Existence of Dominant Solutions in Linear Output Feedback

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Abstract—**The standard LQ-regulator is known to be a dominant controller in the sense that the optimal cost is minimal for all initial states $x_0 \in R^n$. In general, static or dynamic output feedback controllers do not have this dominance property. In this note, it is shown that in the general multivariable case and for the original cost functional, a dynamic output feedback controller using an observer is dominant if, and only if, the observer is perfectly initialized.**

I. Introduction

The standard linear quadratic regulator problem for a system with the state $x(t) \in R^n$, the control $u(t) \in R^m$, and fixed initial time t_0, initial state x_0, and final time t_1 is considered.

Plant:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Cost functional to be minimized:

$$J = x^T(t_1)Fx(t_1) + \int_{t_0}^{t_1} \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}\ dt \tag{3}$$

with

$$F = F^T \geqslant 0,\ Q(t) = Q^T(t) \geqslant 0,\ R(t) = R^T(t) > 0.$$

It is well known that the resulting optimal linear state feedback controller

$$u(t) = -R^{-1}(t)B^T(t)K(t)x(t) \tag{4}$$

involving the matrix Riccati differential equation

$$-\dot{K}(t) = A^T(t)K(t) + K(t)A(t) + Q(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) \tag{5}$$

$$K(t_1) = F \tag{6}$$

is a superior or dominant solution in the sense that the optimal cost

$$J^* = x_0^T K(t_0)x_0 \tag{7}$$

is minimal for every initial state $x_0 \in R^n$. In particular, for any other linear state feedback controller

$$u(t) = -L(t)x(t) \tag{8}$$

the corresponding quadratic cost J_L satisfies the inequality

$$J_L = x_0^T V_L x_0 \geqslant J^* = x_0^T K(t_0)x_0 \qquad \text{for all } x_0 \in R^n \tag{9}$$

or, equivalently, the suboptimal symmetric cost matrix V_L satisfies the matrix inequality

$$V_L \geqslant K(t_0) \tag{10}$$

(partial order of positive semidefinite difference).

In many applications, it is not possible or economically feasible to measure all of the state variables. Therefore, in a vast body of literature, e.g., [1]–[13], the problems of optimal constrained state feedback have been studied. In the cases of static output feedback and dynamic output feedback utilizing the measurement $y(t) \in R^p$,

$$y(t) = C(t)x(t), \tag{11}$$

the optimal control problem (1), (2), (3), (11) is augmented by the following constraints, respectively.

Static output feedback:

$$u(t) = G(t)y(t). \tag{12}$$

Dynamic output feedback:

$$u(t) = G(t)y(t) + H(t)z(t) \tag{13}$$

$$\dot{z}(t) = P(t)z(t) + N(t)y(t) \tag{14}$$

$$z(t_0) = z_0 \tag{15}$$

where $z(t) \in R^s$ is the state of the dynamic compensator or observer (14), (15). It turns out that the optimal solutions to either of the problems are not dominant in the sense of (9) or (10) (with $K(t_0)$ replaced by the cost matrix V^* of the corresponding solution), i.e., the optimal constrained state feedback solutions depend on the initial state x_0 of (2). This is unpleasant since, in view of (11), it is not very practical to assume perfect knowledge of the initial state. Furthermore, tuning the controller gains G or G, H, P, and N, respectively, to x_0 would not be desirable in most cases.

In [1], an optimal controller is chosen which minimizes the maximum (over $x_0 \in R^n$) of the relative deviation of the cost (3) from the optimal cost (7) (attained with unconstrained state feedback).

In [2]–[6], the dependence of the controller on the initial state is removed by assuming a zero-mean random initial state with covariance matrix

$$E\{x_0x_0^T\} = X_0 \tag{16}$$

and by minimizing the expected value of the cost (3)

$$E\{J\} = E\{x_0^T V x_0\} = \text{ tr } \{VX_0\} \tag{17}$$

where V is the resulting positive (semi) definite cost matrix. The covariance matrix X_0 should be considered as a design variable. Usually, $X_0 = I$ is used. The papers [7], [8], [9] contain alternative rederivations of the above-mentioned results. In [11], a computational algorithm is proposed for finding the gain matrices G, H, P, and N.

In [10], structural properties of the system (1), (11) are investigated which guarantee that the optimal static output feedback controller (3), (11) is a dominant or superior solution. The theory is only applicable to systems with rank (B) + rank $(C) > n$.

The problem of finding an improved controller (11) which dominates a given initial design of an output feedback controller of the same type is treated in [12].

The purpose of this paper is to show that the *dominance property* of the LQ-regulator using state feedback is *retained* in the general case of dynamic output feedback provided that a (*full-order* or *reduced-order*) *state observer* is used and provided that this *observer* is *properly initialized.*

In Section II, the problem of finding a superior solution to the standard constrained linear quadratic regulator problem using dynamic output feedback with an observer is treated. In the concluding Section III, the results are summarized and some open research problems are pointed out.

