H^∞-Optimal Control for Singularly Perturbed Systems—Part II: Imperfect State Measurements

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***Abstract*— In this paper we study the H^∞-optimal control of singularly perturbed linear systems under general imperfect measurements, for both finite- and infinite-horizon formulations. Using a differential game theoretic approach, we first show that as the singular perturbation parameter (say, $\epsilon > 0$) approaches zero, the optimal disturbance attenuation level for the full-order system under a quadratic performance index converges to a value that is bounded above by (and in some cases equal to) the maximum of the optimal disturbance attenuation levels for the *slow* and *fast* subsystems under appropriate "slow" and "fast" quadratic cost functions, with the bound being computable independently of ϵ and knowing only the slow and fast dynamics of the system. We then construct a controller based on the slow subsystem only and obtain conditions under which it delivers a desired performance level even though the fast dynamics are completely neglected. The ultimate performance level achieved by this "slow" controller can be uniformly improved upon, however, by a composite controller that uses some feedback from the output of the fast subsystem. We construct one such controller, via a two-step sequential procedure, which uses static feedback from the fast output and dynamic feedback from an appropriate slow output, each one obtained by solving appropriate ϵ-independent lower dimensional H^∞-optimal control problems under some informational constraints. We provide a detailed analysis of the performance achieved by this lower-dimensional ϵ-independent composite controller when applied to the full-order system and illustrate the theory with some numerical results on some prototype systems.**

I. Introduction

One of the important developments in control theory has been the recognition of the close relationship that exists between H^∞-optimal control problems (originally formulated in the frequency domain [1], [2] and then extended to state-space formulations [3]–[8]) and a class of linear-quadratic differential games [9]–[13], which has not only led to simpler derivations of existing results on the former, but also enabled us to develop worst-case (H^∞-optimal) controllers under various information patterns, such as (in addition to perfect and imperfect state measurements) delayed state and sampled state measurements [14], [15]. An up-to-date coverage of this relationship and the derivation of H^∞-optimal controllers under different information patterns can be found in the recent book [16], which also contains an extensive list of references on the topic.

It is by now well known that in both finite- and infinite-horizon formulations one can come arbitrarily close to the H^∞-optimal performance level by designing a dynamic controller, of the same order as that of the plant (which may also include the dynamics of the disturbance, if any), whose construction involves the solutions of two parameterized differential or algebraic Riccati equations, subject to some spectral radius condition. The problem we address in this paper is the possible order reduction of this controller when the plant exhibits open-loop time-scale separation. In the state space, such systems are commonly modeled using the mathematical framework of singular perturbations, with a small parameter, say ϵ, determining the degree of separation between the "slow" and "fast" modes of the system [17]. In this framework, our objective may be rephrased as one of obtaining "approximate" controllers which do not depend on the singular perturbation parameter ϵ and proving that these approximate controllers can be used "reliably" on the original system when $\epsilon > 0$ is sufficiently small.

In Part I of this paper [18], we have already initiated a study on this problem when the controller has access to full state information, where we use the framework of differential games. One of our results in this context has been that the H^∞-optimal performance does not show continuity at $\epsilon = 0$, in the sense that the performance of the original system as $\epsilon \to 0$ is not necessarily the same as the H^∞-optimal performance of the reduced-order (slow) system, even if the fast subsystem is stable. We have actually proven that the former is upper bounded by the maximum of the H^∞-optimal performance of the "slow" and "fast" subsystems, appropriately defined, and that this bound is exact in the infinite-horizon case. We have also constructed composite controllers (from parameterized solutions of slow and fast games), independently of ϵ, which guarantee a desired achievable performance level for the full-order plant when ϵ is sufficiently small.

