general class of policies, if it is not possible to obtain the optimal solution itself.

Such a technique has been developed recently in Ref. 1 for the class of hierarchical decision problems with two decision units and with partial dynamic information. The original dynamic problem (which cannot be solved using the standard techniques of optimization) is converted into two open-loop (static) optimization problems which can be solved using the standard techniques of optimization and optimal control theory, and the optimum performance of the original dynamic decision problem can readily be obtained from these solutions.

Such hierarchical decision problems are also known as Stackelberg games, which have attracted considerable attention in the literature in recent years (see, e.g., Refs. 2–6), and research activities have been concentrated on the case of perfect dynamic information, under which the optimum performance coincides with the minimum value of the cost function of the leader (assuming that the leader has the power of announcing and enforcing an appropriate incentive scheme). Reference 3 discusses also Stackelberg games under partial dynamic information and obtains a set of necessary conditions for the leader's continuous-time optimal control problem within the class of differentiable strategies for the leader. Even though such necessary conditions may be useful in characterizing candidate Stackelberg strategies (within the class of differentiable functions), they do not lead to optimum performance bounds, unless these strategies are explicitly computable. The approach of Ref. 1, however, is directed primarily toward obtaining the optimum attainable performance for the leader, without explicitly determining the corresponding strategies or incentive schemes.

In the present paper, we discuss an extension of the new technique of Ref. 1 to decision problems with more than two decision units and hierarchies, and again under partial dynamic information. We treat both cooperative and noncooperative modes of decision making between the units operating at the same level of hierarchy and show, in each case, that the optimum performance can be determined through the solutions of a set of open-loop (static) optimization problems. This theory is developed in Section 3, while Section 4 is devoted to an illustrative numerical example. A precise problem formulation is provided in Section 2.

2. General Formulation and Definitions

In this section, we discuss possible configurations of coordination and control in a hierarchical system which is controlled by $N+1$ decision makers, each with a possibly different objective functional and possibly different dynamic information. We take $N \geq 2$, and observe that every possible configuration can be obtained by appropriate concatenation of the two prototypes displayed in Fig. 1. The first one illustrates the case of two levels of hierarchy: one of the decision makers (DM0), which we may also call the coordinator, is at the top of the hierarchy, and he enforces his decision(s) on the remaining decision makers, who are all at the same (second) level of hierarchy. These N decision makers (called followers) may have cooperation among themselves, or they may act noncooperatively—thus giving rise to different possibilities for an equilibrium solution. The coordinator's role here is to ensure that some optimum performance is attained for the overall system. Accordingly, one of our objectives in this paper is to obtain tight bounds for this optimum performance under both cooperative and noncooperative actions of the followers and when the coordinator observes these actions only partially.

The second configuration, depicted in Fig. 1b, is known as a *linear hierarchy*, since there are $N+1$ levels of decision making and each level is occupied by only one decision maker. The decisions are announced and enforced sequentially, in the order shown in Fig. 1b. The role of DM0 (at the top of the hierarchy) is again to coordinate the actions of the remaining decision makers, by also taking into account the existing hierarchy in the decision making and possible information exchanges between different

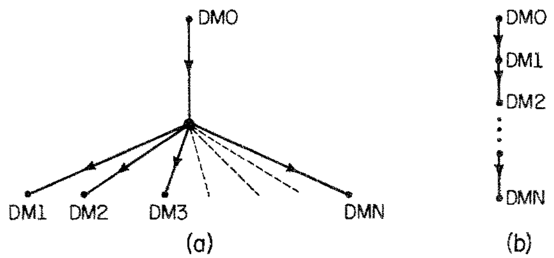

Fig. 1. Two prototypes of possible configurations in hierarchical coordination and control.

levels; he will seek an optimum performance for the overall system, which is also reflected in his objective functional. In the paper, we shall obtain tight bounds for this optimum performance under partial dynamic information.

For the sake of clarity in exposition, we will restrict our attention to the case $N=2$, which, however, introduces no real loss in generality, since the *three decision maker* case captures all the essential features and intricacies of the general problem and provides sufficient insight so that the results can be extended with no major difficulty.

In order to introduce precise definitions for the optimum performance under both types of hierarchy depicted in Fig. 1, we consider a three-person decision problem in *normal* (strategic) form, described by the cost functionals $J_0(\gamma_0, \gamma_1, \gamma_2)$, $J_1(\gamma_0, \gamma_1, \gamma_2)$, $J_2(\gamma_0, \gamma_1, \gamma_2)$, where the strategies γ_0, γ_1, γ_2 belong to *a priori* known strategy spaces Γ_0, Γ_1, Γ_2, respectively. Here, J_0 may be selected dependent on J_1 and J_2 (such as their convex combination), or totally independent of them, with the latter choice reflecting an objective not commensurable with those of the two followers. Under our abstract formulation, both of these cases are covered. Now, depending on the type of hierarchy and the mode of decision making in between the followers, the following three possibilities emerge.

Case (A). Two Levels of Hierarchy and Cooperative Action between the Followers. Consider the hierarchical structure depicted in Fig. 1a, but with $N=2$, and under the stipulation that the two followers act cooperatively. First, we introduce, for each $\gamma_0 \in \Gamma_0$, the *noninferior rational reaction set* of the followers' group by

$$\begin{aligned} R_c(\gamma_0) = \{(\gamma_1^0 \in \Gamma_1, \gamma_2^0 \in \Gamma_2) : \nexists(\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2) \ni J_1(\gamma_0, \gamma_1, \gamma_2) \\ \leq J_1(\gamma_0, \gamma_1^0, \gamma_2^0) \quad \text{and} \quad J_2(\gamma_0, \gamma_1, \gamma_2) \leq J_2(\gamma_0, \gamma_1^0, \gamma_2^0), \\ \text{with strict inequality for at least one } i=1,2.\} \end{aligned} \tag{1}$$

Definition 2.1. The quantity

$$\inf_{\gamma_0 \in \Gamma_0} \sup_{(\gamma_1, \gamma_2) \in R_c(\gamma_0)} J_0(\gamma_0, \gamma_1, \gamma_2) = J_0^* \tag{2}$$

is the *optimum performance* of the coordinator (DM0) in this decision problem, and any strategy $\gamma_0^* \in \Gamma_0$ that achieves this performance level, i.e.,

$$\sup_{(\gamma_1, \gamma_2) \in R_c(\gamma_0^*)} J_0(\gamma_0^*, \gamma_1, \gamma_2) = J_0^* \tag{3}$$

is a *hierarchical equilibrium strategy* for DM0.

