Quantization in H^∞ Parameter Identification

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***Abstract*—We consider the design of quantizers for parameter identification under communication constraints. The objective is to find coarse quantization methods that achieve reasonable performance based on an H^∞ criterion. We make use of feedback in communication and show that the problem induces a class of quantizers similar to logarithmic ones. We then exhibit through examples that the proposed identifier is advantageous over a conventional prefiltering scheme, especially when quantization error is large.**

***Index Terms*—H^∞ synthesis, quantization, robust identification.**

I. Introduction

Networks play an increasingly important role in the communication for control systems that are physically distributed and consist of large numbers of components. In the design of such systems, it is essential to consider the characteristics of the communication so that the control objectives are met under various conditions of the network. One interesting and recent approach is to develop control schemes which require very little communication. For more on this general subject, see, e.g., [6] and the references therein.

In this paper, we consider the problem of parameter identification based on quantized measurements. The general system setup is as follows: The plant to be identified is in a remote location, where little computation is available for identification to take place. Hence, the plant measurements are transmitted through a wireless channel to an identifier. Because of the limited bandwidth, the measurements are sampled and quantized, but possibly very coarsely; this means that the information received by the identifier may be a crude approximation of the actual measurements. Now, the question is how crude can this information be to still obtain acceptable performance. This setup can arise when micro sensors with simple components are used to monitor a system. Each sensor may have access to only partial measurement and/or may not have sufficient computation capability to do identification on its own. Hence, the identification is done at a base station where more resources for computation and power are available; this structure motivates us to employ feedback in the communication from the base station back to the plant side.

Quantization effects on control systems have gained much attention in recent years, especially in stabilization problems (see, e.g., [3], [4], [7], [8], [11], [12], and [15]). In [16], such studies have been extended to identification problems over networks in a discrete-time least-squares setup for MA models. On the other hand, quantized identification has been studied from other perspectives. In [10], quantized least-squares techniques are developed to reduce the computational cost incurred in optimal schemes. In [9], for errors-in-variable models, an identification algorithm based on binary series is proposed. A stochastic gradient algorithm is studied in [17] for finite impulse response models using coarsely quantized measurements.

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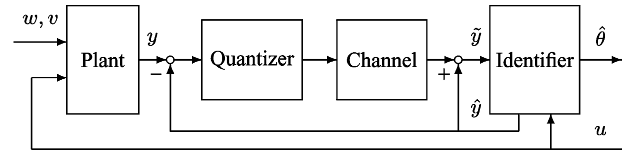


Fig. 1. Parameter identification over a network.

Here, we consider the robust parameter identification scheme proposed in [2], [13] and, in particular, study how quantization affects the performance in this framework. Our first goal is to design an identifier and a static quantizer so that the unknown parameters are estimated in an L^2 setting. We allow feedback in the communication so that estimation from the identifier is used to reduce the number of levels in quantization. We show that the H^∞ identification problem induces a class of quantizers similar to logarithmic ones; such quantizers have been studied for stabilization problems (e.g., [3], [4], and [8]). To deal with the persistent presence of quantization errors, we also study a finite horizon version and allow small estimation error. We then provide a third-order system example and compare the performances between the proposed scheme and a more classical one based on prefiltering [14].

This paper is organized as follows: In Section II, we formulate the quantized identification problem. In Section III, we present the main result and construct the identifier and the quantizer. A more practical version of the problem is considered in Section IV. In Section V, a numerical example is given to illustrate the results. Finally, in Section VI, we give the conclusion.

II. Problem Formulation

In this section, we first describe the setup for the design problem of a quantizer and an identifier, and then formulate it as an H^∞ disturbance attenuation problem.

We consider a situation where the plant to be identified is equipped with micro sensors which communicate with the identifier via wireless channels. The identifier is located at a base station and has access to more computation/communication resources. Thus, information can be fed back to the sensors with much wider bandwidth where quantization effects can be ignored. The motivation of this study is to reduce the amount of transmission by the sensors and to exploit the system structure.

Specifically, we consider the system depicted in Fig. 1. The plant to be identified has the unknown parameters θ, the input u, and the disturbance/noise v, w. The measurement y is transmitted over a discrete noiseless channel to the identifier side, where the estimate $\hat{\theta}$ of the parameter vector θ, and the estimate $\hat{y}$ of the output y are computed. We employ a communication scheme with feedback, and in particular, the estimated output $\hat{y}$ is sent to the plant side without any error. Thus, the error $y - \hat{y}$ in output estimation is quantized to be sent to the identifier.

Here, we assume that the input u is known to the identifier. For example, this input may be generated at the base station and then sent to the plant, possibly prior to operation. Moreover, the focus in this study is quantization effects; thus, it is assumed that sampling occurs implicitly whenever the quantized measurement takes a different value. We will see that limited information caused by coarse quantization alone can be harmful for the performance in identification.

The details of the system in Fig. 1 are described as follows: The plant is an SISO linear time-invariant system whose transfer function is given by

$$P(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{1}$$

where $m < n$. The coefficients $a_{n-1}, \ldots, a_0, b_m, \ldots, b_0 \in \mathbb{R}$ are unknown and are to be identified. In particular, we study the system (1) in the linearly parameterized state equation form with additive disturbance and noise, given by

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2(y(t), u(t))\theta + w(t) \\ y(t) &= Cx(t) + v(t) \end{aligned} \tag{2}$$

where $\theta := [-a_{n-1} \ldots -a_0 b_m \ldots b_0]^T \in \mathbb{R}^r$ is the vector of unknown parameters with $r := n + m + 1$; $x(t) \in \mathbb{R}^n$ is the unknown state with an initial condition $x(0) = x_0$, $u(t) \in \mathbb{R}^m$ is the known bounded input, and $w(t) \in \mathbb{R}^n$ is the unknown disturbance. The measurement $y(t) \in \mathbb{R}$ is corrupted by the noise $v(t)$. In (2), the matrices are defined as follows:

$$\begin{aligned} A_1 &:= \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n} \\ A_2(y, u) &:= A_{21} y + A_{22} u \in \mathbb{R}^{n \times r} \\ C &:= [1 \quad 0 \quad \cdots \quad 0] \in \mathbb{R}^{1 \times n} \end{aligned}$$

where

$$A_{21} := [I_n \quad 0], \quad A_{22} := \begin{bmatrix} 0 & 0 \\ 0 & I_{m+1} \end{bmatrix}.$$

Note that the matrix $A_2(y, u) \in \mathbb{R}^{n \times r}$ is linear in the two arguments. From the state-space equation (2), the input-output behavior of the plant (1) can be recovered if there is no disturbance/noise ($w = 0$ and $v = 0$). Now, letting $a := [-a_{n-1} \cdots -a_0]^T \in \mathbb{R}^n$, we assume $|a| \leq \bar{a}$, where $\bar{a}$ is a positive scalar. Further, we assume that w and v are both in $L^2 \cap L^\infty$.