In the present paper, we develop counterparts of these results for the imperfect measurements case. Some of our conclusions are similar in spirit to those of [18] (though details of proofs are much more involved), while some others are quite different. For a brief preview of our main results, let us introduce the quantity $\gamma_I^*(\epsilon)$ to denote the H^∞-optimum performance of the full-order system under imperfect state measurements and the quantities γ_{Is} and γ_{If} to denote the

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H^∞-optimum performances of (appropriately defined) reduced slow and fast subsystems, respectively. We first show that γ_I^* (0) is bounded above by $\max\{\gamma_{Is}, \gamma_{If}\}$. We then obtain a composite controller, independent of the singular perturbation parameter, under which the associated differential game has a bounded upper value, and a desired achievable H^∞-performance bound is attained for the full-order problem. In the perfect state measurements case, the construction of the composite controller involved a parallel procedure, while here it is sequential: Obtain first a static output feedback using the fast dynamics and then a dynamic output feedback which uses the dynamics of the "reduced" slow game.

The problem of designing controllers for singularly perturbed systems subject to unknown disturbances has been studied before in the literature, notably in papers [19]–[21], where the objective has been to obtain composite controllers that guarantee stability of the overall (possibly nonlinear) system. The main approach of the authors in these papers has involved the construction of appropriate Lyapunov functions, in terms of which a class of stabilizing controllers has been characterized. No optimality properties, however, have been associated with these controllers, which is our main concern in this paper. Yet another paper that deals with uncertain (linear) systems which exhibit time-scale separation is [22], which obtains a two-frequency-scale decomposition for H^∞-disk problems, but does not address the issue of optimal controller design.

The balance of the present paper is organized as follows. In the next section, we formulate the singularly perturbed H^∞-optimal control problem with imperfect state measurements and identify the associated linear-quadratic differential game. We also provide in that section the solution to the full-order problem, for both finite and infinite horizons. In Section III, we identify appropriate slow and fast subsystems and define the associated differential games. The optimality of the slow controller and its robustness to order reduction are studied in Section IV. The main results are provided in Section V, where we develop composite controllers for both the finite- and infinite-horizon cases and obtain precise performance bounds attained by these controllers. Section VI presents some numerical results to illustrate the theory, and Section VII provides a discussion on some immediate extensions. The paper ends with five appendixes, which provide details of some of the derivations given in the main body of the paper.

II. Problem Formulation

The system under consideration, with slow and fast dynamics, is described in the standard "singularly perturbed" form by

$$\begin{cases} \dot{x}_1 = A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u + D_1(t)w \\ \epsilon\dot{x}_2 = A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u + D_2(t)w \\ y = C_1(t)x_1 + C_2(t)x_2 + E(t)w \end{cases} \tag{2.1}$$

where $x' := (x_1', x_2')$ is the n-dimensional state vector, with x_1 of dimension n_1 and x_2 of dimension $n_2 := n - n_1$; y is the measured output; u is the control input, and w is the disturbance, each belonging to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_u$ and $\mathcal{H}_w$, respectively, defined on the time interval $[t_0, t_f]$. The initial condition for the system is

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} := x_0$$

where x_0 is unknown. The control input u is generated by a control policy μ_I, according to

$$u(t) = \mu_I(t, y_{[t_0,t]}) \tag{2.2}$$

where $\mu_I : [t_0, t_f] \times \mathcal{H}_y \to \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in y, further satisfying the given casualty condition. Let us denote the class of all these controllers by $\mathcal{M}_I$. With this system, we associate the standard quadratic performance index

$$L(u, w, x_0) = |x(t_f)|^2_{Q_f} + \int_{t_0}^{t_f} (|x(t)|^2_{Q(t)} + 2x'(t)P(t)u(t) + |u(t)|^2_{R(t)})\, dt$$
$$Q_f \geq 0, \qquad Q(\cdot) \geq 0, \qquad R(\cdot) > 0 \tag{2.3}$$

where Q_f will show dependence on $\epsilon > 0$ (to be clarified later) and

$$\begin{bmatrix} Q & P \\ P' & R \end{bmatrix} \geq 0.$$

Let us also introduce the notation $J(\mu_I, w, x_0)$ to denote $L(u, w, x_0)$, with u given by (2.2). Given a $Q_0 > 0$, which will show some dependence on ϵ, the H^∞-optimal control problem is the minimization of the quality