Note that, unless $R_c(\gamma_0)$ is a singleton, the realized performance level for DM0 may in fact be lower than J_0^*; hence, J_0^* stands out as providing a security level for the cost of DM0.

Case (B). Two Levels of Hierarchy and Noncooperative Action between the Followers. Consider again the hierarchical structure of Fig. 1a with $N=2$, but this time under the noncooperative mode of action. Specifically, we stipulate that, for each announced strategy of the coordinator (DM0), the followers choose their decisions under the Nash equilibrium solution concept. Hence, the *rational reaction set* of the followers' group, for each $\gamma_0 \in \Gamma_0$, would be

$$\begin{aligned} R_n(\gamma_0) = \{(\gamma_1^0 \in \Gamma_1, \gamma_2^0 \in \Gamma_2) : J_1(\gamma_0, \gamma_1^0, \gamma_2^0) \leq J_1(\gamma_0, \gamma_1, \gamma_2^0), \\ J_2(\gamma_0, \gamma_1^0, \gamma_2^0) \leq J_2(\gamma_0, \gamma_1^0, \gamma_2), \forall \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}. \end{aligned} \tag{4}$$

We further restrict attention to *admissible* Nash pairs and define the *admissible rational reaction set* of the followers' group as

$$\begin{aligned} R_{an}(\gamma_0) = \{(\gamma_1^0, \gamma_2^0) \in R_n(\gamma_0) : \nexists (\gamma_1, \gamma_2) \in R_n(\gamma_0) \ni J_1(\gamma_0, \gamma_1, \gamma_2) \\ \leq J_1(\gamma_0, \gamma_1^0, \gamma_2^0) \quad \text{and} \quad J_2(\gamma_0, \gamma_1, \gamma_2) \leq J_2(\gamma_0, \gamma_1^0, \gamma_2^0), \\ \text{with strict inequality for at least one } i = 1, 2\}. \end{aligned} \tag{5}$$

Definition 2.2. The quantity

$$\inf_{\gamma_0 \in \Gamma_0} \sup_{(\gamma_1, \gamma_2) \in R_{an}(\gamma_0)} J_0(\gamma_0, \gamma_1, \gamma_2) = J_0^* \tag{6}$$

is the *optimum performance* of the coordinator (DM0) in this decision problem, and any strategy $\gamma_0^* \in \Gamma_0$ that achieves this performance level, i.e.,

$$\sup_{(\gamma_1, \gamma_2) \in R_{an}(\gamma_0^*)} J_0(\gamma_0^*, \gamma_1, \gamma_2) = J_0^*, \tag{7}$$

is a *hierarchical equilibrium strategy* for DM0.

Remark 2.1. As in the previous case, the realized performance level for DM0 may in fact be lower than J_0^* when $R_{an}(\gamma_0^*)$ is not a singleton, this being the case when the Nash solution obtained at the second level is nonunique. The nonuniqueness is due not only to the structural properties of the cost functionals, but also to the informational properties of the problem (Refs. 7–8). This latter type of nonuniqueness can, however, be avoided by either restricting attention to only open-loop policies for the followers or by requiring the Nash solution to satisfy some further properties (Ref. 8). This point will be discussed further in the following sections.

Case (C). Three Levels of Hierarchy. In the case of the linear hierarchy depicted in Fig. 1b, a definition of hierarchical equilibrium[3] has been given in Ref. 9. We first define, for each $(\gamma_0 \in \Gamma_0, \gamma_1 \in \Gamma_1)$ the *rational reaction set* of DM2 by

$$R_2(\gamma_0; \gamma_1) \triangleq \{\gamma_2^0 \in \Gamma_2 : J_2(\gamma_0, \gamma_1, \gamma_2^0) \leq J_2(\gamma_0, \gamma_1, \gamma_2), \forall \gamma_2 \in \Gamma_2\}, \tag{8}$$

and then introduce, for each $\gamma_0 \in \Gamma_0$, the *rational reaction set* of DM1 by

$$\begin{aligned} R_1(\gamma_0) = \{\gamma_1^0 \in \Gamma_1 : & \sup_{\gamma_2 \in R_2(\gamma_0; \gamma_1^0)} J_1(\gamma_0, \gamma_1^0, \gamma_2) \\ & \leq \sup_{\gamma_2 \in R_2(\gamma_0; \gamma_1)} J_1(\gamma_0, \gamma_1, \gamma_2), \forall \gamma_1 \in \Gamma_1\}. \end{aligned} \tag{9}$$

Definition 2.3. The quantity

$$\inf_{\gamma_0 \in \Gamma_0} \sup_{\gamma_1 \in R_1(\gamma_0)} \sup_{\gamma_2 \in R_2(\gamma_0; \gamma_1)} J_0(\gamma_0, \gamma_1, \gamma_2) = J_0^* \tag{10}$$

is the *optimum performance* of DM0 in this decision problem, and any strategy $\gamma_0^* \in \Gamma_0$ that achieves this performance level, i.e.,

$$\sup_{\gamma_1 \in R_1(\gamma_0^*)} \sup_{\gamma_2 \in R_2(\gamma_0^*, \gamma_1)} J_0(\gamma_0^*, \gamma_1, \gamma_2) = J_0^*, \tag{11}$$

is a *hierarchical equilibrium strategy* for DM0.