Over the channel, the identifier receives the (decoded) measurement $\tilde{y}(t)$ at each time t. Based on this information, the identifier, denoted by δ, calculates the estimate $\hat{\theta}(t) \in \mathbb{R}^{n+m+1}$ of the parameter vector θ as well as the estimate $\hat{x}$ of the state x

$$\left(\hat{\theta}(t), \hat{x}(t)\right) := \delta(t, \tilde{y}_{[0,t]}, u_{[0,t]}),$$

where δ is piecewise continuous in t and Lipschitz continuous in $\tilde{y}_{[0,t]}$ and $u_{[0,t]}$. In addition, we introduce the notation for the estimated output as $\hat{y} := C\hat{x}$.

As we mentioned above, we consider a communication scheme with feedback. Specifically, as the most recent information, the identifier uses $\tilde{y}$ given by

$$\tilde{y}(t) := \hat{y}(t) + q(y(t) - \hat{y}(t)) \tag{3}$$

where q is the quantizer, which is a piecewise-constant function. Notice that in (3), the second term $q(y(t)-\hat{y}(t))$ is sent over the channel, while $\hat{y}(t)$ is available locally at the identifier.

Under this quantization method, we employ the so-called logarithmic quantizers, which have been found useful in stabilization problems (e.g., [3], [4], and [8]). Quantization of this type is *fine* around the origin of the input, but becomes coarser further away from the origin. Since, in our scheme, it is the estimation error or the innovation that is quantized, it is natural to obtain more accurate information when the error is small in magnitude; on the other hand, coarse information suffices when the error is large.

A logarithmic quantizer q is defined as follows: Take $\eta > 1$ and let $\rho := (\eta - 1)/(\eta + 1)$. Then

$$q(z) := \begin{cases} 0, & \text{if } z = 0, \\ \pm(1+\rho)\eta^j, & \text{if } z \in \pm[\eta^j, \eta^{j+1}], \, j \in \mathbb{Z}. \end{cases} \tag{4}$$

This is a sector-type, discontinuous nonlinearity and, in fact, satisfies the following inequality:

$$|z - q(z)| \leq \rho |z|. \tag{5}$$

Notice that for a piecewise constant function to be a sector nonlinearity, it takes an infinite number of discrete levels around the origin.

A much simpler quantization scheme is to quantize y directly (without feeding back $\hat{y}$) and use $\tilde{y}(t) = q(y(t))$ with some q. We find however that under our scheme, this results in uniform quantizers having small quantization steps. This is a consequence of the need to keep the quantization error small for any input to the quantizer regardless of its magnitude. Such an extension is straightforward and thus is not presented in the paper. We also note that quantization of the innovation has been found useful for the purpose of control (see, e.g., [18]).

We now formulate the worst-case disturbance attenuation problem in an L^2 setting. The estimate $\hat{\theta}$ should exhibit a certain level of performance in the presence of the unknown disturbance/noise and the quantization error whose structure is known to the identifier. Though the estimation may not converge to the true value of θ, it should converge to a neighborhood of θ with an *a priori* known bound while maintaining a good transient response.

For an identifier δ and a quantizer q, the performance index is given by the equation shown at the bottom of the page, where $|\cdot|_R$ is the Euclidean norm weighted by R. The weights are assumed to satisfy $Q(t) \geq 0$ for all $t \geq 0$, $\mu > 0$, $Q_0 > 0$, and $P_0 > 0$. The vectors $\bar{x}_0 \in \mathbb{R}^n$ and $\bar{\theta}_0 \in \mathbb{R}^l$ represent the initial estimates by the designer for $x(0)$ and θ, respectively.

The main problem of this paper is as follows: Let $T = \infty$. Given the performance level $\gamma > 0$, design an identifier δ and a quantizer q, if they exist, so that the performance index satisfies $L_\infty(\delta, q) < \gamma$. In this problem, the quantizer may take an infinite number of quantization levels; this assumption will be relaxed later by considering a finite horizon case.

The difficulty in this identification problem lies in the information structure of the overall system. The quantizer has access only to the measurement y and neither to the parameter vector θ nor the states in the identifier, e.g., $\hat{\theta}$ and the estimated state $\hat{x}$, which will be introduced in the design procedure. This is a reasonable structure in the context of identification over networks using simple components at the plant side.

$$L_T(\delta, q) := \sup_{\substack{x_0, \theta, \\ w_{[0,T]}, \\ v_{[0,T]}}} \frac{\int_0^T |\theta - \hat{\theta}(t)|^2_{Q(t)} dt}{\int_0^T [|w(t)|^2 + \mu^2 |v(t)|^2] dt + |\theta - \bar{\theta}_0|^2_{Q_0} + |x_0 - \bar{x}_0|^2_{P_0}}$$

III. Structured Cost-to-Come Function Approach

In this section, we introduce a class of functions called structured cost-to-come functions. The quantized identification problem is solved by finding an appropriate function in this class.

We employ here a game theoretic approach. To this end, we first associate the original problem with a soft constrained differential game which has a cost function of the following form:

$$\begin{aligned} &J_{\gamma,T}(\delta, q; x_0, \theta, w_{[0,T]}, v_{[0,T]}) \\ &:= \int_0^T \left[|\theta - \hat{\theta}(t)|^2_{Q(t)} - \gamma^2 \left(|w(t)|^2 + \mu^2 |v(t)|^2 \right) \right] dt \\ &\quad - \gamma^2 \left[|\theta - \bar{\theta}_0|^2_{Q_0} + |x_0 - \bar{x}_0|^2_{P_0} \right]. \end{aligned} \tag{6}$$

In this game, the identifier δ and the quantizer q serve as the minimizer while the maximizers are $(x_0, \theta, w_{[0,T]}, v_{[0,T]})$.