$$\sup_{w\in\mathcal{H}_w;\; x_0\in\mathcal{H}_x} \{[J(\mu_I, w, x_0)]^{1/2}/[\|w\|^2 + |x_0|^2_{Q_0}]^{1/2}\} \tag{2.4}$$

over all permissible controllers μ_I, or if a minimum does not exist, the derivation of a controller μ_I that will assure a performance within a given neighborhood of the infimum of (2.4). Let us denote this infimum by $\gamma_I^*(\epsilon)$, i.e.,

$$\inf_{\mu_I\in\mathcal{M}_I} \sup_{w\in\mathcal{H}_w;\; x_0\in\mathcal{H}_x} \{[J(\mu_I, w, x_0)]^{1/2}/[\|w\|^2 + |x_0|^2_{Q_0}]^{1/2}\} := \gamma_I^*(\epsilon) \tag{2.5}$$

where we explicitly show the dependence of γ_I^* on the singular perturbation parameter $\epsilon > 0$.

For each $\epsilon > 0$, we can associate a soft-constrained linear-quadratic differential game with this worst-case design problem (see [16, ch. 5, sect. 1]), which has the cost function

$$L_\gamma(u, w, x_0) = L(u, w, x_0) - \gamma^2\{\|w\|^2 + |x_0|^2_{Q_0}\} \tag{2.6}$$

where "u" is the minimizer and "w, x_0" the maximizer. The performance level $\gamma_I^*(\epsilon)$ in (2.5) is then the "smallest" value of $\gamma \geq 0$ under which the differential game with state equation (2.1) and cost function (2.6) has a bounded upper value, when u is chosen according to (2.2).

Even though the problem formulated above has been solved completely for every $\epsilon > 0$ (see, e.g., [16, ch. 5]), the computation of $\gamma_I^*(\epsilon)$ and the construction of a corresponding

H^{∞}-optimal or suboptimal controller for small values of $\epsilon > 0$ present serious difficulties, due to numerical stiffness. To remedy this, we pose in this paper the question of whether $\gamma_I^*(\epsilon)$ and the H^{∞}-optimal controller can be determined for small value of $\epsilon > 0$ by solving well-behaved ϵ-independent smaller-order problems. Another question of interest is whether the fast subsystem dynamics can be neglected completely in the design of such controllers and, if so, under what conditions.

Before studying these questions, we first present the solution to the full-order problem, under three basic assumptions:

Assumption 1: Q_f, Q_0, and $Q(\cdot)$ in (2.3) and (2.4) are partitioned as

$$Q_f = \begin{bmatrix} Q_{f11} & \epsilon Q_{f12} \\ \epsilon Q_{f21} & \epsilon Q_{f22} \end{bmatrix}; \qquad Q_0 = \begin{bmatrix} Q_{011} & \epsilon Q_{012} \\ \epsilon Q_{021} & \epsilon Q_{022} \end{bmatrix};$$
$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$$

where in each case the 11-block is of dimension $n_1 \times n_1$ and the 22-block is of dimension $n_2 \times n_2$. Also, we partition the matrix P in (2.3) as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

in a way compatible with the partitioning of the Q's.

Assumption 2: $A_{ij}(t)$, $P_i(t)$, $Q_{ij}(t)$, $B_i(t)$, $D_i(t)$, $C_i(t)$, $E(t)$ $(i = 1, 2\ j = 1, 2)$ are continuously differentiable on $t \in [t_0, t_f]$.

Assumption 3: Matrix functions $A_{22}(t), Q_{22}(t), N(t), D_2(t)D_2'(t)$

$$\begin{bmatrix} Q_{22}(t) & P_2(t) \\ P_2'(t) & R(t) \end{bmatrix}; \quad \text{and} \quad \begin{bmatrix} D_2(t)D_2'(t) & L_2(t) \\ L_2'(t) & N(t) \end{bmatrix}$$

are invertible for all $t \in [t_0, t_f]$, where $N(t) := E(t)E'(t)$, $L_2(t) := D_2(t)E'(t)$.