Our objective in this paper is to obtain tight bounds for J_0^* in the three cases covered by Definitions 2.1–2.3 and when the information available to the decision makers is dynamic in nature (especially from DM0's point of view). The underlying system that gives rise to the normal form utilized above is assumed to be *deterministic*, but the dynamic information available to the decision makers is only *partial*. More specifically, we let η_i denote the dynamic information available to DMi, $i = 0, 1, 2$, throughout the decision process; and we let y_i denote the value of this information (which may take values in finite- or infinite-dimensional vector spaces) after all three decision makers have chosen their strategies. Then, we have the loop relations

$$\begin{aligned} u_i &= \gamma_i(y_i), \\ y_i &= \eta_i(u_0, u_1, u_2), \qquad i = 0, 1, 2, \end{aligned} \tag{12}$$

where u_i denotes the decision (control) value of DMi, and any dependence

[3] If there is some further structure on the dynamic decision problem, such as a difference equation describing the evolution of state, it is possible to impose some additional restrictions on the hierarchical equilibrium solution, as in Ref. 10. Since we are working with a general abstract model here, such an extension will not be pursued.

on the *state* is suppressed. Even though this is a very general framework, there are of course some implicit structural restrictions imposed on the loop relations (12) so that they satisfy the principle of causality. Finally, the vectors u_i and y_i are assumed to belong, respectively, to the vector spaces U_i and Y_i, where the latter is taken (without any loss of generality) to be the full range space of η_i; no further structure is imposed on U_i and Y_i.

3. Derivation of Tight Performance Bounds

In the absence of the third decision maker (i.e., with $N=1$), all three definitions collapse into a single one—a case which has been studied thoroughly in Ref. 1. It has been shown in that reference that the original dynamic decision problem can be reduced to two static (open-loop) optimization problems, the solutions of which readily yield J_0^*. In the sequel, we investigate, via novel indirect approaches, derivation of tight bounds for J_0^* for the cases covered by Definitions 2.1–2.3, through the solutions of static optimization problems.

Case (A). Two Levels of Hierarchy and Cooperative Action between the Followers. For the case covered by Definition 2.1, consider the following two steps of optimization.

Step 1. For $\mu \in [0, 1]$, let

$$\bar{J}_\mu(u_0, u_1, u_2) = \mu J_1(u_0, u_1, u_2) + (1-\mu) J_2(u_0, u_1, u_2) \tag{13a}$$

and, for fixed $u_0 \in U_0$, $y_0 \in Y_0$, and $\mu \in [0, 1]$, minimize $\bar{J}_\mu(u_0, u_1, u_2)$ over $S \subset U_1 \times U_2$, where

$$S(u_0, y_0) \triangleq \{(u_1, u_2) \in U_1 \times U_2 \colon y_0 = \eta_0(u_0, u_1, u_2)\}. \tag{13b}$$

Define the (partial) optimal response set

$$R_\mu(u_0, y_0) = \{(u_1^0, u_2^0) \in S \colon \bar{J}_\mu(u_0, u_1^0, u_2^0) \leq \bar{J}_\mu(u_0, u_1, u_2), \forall (u_1, u_2) \in S\}. \tag{13c}$$

Note that here u_0 does not depend functionally on u_1 or u_2; it is any element of U_0 which comprises only open-loop strategies.

Step 2. If $R_\mu(u_0, y_0)$ is a singleton, denote the corresponding unique map by $T_\mu : U_0 \times Y_0 \to U_1 \times U_2$, and consider minimization of $\sup_\mu J_0(u_0, T_\mu(u_0, y_0))$ over $(u_0, y_0) \in U_0 \times Y_0$. Denote any minimizing solution pair by (u_0^0, y_0^0), and denote the corresponding values of μ and J_0 by μ^* and J_0^0, respectively.

If $R_\mu(u_0, y_0)$ is not a singleton, define (u_0^0, y_0^0) and J_0^0 through the inequality

$$J_0^0 \triangleq \sup_{\mu\in[0,1]} \sup_{(u_1,u_2)\in R_\mu(u_0^0, y_0^0)} J_0(u_0^0, u_1, u_2)$$

$$\leq \sup_{\mu\in[0,1]} \sup_{(u_1,u_2)\in R_\mu(u_0,y_0)} J_0(u_0, u_1, u_2), \forall (u_0, y_0)\in U_0\times Y_0. \tag{14}$$

Theorem 3.1. Assume the following:

(i) for each $(u_0, y_0)\in U_0\times Y_0$, the set of noninferior solutions of the vector-valued minimization problem

$$\min_{(u_1,u_2)\in U_1\times U_2} \{J_1(u_0, u_1, u_2), J_2(u_0, u_1, u_2)\} \ni y_0 = \eta_0(u_0, u_1, u_2)$$

coincides with $\bigcup_{\mu\in[0,1]} R_\mu(u_0, y_0)$;

(ii) there exists a pair $(u_0^* = u_0^0, y_0^* = y_0^0)$ satisfying the requirements at Step 2, and the inf sup operations in (14) can be interchanged;

(iii) there exists a $\hat{\gamma}_0\in\Gamma_0$ with the property

$$\inf_{(u_1,u_2)\in U_1\times U_2} J_i(\hat{\gamma}_0, u_1, u_2) > J_i(u_0^*, u_1, u_2), \forall (u_1, u_2)\in R_\mu(u_0^*, y_0^*),$$
$$\forall \mu\in[0, 1], \qquad i = 1, 2, \tag{15}$$

such that the composite strategy

$$\gamma_0^*(y_0) = \begin{cases} u_0^*, & \text{if } y_0 = y_0^*, \\ \hat{\gamma}_0(y_0), & \text{otherwise,} \end{cases} \tag{16}$$

is an element of Γ_0.

Then, J_0^0 obtained at Step 2 is the optimum performance J_0^* of the coordinator DM0 in the decision problem covered by Def. 1. Furthermore, γ_0^* is a hierarchical equilibrium strategy for DM0.

Remark 3.1. In the statement of Theorem 3.1, Assumption (i) basically requires that all Pareto-optimal solutions of the related vector-valued static optimization problem can be obtained by minimizing a convex combination of J_1 and J_2. This places some implicit restrictions on the structure of the cost functionals J_1 and J_2 and the constraint equations, but it is not an unreasonable assumption to make (Ref. 11). In the absence of such an assumption, the problem is still tractable, but then the parametrized scalar-valued optimization problem of Step 1 will have to be replaced by a vector-valued optimization problem. Assumption (iii), on the other hand, ensures that the coordinator is in a position to threaten[4] the followers by increasing their cost values if their past actions do not lead to the value y_0^0

determined above at Steps 1 and 2. Note that the strategy γ_0^* is implementable under the underlying assumption of causality.