Further, to formulate the problem as an affine quadratic minimax controller design, we rewrite the system as follows. Let the new state be $\xi := [\,\theta^T \quad x^T\,]^T$. As θ is constant, it is associated with the trivial dynamics $\dot{\theta} = 0$. Hence, the state equation now becomes

$$\begin{aligned} \dot{\xi}(t) &= \begin{bmatrix} 0 & 0 \\ A_2(y(t), u(t)) & A_1 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} w(t) \\ &=: \bar{A}(y(t), u(t))\xi(t) + \bar{D} w(t) \\ y(t) &= [\,0 \quad C\,]\,\xi(t) + v(t) \\ &=: \bar{C}\xi(t) + v(t). \end{aligned} \tag{7}$$

For this system, the cost in (6) can be expressed as

$$\begin{aligned} &J_{\gamma,T}(\delta, q; \xi_0, w_{[0,T]}, v_{[0,T]}) \\ &= \int_0^T \left[|\xi(t) - \hat{\xi}(t)|^2_{\bar{Q}(t)} - \gamma^2 \left(|w(t)|^2 + \mu^2 |v(t)|^2 \right) \right] dt \\ &\quad - \gamma^2 |\xi(0) - \bar{\xi}_0|^2_{\bar{Q}_0} \end{aligned} \tag{8}$$

where the new weights are $\bar{Q}(t) := \operatorname{diag}\{Q(t), 0\}$ and $\bar{Q}_0 := \operatorname{diag}\{Q_0, P_0\}$, the estimated state is $\hat{\xi}(t) := [\hat{\theta}(t)^T \ \hat{x}(t)^T]^T$, and its initial value is $\hat{\xi}(0) = \bar{\xi}_0 := [\bar{\theta}_0^T \ \bar{x}_0^T]^T$.

Our design of the identifier and the quantizer is based on the construction of *structured cost-to-come functions*[1]. Such functions can be viewed as a generalization of the cost-to-come, which have been found useful especially in nonlinear H^∞ control and robust parameter identification. See, e.g., [2] and [13].

In the present context, structured cost-to-come functions are defined as follows.

Definition 3.1: For any given δ and q, we say that a function $\bar{W}(t, \xi; \tilde{y}_{[0,t]}, u_{[0,t]})$ is a structured cost-to-come function if it satisfies the following three conditions.

1) It is continuously differentiable in both t and ξ.
2) For any given $(t, \tilde{y}_{[0,t]}, u_{[0,t]}, w)$ and any (ξ, v) such that $\tilde{y}(t) = \bar{C}\hat{\xi}(t) + q(\bar{C}\xi + v - \bar{C}\hat{\xi}(t))$, where $\hat{\xi}(t) = \delta(t, \tilde{y}_{[0,t]}, u_{[0,t]})$, it holds that

$$\begin{aligned} &-\frac{\partial \bar{W}}{\partial t}(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]}) \\ &-\frac{\partial \bar{W}}{\partial \xi}(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]})(\bar{A}(\bar{C}\xi + v, u(t))\xi + \bar{D}w) \\ &+ |\xi - \hat{\xi}(t)|^2_{\bar{Q}(t)} - \gamma^2[|w|^2 + \mu^2|v|^2] \le 0. \end{aligned} \tag{9}$$

3) $\bar{W}(0, \xi) \ge -\gamma^2 |\xi - \bar{\xi}_0|^2_{\bar{Q}_0}$ for all ξ.

The next lemma will be useful in the sequel. Its proof follows from the definition above.

Lemma 3.2: Given an identifier δ and a quantizer q, suppose that there is a structured cost-to-come function $\bar{W}(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]})$ for the overall system. Then, for any given $(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]})$

$$\bar{W}(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]}) \ge \sup_{\substack{\xi_0, w_{[0,t]}, v_{[0,t]} | \\ \tilde{y}_{[0,t]}, u_{[0,t]}, \xi(t) = \xi}} J_{\gamma,t}(\delta, q; \xi_0, w_{[0,t]}, v_{[0,t]})$$

where the supremum is taken over all ξ_0, $w_{[0,t]}$, and $v_{[0,t]}$ such that the trajectory $\xi_{[0,t]}$ of the plant (7) generated by them satisfies $\tilde{y}(\tau) = \bar{C}\hat{\xi}(\tau) + q(\bar{C}\xi(\tau) + v(\tau) - \bar{C}\hat{\xi}(\tau))$ for all $\tau \in [0, t]$ and $\xi(t) = \xi$; and the estimated state is given by $\hat{\xi}(t) = \delta(t, \tilde{y}_{[0,t]}, u_{[0,t]})$.

In what follows, as such a function, we consider the one given by

$$\bar{W}(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]}) = -\gamma^2 |\xi - \bar{\xi}(t)|^2_{\Sigma(t)} - m(t) \tag{10}$$

where $\Sigma(t) \in \mathbb{R}^{(r+n)\times(r+n)}$, $\bar{\xi}(t) \in \mathbb{R}^{r+n}$, and $m(t) \in \mathbb{R}$ are governed by the differential equations

$$\dot{\Sigma} + \frac{1}{\gamma^2}\bar{Q} + \Sigma\bar{D}\bar{D}^T\Sigma + \Sigma\tilde{\bar{A}} + \tilde{\bar{A}}^T\Sigma - \mu^2\bar{C}^T\bar{C} = -\nu I \tag{11}$$

$$\dot{\bar{\xi}} = \tilde{\bar{A}}\bar{\xi} + \frac{1}{\gamma^2}\Sigma^{-1}\bar{Q}(\bar{\xi} - \hat{\xi}) + \mu^2\Sigma^{-1}\bar{C}^T(\tilde{y} - \bar{C}\bar{\xi}) \tag{12}$$

$$\dot{m} = -|\bar{\xi} - \hat{\xi}|^2_{\bar{Q}} \tag{13}$$

with the initial conditions $\Sigma(0) = \bar{Q}_0$, $\bar{\xi}(0) = \bar{\xi}_0$, and $m(0) = 0$. Here, ν is a positive scalar. We note that we dropped the dependence on t for the matrices and vectors and also used the shorthand $\tilde{\bar{A}} := \bar{A}(\tilde{y}, u)$. Partition the matrix Σ as