II. Dynamic Output Feedback with a Full-Order Observer

The statement of the infimization problem is given in Section II-A. The design parameters are the gain matrix $M(t)$ and the initial state z_0 of the full-order state observer and the gain matrix $L(t)$ of the controller. The results are stated in Section-B. Section II-C contains some remarks on the proof which can be found in [20].

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A. The Optimal Control Problem

The following deterministic linear control system with dynamic output feedback control utilizing a full-order observer is considered.

Plant:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{18}$$

$$x(t_0) = x_0 \tag{19}$$

$$y(t) = C(t)x(t). \tag{20}$$

Observer:

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + M(t)\{y(t) - C(t)\hat{x}(t)\} \tag{21}$$

$$\hat{x}(t_0) = z_0. \tag{22}$$

Controller:

$$u(t) = -L(t)\hat{x}(t). \tag{23}$$

Dimensions:

$$x(t) \text{ and } \hat{x}(t) \in R^n, \; u(t) \in R^m,$$

$$y(t) \in R^p.$$

Ranks:

$$B(t) \in R^{n\times m} \text{ of full rank } m,$$

$$C(t) \in R^{p\times n} \text{ of full rank } p.$$

The optimal control problem consists of finding $z_0 \in R^n$, $L(t) \in R^{m\times n}$, and $M(t) \in R^{n\times p}$ such that the cost functional

$$J = x^T(t_1)Fx(t_1) + \int_{t_0}^{t_1} \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}\, dt \tag{24}$$

$$\text{with } F \geqslant 0, \; Q(t) > 0, \; R(t) > 0$$

is minimized for *all* initial states $x_0 \in R^n$. In other words, a superior or dominant solution is sought.

B. Results

In the case where the initial state x_0 is *unknown*, there is no triple $L(t)$, $M(t)$, and (constant) z_0 which satisfies all of the necessary conditions of the infimum principle [15]. Also, for $M(t)$ held fixed (i.e., for a fixed choice of the observer gain) there is no pair $L(t)$, z_0 satisfying the infimum principle. In other words, in general there is no dominant solution to the optimal control problem of Section II-A if the initial state x_0 is unknown. The results can be summarized as follows.

Theorem: The restricted state feedback problem utilizing a full-order state observer, (18)–(24), has a superior control gain $L(t)$ for every arbitrary choice of the observer gain $M(t)$ if and only if the observer is initialized with the true state x_0. Furthermore, the optimal control gain $L(t)$ is identical to the Riccati gain $R^{-1}(t)B^T(t)K(t)$ of (4)–(6), (8).

Remarks:

1) For any full-order state observer which is correctly initialized with the true initial state x_0, the Riccati gains are optimal with respect to the cost functional (3), (24) for every initial state $x_0 \in R^n$. The optimal cost is $J^* = x_0^T K(t_0)x_0$ and is independent of the observer gain $M(t)$ (because there is no error transient due to the correct initial observer state). Since there is no optimal observer gain it must be chosen by criteria such as noise rejection or observer pole location.

2) It can be proved that the analogous situation arises when a reduced-order state observer is used. By definition [16, p. 329], there exists an initial state of the observer such that the reconstructed state matches the true state for all $t \geqslant t_0$. Again the Riccati gain is superior if and only if the observer is perfectly initialized.

3) These results can be carried over to the time-invariant dynamic output feedback regulator problem with infinite time horizon in a straightforward way.

C. Outline of Proof

The basic steps in the proof of the above theorem are as follows. 1) Introduce the observer error $e(t) = x(t) - \hat{x}(t) \in R^n$. 2) Introduce the augmented state $q(t) \in R^{2n}$, $q(t) = [x^T(t), e^T(t)]^T$. 3) Using (18)–(23) write the cost functional (24) in the form $J = q^T(t_0)\tilde{V}(t_0)q(t_0)$ where $\tilde{V}(t) \in R^{2n\times 2n}$ is the solution of a Lyapunov differential equation. 4) In order to find the desired dominant solution use [15]. For details, the reader is referred to [20].

III. Conclusions

A. Results

Unfortunately, the result of this note is negative: the superiority of the LQ regulator using state feedback does not carry over to dynamic LQ output feedback where a full-order or reduced-order observer is used unless the observer is perfectly initialized.

As was mentioned in the Introduction, for the static output feedback controller (12) the covariance matrix X_0 (16) of the initial state is a design parameter in much the same way as the weighting matrices Q and R in the cost functional (17), (3) are.

It may be interesting to note here that a similar situation with respect to dominance occurs in differential games [17]–[19].

B. Open Areas for Research

The analysis in this note has used the standard quadratic cost functional (3). There may be other cost functionals, structured in a slightly different way, or there may be systems with a special structure where the dominance property of the state feedback is retained in the dynamic output feedback case.

To the authors' knowledge, the use of the covariance matrix X_0 as a design parameter in static output feedback has not been investigated yet. It should be of particular interest to find out whether the robustness of the output feedback regulator can be influenced by the choice of X_0.

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