Let us introduce the notation $A(t)$ and $A_\epsilon(t)$ to denote the partitioned matrices

$$A(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix};$$
$$A_\epsilon(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \frac{1}{\epsilon} A_{21}(t) & \frac{1}{\epsilon} A_{22}(t) \end{bmatrix}$$

where we take $\epsilon > 0$. Similarly, we introduce the partitioned matrices

$$B(t) := \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}; \qquad D(t) := \begin{bmatrix} D_1(t) \\ D_2(t) \end{bmatrix};$$
$$L(t) := \begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix}; \qquad L_\epsilon(t) := \begin{bmatrix} L_1(t) \\ \frac{1}{\epsilon} L_2(t) \end{bmatrix}$$

$$B_\epsilon(t) := \begin{bmatrix} B_1(t) \\ \frac{1}{\epsilon} B_2(t) \end{bmatrix}; \qquad D_\epsilon(t) := \begin{bmatrix} D_1(t) \\ \frac{1}{\epsilon} D_2(t) \end{bmatrix};$$
$$C(t) := [C_1(t) \quad C_2(t)]$$

where $L_1(t) := D_1(t)E'(t)$.

Further, we define the following matrices:

$$\overline{A}(t) := A(t) - B(t)R^{-1}(t)P'(t);$$
$$\overline{A}_\epsilon(t) := A_\epsilon(t) - B_\epsilon(t)R^{-1}(t)P'(t)$$
$$\overline{Q}(t) := Q(t) - P(t)R^{-1}(t)P'(t);$$
$$\overline{S}(t;\gamma) := B(t)R^{-1}(t)B'(t) - \frac{1}{\gamma^2} D(t)D'(t)$$
$$\overline{S}_\epsilon(t;\gamma) := B_\epsilon(t)R^{-1}(t)B_\epsilon'(t) - \frac{1}{\gamma^2} D_\epsilon(t)D_\epsilon'(t);$$
$$\tilde{R}(t;\gamma) := C'(t)N^{-1}(t)C(t) - \frac{1}{\gamma^2} Q(t)$$
$$\tilde{A}(t) := A(t) - L(t)N^{-1}(t)C(t);$$
$$\tilde{A}_\epsilon(t) := A_\epsilon(t) - L_\epsilon(t)N^{-1}(t)C(t)$$
$$\tilde{M}(t) := D(t)D'(t) - L(t)N^{-1}(t)L'(t);$$
$$\tilde{M}_\epsilon(t) := D_\epsilon(t)D_\epsilon'(t) - L_\epsilon(t)N^{-1}(t)L_\epsilon'(t).$$

We already know (see [16, ch. 5, theorem 5.3]) that for each $\epsilon > 0$, there exists a $\hat{\gamma}_I \geq 0$ such that for all $\gamma > \hat{\gamma}_I$ the zero-sum differential game described by (2.1) and (2.6) has a bounded upper value (which in this case is equal to *zero*) and a controller that delivers the upper value is

$$u_I^*(t) = u_I^*(t, y_{[t_0,t]}) = -R^{-1}(B_\epsilon'\tilde{Z}(t;\epsilon) + P')\hat{x}(t), \quad t \geq t_0. \tag{2.7}$$

Here, $\hat{x}(t)$ is the "observer state," generated by the differential equation

$$\dot{\hat{x}} = (\overline{A}_\epsilon - \overline{S}_\epsilon\tilde{Z})\hat{x} + (I - \frac{1}{\gamma^2}\tilde{\Sigma}\tilde{Z})^{-1}(\tilde{\Sigma}C' + L_\epsilon)N^{-1} \cdot (y - C\hat{x} - \frac{1}{\gamma^2} L_\epsilon'\tilde{Z}\hat{x}); \qquad \hat{x}(t_0) = 0. \tag{2.8}$$