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix}, \text{ where } \Sigma_1 \in \mathbb{R}^{r\times r}.$$

We are now ready to construct the identifier and the quantizer for the system to achieve disturbance attenuation. As the identifier, let

$$\delta(t, \tilde{y}_{[0,t]}, u_{[0,t]}) = \hat{\xi}(t) \tag{14}$$

where $\hat{\xi}(t)$ is generated by

$$\begin{aligned} \dot{\hat{\xi}}(t) &= \bar{A}(\tilde{y}(t), u(t))\hat{\xi} + \mu^2\Sigma(t)^{-1}\bar{C}^T(\tilde{y}(t) - \bar{C}\hat{\xi}(t)) \\ \hat{\xi}(0) &= \bar{\xi}_0 \end{aligned}$$

and $\Sigma(t)$ is from (11). Note that $\hat{\xi}(t)$ is the same as $\bar{\xi}(t)$ in (12), and $m(t)$ in (13) is equal to zero for all t.

For the quantizer, let $\bar{\Sigma}_2$, $\bar{\Sigma}_3$, and $\bar{a}$ be positive scalars. Take a quantizer whose quantization error $q(z) - z$ satisfies the following inequality:

$$|q(z) - z| \le \rho_0 |z| \text{ for all } z \in \mathbb{R} \tag{15}$$

where $\rho_0 := \mu\{\nu/[(\mu^2 + \bar{\Sigma}_3\bar{a})^2 + (\bar{\Sigma}_2\bar{a})^2]\}^{1/2}$. Clearly, according to (5), there exists a logarithmic quantizer that meets the bound above.

We now state the main result of the paper.

Theorem 3.3: Given a scalar $\gamma > 0$, construct the identifier in (14) and the quantizer in (15) as above. Suppose that $\Sigma(t)$ satisfies the following conditions: i) $|\Sigma_2(t)| \le \bar{\Sigma}_2$; ii) $|\Sigma_3(t)| \le \bar{\Sigma}_3$; iii) $\Sigma(t) > 0$ for all t and all possible pairs $(\tilde{y}_{[0,\infty]}, u_{[0,\infty]})$ of the output and the input of the plant. Then, the following statements hold.

1) The identifier achieves the performance level γ.
2) Suppose that, in addition, the following persistency of excitation condition holds:

$$\lambda_{\min}\left(\Sigma_1(t) - \Sigma_2(t)\Sigma_3^{-1}(t)\Sigma_2^T(t)\right) \to \infty \ \text{ as } t \to \infty \tag{16}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Then, perfect parameter estimation is achieved asymptotically, that is, $\lim_{t\to\infty}\hat{\theta}(t) = \theta$.

In the theorem, the conditions i)– iii) on the matrix $\Sigma(t)$ are given in a fairly general form. Later, we will comment on how these conditions can be met and also relaxed.

Proof: 1) We first show that $\bar{W}(t,\xi)$ in (10) is a structured cost-to-come function. Clearly, it is continuously differentiable in t and ξ, and further $\bar{W}(0,\xi) = -\gamma^2|\xi - \bar{\xi}_0|^2_{Q_0}$. Hence, it remains to show that the inequality in (9) holds. Fix $(t, \tilde{y}_{[0,t]}, u_{[0,t]}, w)$ and (ξ, v) as specified there. Let

$$V := -\frac{\partial \bar{W}}{\partial t} - \frac{\partial \bar{W}}{\partial \xi}(\bar{A}\xi + \bar{D}w) + |\xi - \hat{\xi}|^2_Q - \gamma^2[|w|^2 + \mu^2|v|^2]$$

where $\bar{A} = \bar{A}(\bar{C}\xi + v, u(t))$. We show below that $V \le 0$.

Noticing that $\bar{A}(y,u)$ is linear in the two arguments, we can write $\bar{A}(\bar{C}\xi + v, u) = \bar{A}(\tilde{y}, u) - eF$, where $e := q(\bar{C}\xi + v - \bar{C}\hat{\xi}) - (\bar{C}\xi + v - \bar{C}\hat{\xi}) = q(y - \hat{y}) - (y - \hat{y})$ and

$$F := \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}.$$

Note that $\tilde{y} = y + e$. Now, we can bound V from above using (12) and (13) and then completing the square for w

$$\begin{aligned}
\frac{1}{\gamma^2}V &= \left|\xi - \bar{\xi}\right|^2_{\dot{\Sigma}} - 2\left(\xi - \bar{\xi}\right)^T \Sigma\dot{\bar{\xi}} + \dot{m} \\
&\quad + 2\left(\xi - \bar{\xi}\right)^T \Sigma(\bar{A}\xi + \bar{D}w) \\
&\quad + \left|(\xi - \bar{\xi}) + (\bar{\xi} - \hat{\xi})\right|^2_{\bar{Q}} - (|w|^2 + \mu^2|v|^2) \\
&\le \left|\xi - \bar{\xi}\right|^2_{\dot{\Sigma} + 1/\gamma^2 \bar{Q} + \Sigma\tilde{\bar{A}} + \tilde{\bar{A}}^T\Sigma + \Sigma\bar{D}\bar{D}^T\Sigma} \\
&\quad - 2\left(\xi - \bar{\xi}\right)^T\left[\mu^2\bar{C}^T(\tilde{y} - \bar{C}\bar{\xi}) + \Sigma eF\xi\right] - \mu^2|v|^2.
\end{aligned}$$

We use $v = (y - \bar{C}\bar{\xi}) - \bar{C}(\xi - \bar{\xi})$, $\tilde{y} = y + e$, and also (11) to obtain

$$\begin{aligned}
\frac{1}{\gamma^2}V \le -\nu\left|\xi - \bar{\xi}\right|^2 - 2e\left(\xi - \bar{\xi}\right)^T\left(\mu^2\bar{C}^T + \Sigma F\xi\right) \\
-\mu^2\left|y - \bar{C}\bar{\xi}\right|^2.
\end{aligned}$$

Express the right-hand side by the components θ and x of ξ and so on, and then it becomes

$$\begin{aligned}
\frac{1}{\gamma^2}V \le -\nu|x - \bar{x}|^2 - 2e(x - \bar{x})^T\left(\Sigma_3 a + \mu^2 C^T\right) - \nu\left|\theta - \bar{\theta}\right|^2 \\
-2e\left(\theta - \bar{\theta}\right)^T\Sigma_2 a - \mu^2|y - C\bar{x}|^2.
\end{aligned}$$

Finally, complete the square for $x - \bar{x}$ and $\theta - \bar{\theta}$, and then apply $|e| \le \rho_0|y - \hat{y}|$, which follows from the inequality (15). These steps would lead us to conclude that $V \le 0$. Hence, (9) is now shown, and therefore $\bar{W}(t,\xi)$ is a structured cost-to-come function.