Proof of Theorem 3.1. Let $\gamma_0^* \in \Gamma_0$ be a hierarchical equilibrium strategy for DM0, in accordance with Definition 2.1. Furthermore, let (γ_1^*, γ_2^*) be a pair of supremizing strategies in (3). Denote the realized values of y_0, y_1, y_2, after these strategies are applied, by y_0^*, y_1^*, y_2^*, respectively, and let

$$\gamma_0^*(y_0^*) = u_0^*, \qquad \gamma_1^*(y_1^*) = u_1^*, \qquad \gamma_2^*(y_2^*) = u_2^*.$$

Now, since $(\gamma_1^*, \gamma_2^*) \in R_c(\gamma_0^*)$, we have from (1):

$$\begin{aligned} &\nexists(\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2) \quad \text{such that} \\ &J_1(\gamma_0^*, \gamma_1, \gamma_2) \leq J_1(\gamma_0^*, \gamma_1^*, \gamma_2^*), \\ &J_2(\gamma_0^*, \gamma_1, \gamma_2) \leq J_2(\gamma_0^*, \gamma_1^*, \gamma_2^*), \\ &\text{with strict inequality in at least one of these.} \end{aligned} \tag{17}$$

This statement is further equivalent to

$$\begin{aligned} &\nexists(u_1 \in U_1, u_2 \in U_2) \quad \text{such that} \\ &y_0^* = \eta_0(u_0^*, u_1, u_2), \\ &J_1(u_0^*, u_1, u_2) \leq J_1(u_0^*, u_1^*, u_2^*), \\ &J_2(u_0^*, u_1, u_2) \leq J_2(u_0^*, u_1^*, u_2^*), \\ &\text{with strict inequality in at least one of these.} \end{aligned} \tag{18}$$

Under the first assumption of Theorem 3.1, this property can be restated as follows:

$$\begin{aligned} &\exists \mu \in [0, 1], \text{ say } \mu^*, \text{ such that } (u_1^*, u_2^*) \text{ minimizes} \\ &\bar{J}_{\mu^*}(u_0^*, u_1, u_2) = \mu^* J_1(u_0^*, u_1, u_2) + (1-\mu^*) J_2(u_0^*, u_1, u_2) \\ &\text{over } U_1 \times U_2 \text{ and subject to } y_0^* = \eta_0(u_0^*, u_1, u_2). \end{aligned} \tag{19}$$

Hence,

$$(u_1^*, u_2^*) \in R_{\mu^*}(u_0^*, y_0^*).$$

Now, the pair (u_0^*, y_0^*) is under the control of DM0, and he can enforce any such solution [the latter one being a governing factor in the constraint $y_0 = \eta_0(u_0, u_1, u_2)$] on the followers under the third condition of Theorem

[4] If there exist overly restrictive hard bounds on the decision variables of the leader, such a threat strategy will not exist, in which case J_0^* will be greater than J_0^0. See Ref. 12 for a discussion of two-person Stackelberg problems with such constraints.

3.1. However, DM0 does not have control over the choice of controls from $R_\mu(u_0, y_0)$, nor does he have control over the choice of any specific value for μ. Therefore, a secured level for J_0 will be

$$\begin{aligned} J_0^* &= \inf_{\gamma_0 \in \Gamma_0} \sup_{(\gamma_1,\gamma_2)\in R_c(\gamma_0)} J_0(\gamma_0, \gamma_1, \gamma_2) \\ &= \inf_{(u_0,y_0)\in U_0\times Y_0} \sup_{\mu\in[0,1]} \sup_{(u_1,u_2)\in R_\mu(u_0,y_0)} J_0(u_0, u_1, u_2) = J_0^0, \end{aligned} \tag{20}$$

where, in going from the second to the third step, we have utilized the equivalence between (17) and (19) and the interchangeability of the inf sup operations. This, then, completes the proof of the theorem. □

We should note that (20), in fact, gives the optimum performance (see Definition 2.1) in terms of the minimax (= maximum) value of an open-loop optimization problem without requiring existence of a hierarchical equilibrium strategy for DM0.

Remark 3.2. One can easily visualize an extension of Theorem 3.1 to the case of N followers, as depicted in Fig. 1a, in which case (20) will be replaced by

$$J_0^* = \inf_{(u_0,y_0)\in U_0\times Y_0} \sup_{\mu\in M} \sup_{(u_1,\ldots,u_N)\in R_\mu(u_0,\, y_0)} J_0(u_0, u_1, \ldots, u_N), \tag{21}$$

where

$$\begin{aligned} R_\mu(u_0, y_0) = \{(u_1^0, \ldots, u_N^0) \in S: \bar{J}_\mu(u_0, u_1^0, \ldots, u_N^0) \\ \leq \bar{J}_\mu(u_0, u_1, \ldots, u_N), \forall (u_1, \ldots, u_N) \in S\}, \end{aligned} \tag{22a}$$

$$S(u_0, y_0) \triangleq \{(u_1, \ldots, u_N) \in U_1 \times \cdots \times U_N: y_0 = \eta_0(u_0, u_1, \ldots, u_N)\}, \tag{22b}$$

$$\bar{J}_\mu(u_0, u_1, \ldots, u_N) = \sum_{i=1}^{N} \mu_i J_i(u_0, u_1, \ldots, u_N), \tag{22c}$$

$$M = \left\{\mu_i \geq 0, i = 1, \ldots, N: \sum_{i=1}^{N} \mu_i = 1\right\}. \tag{22d}$$

Remark 3.3. Even though J_0^*, as given by (20), provides a security level for DM0, it is not necessarily the performance that will be realized, since the followers may not choose the supremizing strategies. There exists, however, also a lower bound for DM0's performance, which is obtained by assuming that the followers will pick their strategies in the indifference region in a manner that is mostly beneficial to DM0. This argument readily leads to the value

$$J_{0*} = \inf_{(u_0,y_0)\in U_0\times Y_0} \inf_{\mu\in[0,1]} \inf_{(u_1,u_2)\in R_\mu(u_0,\, y_0)} J_0(u_0, u_1, u_2), \tag{23}$$

which may also be attained. The actual realized value of J_0 will always lie in the interval $[J_{0*}, J_0^*]$.