The matrix $\tilde{Z}(t;\epsilon)$ is the unique bounded nonnegative definite solution of the backward matrix Riccati differential equation

$$\dot{\tilde{Z}} + \overline{A}_\epsilon'\tilde{Z} + \tilde{Z}\overline{A}_\epsilon - \tilde{Z}\overline{S}_\epsilon\tilde{Z} + \overline{Q} = 0; \qquad \tilde{Z}(t_f) = Q_f \tag{2.9}$$

and $\tilde{\Sigma}(t;\epsilon)$ is the unique bounded positive definite solution of the forward matrix Riccati differential equation

$$\dot{\tilde{\Sigma}} = \tilde{A}_\epsilon\tilde{\Sigma} + \tilde{\Sigma}\tilde{A}_\epsilon' - \tilde{\Sigma}\tilde{R}\tilde{\Sigma} + \tilde{M}_\epsilon; \qquad \tilde{\Sigma}(t_0) = Q_0^{-1}. \tag{2.10}$$

For $\gamma < \hat{\gamma}_I$, either $I - (1/\gamma^2)\tilde{\Sigma}\tilde{Z}$ has at least one negative eigenvalue or one of the two Riccati equations above has a conjugate point in the open interval (t_0, t_f), leading in each case to the conclusion that the soft-constrained game has an unbounded upper value. The level $\hat{\gamma}_I$ is indeed the H^{∞}-optimal performance level defined by (2.5).

The Riccati equation (2.9) also arises in the problem with perfect state measurements, which has been studied extensively in Part I of this paper [18]. It has been shown there that $\tilde{Z}$ admits the partitioning

$$\tilde{Z} := \begin{bmatrix} \tilde{Z}_{11} & \epsilon\tilde{Z}_{12} \\ \epsilon\tilde{Z}_{12}' & \epsilon\tilde{Z}_{22} \end{bmatrix} \tag{2.11}$$

where $\tilde{Z}_{11}, \tilde{Z}_{12}$ and $\tilde{Z}_{22}$, satisfy the following matrix differential equations:

$$\dot{\tilde{Z}}_{11} + \overline{A}'_{11}\tilde{Z}_{11} + \overline{A}'_{21}\tilde{Z}'_{12} + \tilde{Z}_{11}\overline{A}_{11} + \tilde{Z}_{12}\overline{A}_{21} + \overline{Q}_{11} - \tilde{Z}_{11}\overline{S}_{11}\tilde{Z}_{11} - \tilde{Z}_{12}\overline{S}_{21}\tilde{Z}_{11} - \tilde{Z}_{11}\overline{S}_{12}\tilde{Z}'_{12} - \tilde{Z}_{12}\overline{S}_{22}\tilde{Z}'_{12} = 0; \qquad \tilde{Z}_{11}(t_f) = Q_{f11} \tag{2.12}$$

$$\epsilon\dot{\tilde{Z}}_{12} + \epsilon\overline{A}'_{11}\tilde{Z}_{12} + \overline{A}'_{21}\tilde{Z}_{22} + \tilde{Z}_{11}\overline{A}_{12} + \tilde{Z}_{12}\overline{A}_{22} + \overline{Q}_{12} - \epsilon\tilde{Z}_{11}\overline{S}_{11}\tilde{Z}_{12} - \epsilon\tilde{Z}_{12}\overline{S}_{21}\tilde{Z}_{12} - \tilde{Z}_{11}\overline{S}_{12}\tilde{Z}_{22} - \tilde{Z}_{12}\overline{S}_{22}\tilde{Z}_{22} = 0; \qquad \tilde{Z}_{12}(t_f) = Q_{f12} \tag{2.13}$$

$$\epsilon\dot{\tilde{Z}}_{22} + \epsilon\overline{A}'_{12}\tilde{Z}_{12} + \overline{A}'_{22}\tilde{Z}_{22} + \epsilon\tilde{Z}'_{12}\overline{A}_{12} + \tilde{Z}_{22}\overline{A}_{22} + \overline{Q}_{22} - \epsilon^2\tilde{Z}'_{12}\overline{S}_{11}\tilde{Z}_{12} - \epsilon\tilde{Z}_{22}\overline{S}_{21}\tilde{Z}_{12} - \epsilon\tilde{Z}'_{12}\overline{S}_{12}\tilde{Z}_{22} - \tilde{Z}_{22}\overline{S}_{22}\tilde{Z}_{22} = 0; \qquad \tilde{Z}_{22}(t_f) = Q_{f22}. \tag{2.14}$$