To show that the pair δ and q is a solution, we must prove that the cost in (8) satisfies

$$\sup_{\xi_0, w_{[0,\infty)}, v_{[0,\infty)}} J_{\gamma,\infty}(\delta, q; \xi_0, w_{[0,\infty)}, v_{[0,\infty)}) \le 0. \tag{17}$$

Observe that this holds if and only if

$$\sup_{\xi_0, w_{[0,t]}, v_{[0,t]}} J_{\gamma,t}(\delta, q; \xi_0, w_{[0,t]}, v_{[0,t]}) \le 0 \text{ for all } t \ge 0.$$

This inequality can be shown as follows: For each $t \ge 0$, we have

$$\begin{aligned}
&\sup_{\xi_0, w_{[0,t]}, v_{[0,t]}} J_{\gamma,t}(\delta, q; \xi_0, w_{[0,t]}, v_{[0,t]}) \\
&\quad = \sup_{\tilde{y}_{[0,t]}, \xi} \ \sup_{\substack{\xi_0, w_{[0,t]}, v_{[0,t]} | \\ \tilde{y}_{[0,t]}, u_{[0,t]}, \xi_t = \xi}} J_{\gamma,t}(\delta, q; \xi_0, w_{[0,t]}, v_{[0,t]}) \\
&\quad \le \sup_{\tilde{y}_{[0,t]}, \xi} \bar{W}(t, \xi, \tilde{y}_{[0,t]}, u_{[0,t]}) \le 0
\end{aligned}$$

where the first inequality is due to Lemma 3.2 and the last inequality holds because $\Sigma(t) > 0$ by the condition (iii) and because the identifier (14) makes $m(\cdot) \equiv 0$. Consequently, (17) follows.

2) We next prove that asymptotic perfect estimation is achieved under the persistency of excitation condition (16). Since we have $m(\cdot) \equiv 0$, from Lemma 3.2, for every $(\tilde{y}_{[0,\infty]}, u_{[0,\infty]}, \xi_0)$, and every $w_{[0,\infty]}$ and $v_{[0,\infty]}$ in L^2, it holds that

$$\begin{aligned}
-\gamma^2|\xi(t) - \hat{\xi}(t)|^2_{\Sigma(t)} &\ge \int_0^t |\xi(\tau) - \hat{\xi}(\tau)|^2_{Q(\tau)} d\tau \\
-\gamma^2 \int_0^\infty (|w(\tau)|^2 &+ \mu^2|v(\tau)|^2)d\tau - \gamma^2|\xi(0) - \bar{\xi}_0|^2_{Q_0}.
\end{aligned}$$

The inequality above shows that $|\xi(t) - \hat{\xi}(t)|^2_{\Sigma(t)}$ is uniformly bounded for $t \ge 0$. Further, note that $|\xi(t) - \hat{\xi}(t)|^2_{\Sigma(t)} \ge |\theta - \hat{\theta}(t)|_{\Sigma_1(t) - \Sigma_2(t)\Sigma_3^{-1}(t)\Sigma_2^T(t)}$. Therefore, the uniform boundedness of the left-hand side together with the condition (16) implies that $\hat{\theta}(t) \to \theta$ as $t \to \infty$ □

Theorem 3.3 characterizes a tradeoff between the requirement in communication and the performance in identification: More communication, or finer quantization, implies faster identification. The parameter ν has a particular effect: In the quantizer in (15), larger ν means coarser quantization, while in the differential equation for Σ, it is clear that larger ν makes the growth rate of $\Sigma_1 - \Sigma_2\Sigma_3^{-1}\Sigma_2^T$ smaller; from (16), in turn, the parameter estimation becomes slower.

The result above is a quantized version of the scheme in [2] and [13]. There, optimal identification problems are considered, and the so-called cost-to-come functions play a key role for deriving exact solutions. For quantized identification, we show that the function $\bar{W}(t, \xi, y_{[0,\infty)}, u_{[0,\infty)})$ in (9) is a *structured* cost-to-come function; this leads us to a sufficient condition for the problem.

Several remarks are in order. First, the condition iii) in the theorem is difficult to satisfy in general. However, it holds in the special case when $\nu = 0$ by properly choosing the weight $Q(t)$ and the performance level γ in the cost; see [13] for details. This approach can be employed in our setting by taking sufficiently small ν. Next, we comment on the conditions i) and ii). In the Riccati differential equation of Σ in (11), Σ_3 is independent of other entries of Σ

$$\begin{aligned}
\dot{\Sigma}_3 &= -\Sigma_3\Sigma_3 - \Sigma_3 A_1 - A_1^T\Sigma_3 + \mu^2 C^T C - \nu I, \\
\Sigma_3(0) &= P_0.
\end{aligned}$$

This allows it to become constant by letting its initial value be $P_0 = P^{-1}$, where P is a solution to the algebraic Riccati equation $A_1 P + PA_1^T + I - P(\mu^2 C^T C - \nu I)P = 0$. We can then remove the condition ii) by taking $\bar{\Sigma}_3 = |\Sigma_3|$. Further, in this case, the condition i) on Σ_2 can be guaranteed as follows: By (11), its dynamics can be written as

$\dot{\Sigma}_2 = -\Sigma_2(A_1 + \Sigma_3) - A_2(\tilde{y}, u)\Sigma_3$ with $\Sigma_2(0) = 0$. Assume that for some $\alpha > 0$, $A_2(\tilde{y}(t), u(t)) \leq \alpha$ and that $-A_1 - \Sigma_3$ is Hurwitz. Then, it is immediate to find $\bar{\Sigma}_2$ that uniformly bounds $|\Sigma_2(t)|$.