Case (B). Two Levels of Hierarchy and Noncooperative Action between the Followers. This is the case covered by Definition 2.2; and, in order to avoid informational nonuniqueness of the Nash solution at the second level of hierarchy, we assume at the outset that the two followers have access to only open-loop information.[5] Then, the following two steps of optimization will lead to the optimum performance value for DM0.

Step 1. Let $S(u_0, y_0)$ be defined as in (13b). Determine the sets

$$\bar{R}_n(u_0, y_0) \triangleq \{(u_1^0, u_2^0) \in S: J_1(u_0, u_1^0, u_2^0) \leq J_1(u_0, u_1, u_2^0),$$
$$J_2(u_0, u_1^0, u_2^0) \leq J_2(u_0, u_1^0, u_2), \forall (u_1, u_2) \in S\} \tag{24}$$

and

$$\bar{R}_{an}(u_0, y_0) \triangleq \{(u_1^0, u_2^0) \in \bar{R}_n(u_0, y_0): \nexists (u_1, u_2) \in \bar{R}_n(u_0, y_0) \ni J_1(u_0, u_1, u_2)$$
$$\leq J_1(u_0, u_1^0, u_2^0) \quad \text{and} \quad J_2(u_0, u_1, u_2) \leq J_2(u_0, u_1^0, u_2^0),$$
$$\text{with strict inequality for at least one } i = 1, 2\}. \tag{25}$$

Step 2. If $\bar{R}_{an}(u_0, y_0)$ is a singleton for every pair (u_0, y_0), denote the corresponding unique map by $T: U_0 \times Y_0 \to U_1 \times U_2$, and consider the minimization of $J_0(u_0, T(u_0, y_0))$ over $(u_0, y_0) \in U_0 \times Y_0$. Denote any minimizing solution pair by (u_0^0, y_0^0), and the corresponding value of J_0 by

$$J_0^0 = J_0(u_0^0, T(u_0^0, y_0^0)).$$

If $\bar{R}_{an}(u_0, y_0)$ is not a singleton, define (u_0^0, y_0^0) and J_0^0 through the inequality

$$J_0^0 \triangleq \sup_{(u_1,u_2) \in \bar{R}_{an}(u_0^0, y_0^0)} J_0(u_0^0, u_1, u_2)$$
$$\leq \sup_{(u_1,u_2) \in \bar{R}_{an}(u_0, y_0)} J_0(u_0, u_1, u_2), \qquad \forall (u_0, y_0) \in U_0 \times Y_0. \tag{26}$$

Theorem 3.2. Assume the following:

(i) there exists a pair $(u_0^* = u_0', y_0^* = y_0^0)$ satisfying the requirements at Step 2, and the inf sup operations in (26) are interchangeable;

[5] Yet another approach that avoids informational nonuniqueness is to adopt the closed-loop Nash solution concept under the delayed commitment mode of decision making (see Ref. 13, Chapter 6).

(ii) there exists a $\hat{\gamma}_0 \in \Gamma_0$ with the property

$$\inf_{(u_1,u_2)\in U_1\times U_2} J_i(\hat{\gamma}_0, u_1, u_2) > J_i(u_0^*, u_1, u_2),$$
$$\forall (u_1, u_2) \in \bar{R}_{an}(u_0^*, y_0^*), \qquad i = 1, 2, \tag{27}$$

such that the composite strategy

$$\gamma_0^*(y_0) = \begin{cases} u_0^*, & \text{if } y_0 = y_0^*, \\ \hat{\gamma}_0(y_0), & \text{otherwise,} \end{cases} \tag{28}$$

is an element of Γ_0.

Then, J_0^0 obtained at Step 2 is the optimum performance (J_0^*) of the coordinator DM0 in the decision making problem covered by Definition 2.2. Furthermore, γ_0^* is a hierarchical equilibrium strategy for DM0.

Proof. The proof parallels that of Theorem 3.1, with the only difference being that the set of noninferior solutions at the second level of hierarchy is now replaced by the set of all admissible Nash equilibrium solutions. Note that J_0^*, as defined by (6), will be equivalent to

$$J_0^* = \inf_{(u_0,y_0)\in U_0\times Y_0} \; \sup_{(u_1,u_2)\in \bar{R}_{an}(u_0,\, y_0)} J_0(u_0, u_1, u_2), \tag{29}$$

which is also valid even if a hierarchical equilibrium strategy does not exist. □

Remark 3.4. An extension of Theorem 3.2 to the case of N (instead of 2) followers is immediate (as in the case of Theorem 3.1, discussed in Remark 3.2). We simply replace (29) with

$$J_0^* = \inf_{(u_0,y_0)\in U_0\times Y_0} \; \sup_{(u_1,\ldots,u_N)\in \bar{R}_{an}(u_0,y_0)} J_0(u_0, u_1, \ldots, u_N), \tag{30}$$

where $\bar{R}_{an}(u_0, y_0)$ is now defined as a subset of $S(u_0, y_0)$ given by (22b), comprising all admissible Nash equilibrium solutions. We omit further details of this extension.

Remark 3.5. A counterpart of Remark 3.3 will also be valid here. The actual realized value of J_0 will always lie in the interval $[J_{0*}, J_0^*]$, where

$$J_{0*} = \inf_{(u_0,y_0)\in U_0\times Y_0} \; \inf_{(u_1,u_2)\in \bar{R}_{an}(u_0,\, y_0)} J_0(u_0, u_1, u_2), \tag{31}$$

and J_0^* is given by (29).

Case (C). Three Levels of Hierarchy. This is the case covered by Definition 2.3 and depicted in Fig. 1b. Consider the following three steps of optimization.