By a standard duality argument it follows that $\tilde{\Sigma}$ admits the partitioning

$$\tilde{\Sigma} := \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}'_{12} & \frac{1}{\epsilon}\tilde{\Sigma}_{22} \end{bmatrix} \tag{2.15}$$

where $\tilde{\Sigma}_{11}, \tilde{\Sigma}_{12}$, and $\tilde{\Sigma}_{22}$ satisfy the following matrix differential equations:

$$\dot{\tilde{\Sigma}}_{11} = \tilde{A}_{11}\tilde{\Sigma}_{11} + \tilde{A}_{12}\tilde{\Sigma}'_{12} + \tilde{\Sigma}_{11}\tilde{A}_{11} + \tilde{\Sigma}_{12}\tilde{A}'_{12} + \tilde{M}_{11} - \tilde{\Sigma}_{11}\tilde{R}_{11}\tilde{\Sigma}_{11} - \tilde{\Sigma}_{12}\tilde{R}_{21}\tilde{\Sigma}_{11} - \tilde{\Sigma}_{11}\tilde{R}_{12}\tilde{\Sigma}'_{12} - \tilde{\Sigma}_{12}\tilde{R}_{22}\tilde{\Sigma}'_{12}; \quad \tilde{\Sigma}_{11}(t_0) = Q_{011}^{-1} + O(\epsilon) \tag{2.16}$$

$$\epsilon\dot{\tilde{\Sigma}}_{12} = \epsilon\tilde{A}_{11}\tilde{\Sigma}_{12} + \tilde{A}_{12}\tilde{\Sigma}_{22} + \tilde{\Sigma}_{11}\tilde{A}'_{21} + \tilde{\Sigma}_{12}\tilde{A}'_{22} + \tilde{M}_{12} - \epsilon\tilde{\Sigma}_{11}\tilde{R}_{11}\tilde{\Sigma}_{12} - \epsilon\tilde{\Sigma}_{12}\tilde{R}_{21}\tilde{\Sigma}_{12} - \tilde{\Sigma}_{11}\tilde{R}_{12}\tilde{\Sigma}_{22} - \tilde{\Sigma}_{12}\tilde{R}_{22}\tilde{\Sigma}_{22};$$
$$\tilde{\Sigma}_{12}(t_0) = -Q_{011}^{-1}Q_{012}Q_{022}^{-1} + O(\epsilon) \tag{2.17}$$

$$\epsilon\dot{\tilde{\Sigma}}_{22} = \epsilon\tilde{A}_{21}\tilde{\Sigma}_{12} + \tilde{A}_{22}\tilde{\Sigma}_{22} + \epsilon\tilde{\Sigma}'_{12}\tilde{A}'_{21} + \tilde{\Sigma}_{22}\tilde{A}'_{22} + \tilde{M}_{22} - \epsilon^2\tilde{\Sigma}'_{12}\tilde{R}_{11}\tilde{\Sigma}_{12} - \epsilon\tilde{\Sigma}_{22}\tilde{R}_{21}\tilde{\Sigma}_{12} - \epsilon\tilde{\Sigma}'_{12}\tilde{R}_{12}\tilde{\Sigma}_{22} - \tilde{\Sigma}_{22}\tilde{R}_{22}\tilde{\Sigma}_{22}; \quad \tilde{\Sigma}_{22}(t_0) = Q_{022}^{-1} + O(\epsilon). \tag{2.18}$$

Of course, the preceding analysis is valid for all γ larger than $\sup\{\gamma_I^*(\epsilon), 0 < \epsilon \leq \epsilon_0\}$, where ϵ_0 is some prechosen small (positive) scalar.