IV. Design of Logarithmic Quantizers With Deadbands

In this section, we aim at designing a logarithmic-like quantizer that has a deadband and hence takes only finite values around the origin. In doing so, since the quantization error may be persistent, we relax the problem to one with a finite horizon performance index. The goal is to guarantee disturbance attenuation while the error in estimation is large.

Let $r > 0$ be a positive scalar. The problem of this section is to find an identifier and a quantizer which achieve a given γ level for the cost in (6), where the terminal time T is determined by

$$T = \min\{t \geq 0 : |\theta - \hat{\theta}(t)| \leq r\}. \tag{18}$$

That is, it is the time when the error in estimation becomes small enough and reaches the given level r. We now allow small error in the estimation, represented by the scalar r.

We use the identifier in (14) and construct the quantizer as follows. Let $\bar{\Sigma}_2$, $\bar{\Sigma}_3$, and $\bar{a}$ be as in the previous section, and let R be a positive scalar. Take a quantizer satisfying

$$|q(z) - z| \leq \frac{\nu}{(\mu^2 + \bar{\Sigma}_3\bar{a})^2}\Big\{-R\bar{\Sigma}_2\bar{a} + \Big[(R\bar{\Sigma}_2\bar{a})^2 + \frac{1}{\nu}(\mu^2 + \bar{\Sigma}_3\bar{a})^2(\nu r^2 + \mu^2|z|)^2\Big]^{1/2}\Big\} \tag{19}$$

for all $z \in \mathbb{R}$. As shown in an example later, logarithmic-like quantizers with deadbands satisfy this bound. For such quantizers, we have the following result.

Proposition 4.1: Given a scalar $\gamma > 0$, construct the identifier in (14) and the quantizer in (19) as above. Suppose that $\Sigma(t)$ satisfies the conditions i)– iii) in Theorem 3.3 for all t and all possible pairs $(\tilde{y}_{[0,\infty]}, u_{[0,\infty]})$. Moreover, suppose that $|\theta - \hat{\theta}(t)| \leq R$ for all t. Then, the following statements hold.

1) The performance level of γ is achieved under the identification scheme, where the terminal time T is given in (18).
2) Further, if the persistency of excitation condition (16) holds, then the terminal time T is finite.

Proof: The proof is similar to the one for Theorem 3.3. However, instead of showing that $\bar{W}(t, \xi, y_{[0,T]}, u_{[0,T]})$ is a structured cost-to-come function and in particular (9) holds, we prove a slightly relaxed condition as follows: For any given $(t, \tilde{y}_{[0,t]}, u_{[0,t]}, w)$ and any (ξ, v) such that $\tilde{y}(t) = \bar{C}\hat{\xi}(t) + q(\bar{C}\xi + v - \bar{C}\hat{\xi}(t))$ and $|\theta - \hat{\theta}(t)| \in [r, R]$, the inequality (9) holds. This condition implies the inequality in Lemma 3.2 while $t \leq T$, that is, by definition of the terminal time T, while $|\theta - \hat{\theta}(t)| \in [r, R]$. The existence of a finite T such that $|\theta - \hat{\theta}(T)| \leq r$ follows similarly to the proof of asymptotic estimation in Theorem 3.3. □

Remark 4.2: The bound (19) on quantization error is useful to avoid fast sampling in communication. One way is to employ two quantizers satisfying the bound, but take different output values. A hysteresis effect can be realized by switching between them each time the quantized output value changes; this method has been employed in a quantized adaptive control scheme [5].

V. A Numerical Example

A third-order system example is presented to demonstrate the performance of the proposed identification scheme. We took as the plant the transfer function

$$P(s) = \frac{1}{s^3 + 2s^2 + 3s + 4}$$

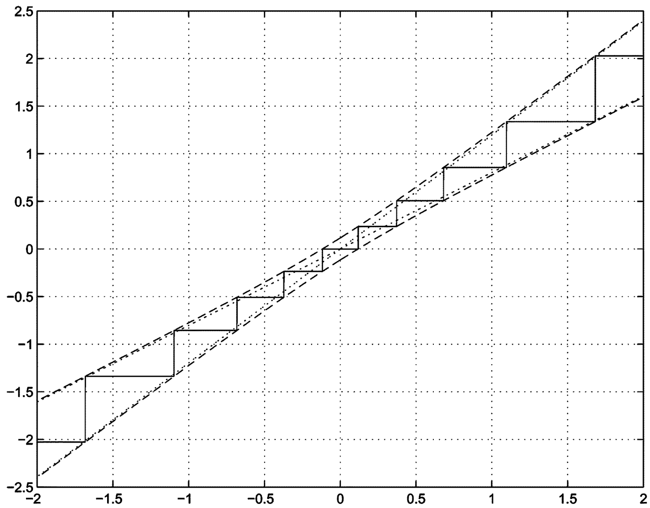


Fig. 2. Logarithmic-like quantizer.

where the parameters to be identified are $\theta = [-2 -3 -4\ 1]^T$. We designed a quantizer by following Section IV. For all simulations, the input was $u(t) = 2\sin(0.4t) + \sin(1.5t) + 0.5\sin(4t)$.