Step 1. Minimize $J_2(u_0, u_1, u_2)$ over $S_2 \subset U_2$, where

$$S_2(u_0, y_0, u_1, y_1) \triangleq \{u_2 \in U_2 \colon y_0 = \eta_0(u_0, u_1, u_2), y_1 = \eta_1(u_0, u_1, u_2)\},$$
$$u_0 \in U_0, \qquad y_0 \in Y_0, \qquad u_1 \in U_1, \qquad y_1 \in Y_1. \tag{32}$$

Assume that a solution exists and is unique, and denote the corresponding map by

$$T_2 : U_0 \times Y_0 \times U_1 \times Y_1 \to U_2.$$

Step 2. Now, consider the minimization of $J_1(u_0, u_1, T_2(u_0, y_0, u_1, y_1))$ over $S_1 \subset U_1 \times Y_1$, where

$$S_1(u_0, y_0) \triangleq \{(u_1, y_1) \in U_1 \times Y_1 \colon y_0 = \eta_0(u_0, u_1, T_2(u_0, y_0, u_1, y_1))\},$$
$$u_0 \in U_0,\ y_0 \in Y. \tag{33}$$

Let

$$\bar{R}_1(u_0, y_0) = \{(u_1^0, y_1^0) \in S_1 \colon J_1(u_0, u_1^0, T_2(u_0, y_0, u_1^0, y_1^0))$$
$$\leq J_1(u_0, u_1, T_2(u_0, y_0, u_1, y_1)), \forall (u_1, y_1) \in S_1\}. \tag{34}$$

Step 3. If $\bar{R}_1(u_0, y_0)$ is a singleton for every pair $(u_0, y_0) \in U_0 \times Y_0$, denote the corresponding unique map by

$$T_1 : U_0 \times Y_0 \to U_1 \times Y_1$$

and its restriction to U_1 by

$$T_{11} : U_0 \times Y_0 \to U_1.$$

Consider the minimization of $J_0(u_0, T_{11}(u_0, y_0), T_2(u_0, y_0, T_1(u_0, y_0))$ over $(u_0, y_0) \in U_0 \times Y_0$; denote any minimizing solution pair by (u_0^0, y_0^0), and the corresponding value of J_0 by

$$J_0^0 = J_0(u_0^0, T_{11}(u_0^0, y_0^0), T_2(u_0, y_0, T_1(u_0, y_0))).$$

If $\bar{R}_1(u_0, y_0)$ is not a singleton, define (u_0^0, y_0^0) and J_0^0 through the inequality

$$J_0^0 \triangleq \sup_{(u_1, y_1) \in \bar{R}_1(u_0^0, y_0^0)} J_0(u_0^0, u_1, T_2(u_0^0, y_0^0, u_1, y_1))$$
$$\leq \sup_{(u_1, y_1) \in \bar{R}_1(u_0, y_0)} J_0(u_0, u_1, T_2(u_0, y_0, u_1, y_1)), \qquad \forall (u_0, y_0) \in U_0 \times Y_0. \tag{35}$$

Theorem 3.3. Assume the following:

(i) there exists a pair $(u_0^* = u_0^0, y_0^* = y_0^0)$ satisfying the requirements at Step 3, and the inf sup operations in (35) are interchangeable;

(ii) there exists a $\hat{\gamma}_0 \in \Gamma_0$ with the property

$$\inf_{(u_1,u_2)\in U_1\times U_2} J_i(\hat{\gamma}_0, u_1, u_2) > J_i(u_0^*, u_1^*, T_2(u_0^*, y_0^*, u_1^*, y_1^*)),$$
$$\forall u_1^* \in \bar{R}_1(u_0^*, y_0^*), i = 1, 2, \tag{36}$$

where y_1^* satisfies the causal relationship

$$y_1^* = \eta_1(u_0^*, u_1^*, T_2(u_0^*, y_0^*, u_1^*, y_1^*)),$$

such that the composite strategy

$$\gamma_0^*(y_0) = \begin{cases} u_0^*, & \text{if } y_0 = y_0^*, \\ \hat{\gamma}_0(y_0), & \text{otherwise,} \end{cases} \tag{37}$$

is an element of Γ_0;

(iii) there exists a $\hat{\gamma}_1 \in \Gamma_1$ with the property

$$\inf_{u_2\in U_2} J_2(u_0^*, \hat{\gamma}_1, u_2) > J_2(u_0^*, u_1^*, T_2(u_0^*, y_0^*, u_1^*, y_1^*)), \tag{38}$$

where y_1^* is as given before, such that the composite strategy

$$\gamma_1^*(y_1) = \begin{cases} u_1^*, & \text{if } y_1 = y_1^*, \\ \hat{\gamma}_1(y_1), & \text{otherwise,} \end{cases} \tag{39}$$

is an element of Γ_1.

Then, J_0^0 obtained at Step 3 is the optimum performance (J_0^*) of the coordinator DM0 in the decision making problem covered by Definition 3.3. Furthermore, γ_0^* is a hierarchical equilibrium strategy for DM0.

Proof. Let $\gamma_0^* \in \Gamma_0$ be hierarchical equilibrium strategy for DM0, in accordance with Definition 3.3. Furthermore, let (γ_1^*, γ_2^*) be a pair of supremizing strategies in (11). Denote the realized values of y_0, y_1, y_2, after these strategies are applied, by y_0^*, y_1^*, y_2^*, respectively; and let

$$\gamma_0^*(y_0) = u_0^*, \qquad \gamma_1^*(y_1^*) = u_1^*, \qquad \gamma_2^*(y_2^*) = u_2^*.$$

Now, since $\gamma_2^* \in R_2(\gamma_0^*, \gamma_1^*)$, we have, from (8),

$$J_2(\gamma_0^*, \gamma_1^*, \gamma_2^*) \leq J_2(\gamma_0^*, \gamma_1^*, \gamma_2), \qquad \forall \gamma_2 \in \Gamma_2.$$

This inequality is equivalent to the following statement:

$u_2^* = \gamma_2^*(y_2^*)$ minimizes $J_2(u_0^*, u_1^*, u_2)$ over $u_2 \in U_2$

and subject to the constraints

$y_0^* = \eta_0(u_0^*, u_1^*, u_2)$, $y_1^* = \eta_1(u_0^*, u_1^*, u_2)$.