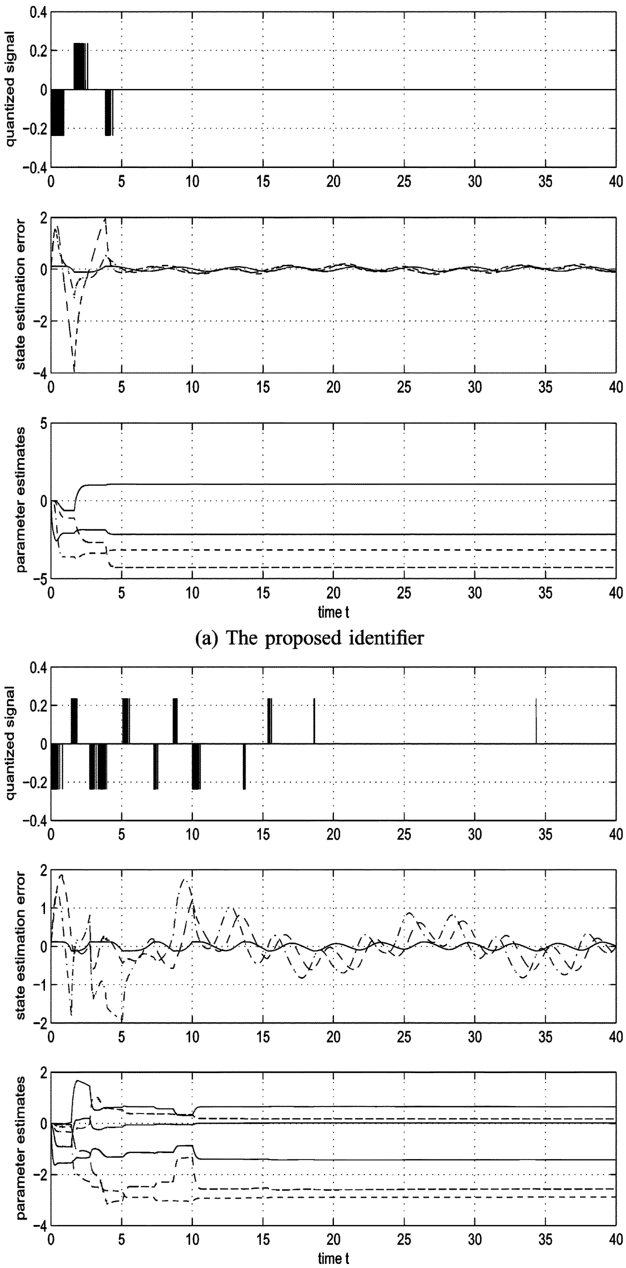
The weights in the cost were taken as $Q(t) = \mu^2\Phi(t)^T C^T C\Phi(t)$, $Q_0 = 0.01I$, and $P_0 = P^{-1}$, where $\Phi(t) := \Sigma_3(t)^{-1}\Sigma_2(t)^T$ and P is a solution of the Riccati equation $A_1 P + P A_1^T + I - P(\mu^2 e_1 e_1^T - \nu I)P = 0$. It is shown in [13] that with this choice of $Q(t)$, if there is no quantization, the optimal performance level always is $\gamma_{\mathrm{opt}} = 1$. Hence, in our design, we used a slightly larger value $\gamma = 1.2$. Also, the choice of P_0 above makes the matrix Σ_3 in (14) time invariant as mentioned in Section III, and also $-A_1 - \Sigma_3$ is Hurwitz.

Other parameters in the identifier (14) are $\nu = 0.002$, $\mu = 10$, $\bar{\Sigma}_2 = 2$, $R = 2$, and $\bar{a} = 6$. We used these parameters for most part of the design, except for that of the quantizer. To make q coarser than the one given in Proposition 4.1, we replaced the value $\nu = 0.002$ with $\nu = 10$. A graph of the quantizer is shown in Fig. 2. The dashed lines are the envelope given by (19), and the dotted lines are the sector-type bound from (15).

For comparison, we also constructed a prefiltering-based identifier (see, e.g., [14]). The prefilter was $1/(s+3)^3$, while the initial condition for the covariance matrix was $10^5 I$. Other parameters were chosen so that the performance without quantization is similar to that of our design. Note that for this scheme, there are extra parameters (equal to 0) to be identified.

We made two sets of simulations for the two identifiers. The first one is without any disturbance/noise. The initial conditions for the state and the estimated state were chosen to be the same $x_0 = \hat{x}_0 = [2\ 0\ 0]^T$. The responses of the proposed identifier and the conventional one are shown in Figs. 3(a) and (b), respectively. In both figures, the first plot shows the quantized signal $q(y - \hat{y})$ in (4), the second one is the state estimation errors $x_i(t) - \hat{x}_i(t)$, $i = 1, 2, 3$, and the third one is the parameter estimates $\theta_i(t)$, $i = 1, \ldots, r$. We see that the proposed scheme obtains very good parameter identification and state estimation. Moreover, after $t = 5$, the quantized signal $q(\tilde{y} - \hat{y})$ does not change, showing that little communication is needed. In contrast, the overall performance in Fig. 3(b) is somewhat poor, and more communication is required.

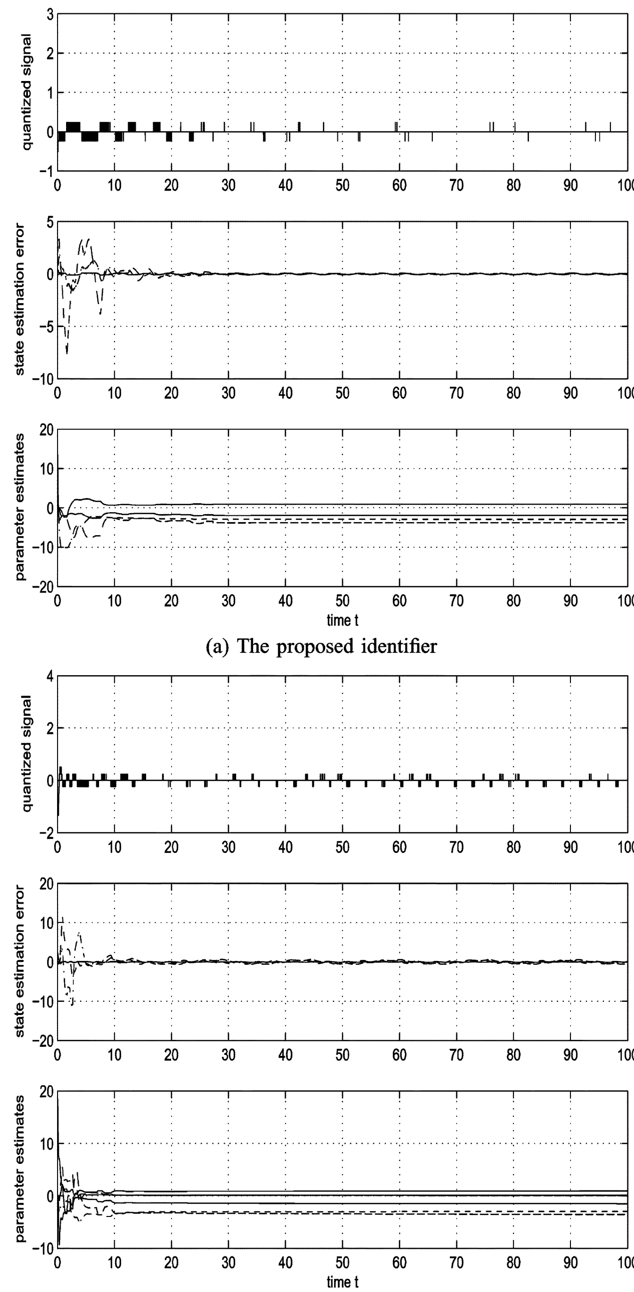
In the second set of simulations, we added disturbance and noise in the plant. We took w and v as Gaussian white noises with mean 0 and variances 0.08 and 0.02, respectively. In addition, we used different initial conditions as $x_0 = [2\quad -1\quad 1]^T$ for the plant and $\hat{x}_0 = 0$



(a) The proposed identifier

(b) The prefilter-based identifier

Fig. 3. Responses of the identifiers without any disturbance/noise.



(a) The proposed identifier

(b) The prefilter-based identifier

Fig. 4. Responses of the identifiers with disturbance/noise and $x_0 \neq \hat{x}_0$.

for the estimator. The plots are given in Fig. 4 in a similar fashion. We observe that in both cases, there are more changes in the quantized signal and the estimation takes longer. However, the proposed scheme obtains almost perfect identification, while the conventional one shows some errors.

VI. Conclusion

We considered the problem of H^∞ parameter identification over a network channel. The focus was on quantization, and we considered a communication scheme where feedback of information from the identifier is used. The resulting logarithmic-like quantizers showed satisfactory performance in the numerical results. Future research includes problems in discrete time and also extensions to adaptive control settings.

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An Efficient Maximization Algorithm With Implications in Min-Max Predictive Control

T. Alamo, D. Muñoz de la Peña, and E. F. Camacho

***Abstract*—In this technical note, an algorithm for binary quadratic programs defined by matrices with band structure is proposed. It was shown in the article by T. Alamo, D. M. de la Peña, D. Limon, and E. F. Camacho, "Constrained min-max predictive control: Modifications of the objective function leading to polynomial complexity," *IEEE Tran. Autom. Control*, vol. 50, pp. 710–714, May 2005, that this class of problems arise in robust model predictive control when min-max techniques are applied. Although binary quadratic problems belongs to a class of NP-complete problems, the computational burden of the proposed maximization algorithm for band matrices is polynomial with the dimension of the optimization variable and exponential with the band size. Computational results and comparisons on several hundred test problems demonstrate the efficiency of the algorithm.**

***Index Terms*—Band matrices, binary quadratic programming, combinatorial optimization, min-max techniques, model predictive control.**

I. INTRODUCTION

The objective of binary quadratic programming (BQP) is to find a binary vector that maximizes a quadratic function. These kind of problems belong to a class of NP-complete combinatorial problems that have many interesting applications. Capital budgeting and financial analysis [20], traffic message problems [9], and machine scheduling [3] can be formulated as BQP problems. In model predictive control (MPC), BQP problems arise when min-max techniques are applied to linear systems with bounded additive uncertainties; see [7], [15], and [17]. In this case, a BQP problem has to be solved in order to evaluate the inner maximization problem at each iteration of the minimization algorithm.

There is a vast literature on BQP that goes back to the 1970's, see [10], [19], and [11] and the references therein. Solution approaches include linear programming-based methods [4], branch and bound with preprocessing [18], eigenvalue-based approaches [16], and semidefinite relaxations [11]. These techniques deal with large scale (possibly sparse) problems and are able to solve problems with hundred of variables. However, in general are not appropriate for application to MPC because in this case, a high number of low order (less than 50 variables) BQP problems have to be solved. Therefore, techniques that depend on linear or semi-definite programming solvers are too cumbersome. This make the implementation of min-max controllers a hard issue.

In [2], several approximate techniques were proposed to solve the inner maximization problem in polynomial time. In particular, it was proved that the maximization problem can be approximated by an BQP problem with band matrix, i.e., the matrix $M = \{m_{ij}\}$ satisfies $m_{ij} = 0$ if $|i-j| \geq L$ where L is the band size. This problem is highly structured and can be solved efficiently. In this note the algorithm that solves this class of problems is presented. The computational burden of the proposed maximization algorithm is polynomial with the dimension of the optimization variable and exponential with L, the band size. The algorithm uses no multiplications or divisions and it is appropriate to implement the inner optimization of the min-max problem. The primary goal is to provide a detailed description and computational experiments of the algorithm for BQP problems with band structure. The results in this note complements the previous work [2].

The paper is organized as follows: In Section II some preliminary notation is introduced. The problem formulation is presented in Section III. Section IV presents the main contribution of the paper. An example of application is shown is Section V. The computational burden of the proposed algorithm is analyzed in Section VI. The paper ends with a section of conclusions.

II. PRELIMINARY NOTATION

Given a vector $w, w(k)$ denotes the kth component of the vector. Vector $\bar{0}_n$ is a vector of zeros of dimension n. Given $n, B_n = \{w \in \mathbb{R}^n : w(k) \in \{-1, 1\}, k = 1, \ldots, n\}$ denotes the set of vertices of the unit-hypercube in $\mathbb{R}^n$. Given a vector $w \in \mathbb{R}^n$ and an integer k $(1 \leq k \leq n), s_k(w) = [w(n+1-k), \ldots, w(n-1), w(n)]^T \in \mathbb{R}^k$ denotes the suffix of length k of w.

III. PROBLEM FORMULATION

The objective of binary quadratic programming (BQP) is to find, given a symmetric matrix M and a vector q, a binary vector w of dimension N that maximizes:

$$F(w) = w^T M w + q^T w. \tag{1}$$

In this paper an algorithm to solve efficiently BQP problems in which the matrix $M = \{m_{ij}\}$ has a band structure, that is, $m_{ij} = 0$ if $|i - j| \geq L$, where L is the band size, is presented. The algorithm exploits the structure of the matrix to build a set of 2^L hypotheses that contains the maximum. To build this set and evaluate the maximum, the algorithm requires a number of evaluations of $F(w)$ equal to $(N -$

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