Hence,

$$u_2^* \in S_2(u_0^*, y_0^*, u_1^*, y_1^*);$$

and, by the assumption made at Step 1, u_2^* is uniquely defined by

$$u_2^* = T_2(u_0^*, y_0^*, u_1^*, y_1^*).$$

Now, the pair (u_1^*, y_1^*) is partially under the control of DM1, in the sense that he can enforce any such solution on DM2 provided that the constraints

$$y_0^* = \eta_0(u_0^*, u_1^*, T_2(u_0^*, y_0^*, u_1^*, y_1^*)),$$
$$y_1^* = \eta_1(u_0^*, u_1^*, T_2(u_0^*, y_0^*, u_1^*, y_1^*))$$

are satisfied. Therefore, the pair (u_1^*, y_1^*) has the property that it minimizes

$$J_1(u_0^*, u_1, T_2(u_0^*, y_0^*, u_1, y_1))$$

over $(u_1, y_1) \in U_1 \times Y_1$ and subject to

$$y_0^* = \eta_0(u_0^*, u_1, T_2(u_0^*, y_0^*, u_1, y_1)),$$
$$y_1 = \eta_1(u_0^*, u_1, T_2(u_0^*, y_0^*, y_0^*, u_1, y_1)).$$

The latter constraint is, in fact, an identity because of the choice of T_2, and hence

$$(u_1^*, y_1^*) \in \bar{R}_1(u_0^*, y_0^*).$$

Furthermore, this is an enforceable solution on DM2, by the third assumption of Theorem 3.3.

The pair (u_0^*, y_0^*), on the other hand, is completely under the control of DM0, which he can enforce directly on DM1 and DM2 under Assumption (ii) of Theorem 3.3, and also indirectly on DM2 under Assumption (iii). But he has no control over DM1's choice from $\bar{R}_1(u_0^*, y_0^*)$, and therefore a secured level for J_0 is

$$\begin{aligned} J_0^* &= \inf_{\gamma_0 \in \Gamma_0} \sup_{\gamma_1 \in R_1(\gamma_0)} \sup_{\gamma_2 \in R_2(\gamma_0, \gamma_1)} J_0(\gamma_0, \gamma_1, \gamma_2) \\ &= \inf_{(u_0, y_0) \in U_0 \times Y_0} \sup_{(u_1, y_1) \in \bar{R}_1(u_0, y_0)} J_0(u_0, u_1, T_2(u_0, y_0, u_1, y_1)) = J_0^0. \end{aligned} \tag{40}$$

This completes the proof of the theorem. □

It should be noted that the latter expression (40) provides an optimum performance level for DM0 without requiring existence of a hierarchical equilibrium strategy.

Remark 3.6. The assumption made at Step 1, concerning the uniqueness of the solution of the static optimization problem, seems to be essential for the decomposition of the original dynamic problem into a sequence of static optimization problems, as presented here.

Remark 3.7. As a counterpart of Remarks 3.3 and 3.5, the lower bound on the realized value of J_0 will be

$$J_{0^*} = \inf_{(u_0, y_0) \in U_0 \times Y_0} \; \inf_{(u_1, y_1) \in \bar{R}_1(u_0, y_0)} J_0(u_0, u_1, T_2(u_0, y_0, u_1, y_1)). \tag{41}$$

4. Illustrative Numerical Example

To illustrate the results of Section 3, consider the three-stage dynamic decision problem described, in extensive form, by the scalar state equation

$$\begin{aligned} x(1) &= x(0) + u_2, \qquad x(0) = 1, \\ x(2) &= x(1) + u_1, \\ x(3) &= x(2) + u_0, \end{aligned} \tag{42}$$

and the state observation equations

$$y_1 = x(1), \qquad y_0 = x(2).$$

In other words, DM2 acts at Stage 0, with control variable u_2, and knows only the value of $x(0)$, which is also known by DM1 and DM0. DM1 acts at Stage 1, with control variable u_1, and knows $x(1)$. Finally, DM0 acts at Stage 2, with control variable u_0, and has access to $x(2)$ [but not to $x(1)$]. The cost functions are taken as

$$\begin{aligned} J_2 &= [x(3)]^2 + 2[u_2]^2, \\ J_1 &= [x(3)]^2 + [u_1]^2, \\ J_0 &= \tfrac{1}{2}[J_1 + J_2] + [u_0]^2. \end{aligned} \tag{43}$$

By eliminating the state variables, we obtain the equivalent representation

$$\begin{aligned} J_2 &= [1 + u_0 + u_1 + u_2]^2 + 2[u_2]^2, \\ J_1 &= [1 + u_0 + u_1 + u_2]^2 + [u_1]^2, \\ J_0 &= [1 + u_0 + u_1 + u_2]^2 + [u_0]^2 + \tfrac{1}{2}[u_1]^2 + [u_2]^2, \end{aligned} \tag{44}$$

where

$$u_0 = \gamma_0(y_0), \qquad u_1 = \gamma_1(y_1), \qquad u_2 = \text{const},$$

and

$$y_0 = \eta_0(u_1, u_2) = 1 + u_1 + u_2,$$
$$y_1 = \eta_1(u_2) = 1 + u_2. \tag{45}$$

The optimum performances for DM0, in the three cases treated in Section 3, is computed as follows.

Case (A). *Step 1.* For $\mu \in [0, 1]$, the minimum of

Step 1. For $\mu \in [0, 1]$, the minimum of

$$\bar{J}_\mu = [1 + u_0 + u_1 + u_2]^2 + \mu[u_1]^2 + 2(1-\mu)[u_2]^2$$

over

$$S = \{(u_1, u_2) \in \mathscr{R} \times \mathscr{R} : y_0 = 1 + u_1 + u_2\}, \tag{46}$$

and for fixed $(u_0, y_0) \in \mathscr{R} \times \mathscr{R}$, is attained uniquely at

$$u_1^0 = 2(1-\mu)(y_0 - 1)/(2-\mu), \qquad u_2^0 = \mu(y_0 - 1)/(2-\mu).$$

Therefore,

$$T_\mu(u_0, y_0) = [2(1-\mu), \mu](y_0 - 1)/(2-\mu).$$

Step 2. We have to consider the minimization of

$$\max_{\mu \in [0,1]} \{[u_0 + y_0]^2 + [u_0]^2 + 2[(1-\mu)^2/(2-\mu)^2](y_0 - 1)^2$$
$$+ [\mu^2/(2-\mu)^2](y_0 - 1)^2\}$$

over $(u_0, y_0) \in \mathscr{R} \times \mathscr{R}$. This min-max optimization problem indeed admits a saddle point which is given by

$$u_0^* = -\tfrac{1}{3}, \qquad y_0^* = \tfrac{2}{3}, \qquad \mu^* = 1,$$

and consequently,

$$J_0^0 = \tfrac{1}{3} = J_0^*.$$

A lower bound on the realized value of J_0, as given by (23), is obtained by taking

$$u_0 = -\tfrac{1}{5}, \qquad y_0 = \tfrac{2}{5}, \qquad \mu = \tfrac{1}{2},$$

leading to

$$J_{0*} = \tfrac{1}{5},$$

which is, in fact, the lowest possible value that J_0 can attain. An appealing hierarchical equilibrium strategy $\gamma_0^*(y_0)$ for DM0 would, of course, be the one that secures the cost level of

$$J_0^* = \tfrac{1}{3}$$

in case the followers select

$$\mu = 1,$$

but also realizes the lower bound

$$J_{0*} = \tfrac{1}{5}$$

in case the followers' choice is

$$\mu = \tfrac{1}{3}.$$

It is not known, at this stage, whether such a strategy exists for DM0 and how it can be computed.

It should be noted that all conditions of Theorem 3.1 are fulfilled in this example: Assumption (i) holds because both J_1 and J_2 are strictly convex on $S(u_0, y_0)$; Assumption (ii) holds as shown above (since an explicit saddle-point solution has been obtained); and Assumption (iii) holds by taking

$$\hat{\gamma}_0(y_0) = Ky_0,$$

where $K > 0$ is a sufficiently large number.

Case (B). Here, we consider only open-loop Nash solutions at the second level of hierarchy, in order to avoid any informational nonuniqueness.

Step 1. On the constraint set $S(u_0, y_0)$,

$$J_1 = [u_0 + y_0]^2 + [u_1]^2, \qquad J_2 = [u_0 + y_0]^2 + 2[u_2]^2,$$

with

$$u_1 + u_2 = y_0 - 1.$$

Therefore,

$$\bar{R}_n(u_0, y_0) = S(u_0, y_0),$$

where the latter is given by (46). Furthermore,

$$\bar{R}_{an}(u_0, y_0) = \{(u_1, u_2) \in S: \min(y_0 - 1, 0) \leq u_1 \leq \max(y_0 - 1, 0)\}.$$

Step 2. On $\bar{R}_{an}(u_0, y_0)$, J_0 can be written as

$$\hat{J}_0 = [u_0 + y_0]^2 + [u_0]^2 + \tfrac{1}{2}[u_1]^2 + [y_0 - 1 - u_1]^2,$$

with

$$\underline{u}_1 \triangleq \min(y_0 - 1, 0) \leq u_1 \leq \max(y_0 - 1, 0) \triangleq \bar{u}_1.$$

It now easily follows that

$$\sup_{(u_1,u_2)\in \bar{R}_{an}} J_0 = \max_{\underline{u}_1 \le u_1 \le \bar{u}_1} \hat{J}_0 = [u_0 + y_0]^2 + [u_0]^2 + [y_0 - 1]^2,$$

and hence Ineq. (26) leads to the unique solution

$$u_0^* = -\tfrac{1}{3}, \qquad y_0^* = \tfrac{2}{3},$$

with the optimum performance level being

$$J_0^0 = \tfrac{1}{3} = J_0^*.$$

Note that this is the same as the optimum performance level obtained in Case (A) under cooperative action of the followers. Moreover, the value of J_{0*} is also the same as in Case (A), i.e.,

$$J_{0*} = \tfrac{1}{5}.$$

Both conditions of Theorem 3.2 are satisfied in this example; the second one can be shown again by taking

$$\hat{\gamma}_1(y_0) = K y_0,$$

where $K > 0$ is sufficiently large.

Case (C). *Step 1.* The set $S_2(u_0, y_0, u_1, y_1)$ is a singleton

$$S_2 = \{u_2 = y_1 - 1\},$$

provided that a hierarchical equilibrium solution exists for the problem. Note that we are not using the second constraint that involves η_0, since it is redundant. Hence, the minimization problem at this step is a trivial one, leading to

$$u_2 = T_2(y_1) = y_1 - 1.$$

Step 2. The constraint set S_1 is

$$S_1 = \{(u_1, y_1) \in \mathscr{R} \times \mathscr{R} : u_1 = y_0 - y_1\},$$

and on this set $J_1(u_0, u_1, T_2(y_1))$ can be written as

$$\hat{J}_1(u_0, u_1, y_0) = [u_0 + y_0]^2 + [u_1]^2,$$

whose unique minimum is attained at

$$u_1^0 = T_{11}(u_0, y_0) = 0, \qquad y_1^0 = T_{12}(u_0, y_0) = y_0.$$

Step 3. We now consider the minimization of

$$J_0(u_0, T_{11}, T_2(u_0, y_0, T_{11}, T_{12}(y_0))) = [u_0 + y_0]^2 + [u_0]^2 + [y_0 - 1]^2,$$

over $(u_0, y_0) \in \mathcal{R} \times \mathcal{R}$. The unique solution is

$$u_0^* = -\tfrac{1}{3}, \qquad y_0^* = \tfrac{2}{3},$$

with the corresponding optimum performance level being

$$J_0^0 = \tfrac{1}{3} = J_0^*,$$

which is again the same as the ones obtained in the other two cases. However, the performance here is more robust, since the lower bound is also

$$J_{0^*} = \tfrac{1}{3}.$$

All conditions of Theorem 3.3 are satisfied here, with

$$\hat{\gamma}(y_1) = K_1 y_1, \qquad \hat{\gamma}_0(y_0) = K_0 y_0,$$

where $K_1 > 0$, $K_0 > 0$ are sufficiently large numbers.

5. Conclusions

This paper has presented an indirect method for derivation of performance bounds for $(N+1)$-person deterministic hierarchical decision problems with partial dynamic information, $N+1>2$. This indirect method converts the original, highly nontrivial dynamic problem into a sequence of static optimization problems, whose solutions yield the optimum performance bounds sought. Even though the derivation of appealing hierarchical equilibrium strategies still remains a challenging task[6] for the general class of problems, the bounds obtained in this paper serve to compare performances of some *simple* suboptimal strategies for the coordinator against the best performance achievable.

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