For the infinite-horizon case (i.e., as $t_f \to \infty$ and $t_0 \to -\infty$, as well as when $t_f = \infty$ and $t_0 = -\infty$), we take A, B, D, C, E, P, Q, and R to be time-invariant, $Q_f = 0$ and $Q_0 = \infty$, and require that $x(t) \to 0$ as $t \to -\infty$ and $t \to \infty$. Furthermore, we assume that (A_ϵ, B_ϵ) and (A_ϵ, D_ϵ) are controllable, and (A_ϵ, Q) and (A_ϵ, C) are observable. Then, for each $\epsilon > 0$, there exists a $\hat{\gamma}_{I\infty}$ such that, for all $\gamma > \hat{\gamma}_{I\infty}$, the infinite-horizon soft-constrained game has a finite upper value, achieved by the controller

$$u_I^*(t) = \mu_{I\infty}^*\left(t, y_{(-\infty,t]}\right) = -R^{-1}(B'_\epsilon\tilde{Z}_\infty(\epsilon) + P')\hat{x}(t) \tag{2.19}$$

where $\hat{x}(t)$ is generated by the following differential equation:

$$\dot{\hat{x}} = (\overline{A}_\epsilon - \overline{S}_\epsilon\tilde{Z}_\infty)\hat{x} + \left(I - \frac{1}{\gamma^2}\tilde{\Sigma}_\infty\tilde{Z}_\infty\right)^{-1} \cdot (\Sigma_\infty C' + L_\epsilon)N^{-1} \cdot \left(y - C\hat{x} - \frac{1}{\gamma^2}L'_\epsilon\tilde{Z}_\infty\hat{x}\right); \quad \hat{x}(-\infty) = 0. \tag{2.20}$$

The matrix $\tilde{Z}_\infty(\epsilon)$ is the minimal positive definite solution of the generalized algebraic Riccati equation (GARE)

$$\overline{A}'_\epsilon\tilde{Z}_\infty + \tilde{Z}_\infty\overline{A}_\epsilon - \tilde{Z}_\infty\overline{S}_\epsilon\tilde{Z}_\infty + \overline{Q} = 0 \tag{2.21}$$

and $\tilde{\Sigma}_\infty(\epsilon)$ is the minimal positive definite solution of the dual GARE

$$\tilde{A}_\epsilon\tilde{\Sigma}_\infty + \tilde{\Sigma}_\infty\tilde{A}'_\epsilon - \tilde{\Sigma}_\infty\tilde{R}\tilde{\Sigma}_\infty + \tilde{M}_\epsilon = 0. \tag{2.22}$$

The level $\hat{\gamma}_{I\infty}$ is again the H^∞-optimal performance level of the infinite-horizon disturbance attenuation problem, and for $\gamma < \hat{\gamma}_{I\infty}$, the soft-constrained game has infinite upper value.

The GARE (2.21) has also been studied extensively in [18], where we have shown that the solution $\tilde{Z}_\infty$ admits a partitioning that is identical with (2.11). Substituting this structure into the GARE leads to the following coupled algebraic Riccati equations for $\tilde{Z}_{\infty 11}, \tilde{Z}_{\infty 12}$, and $\tilde{Z}_{\infty 22}$:

$$\overline{A}'_{11}\tilde{Z}_{\infty 11} + \overline{A}'_{21}\tilde{Z}'_{\infty 12} + \tilde{Z}_{\infty 11}\overline{A}_{11} + \tilde{Z}_{\infty 12}\overline{A}_{21} + \overline{Q}_{11} - \tilde{Z}_{\infty 11}\overline{S}_{11}\tilde{Z}_{\infty 11} - \tilde{Z}_{\infty 12}\overline{S}_{21}\tilde{Z}_{\infty 11} - \tilde{Z}_{\infty 11}\overline{S}_{12}\tilde{Z}'_{\infty 12} - \tilde{Z}_{\infty 12}\overline{S}_{22}\tilde{Z}'_{\infty 12} = 0 \tag{2.23}$$

$$\epsilon\overline{A}'_{11}\tilde{Z}_{\infty 12} + \overline{A}'_{21}\tilde{Z}_{\infty 22} + \tilde{Z}_{\infty 11}\overline{A}_{12} + \tilde{Z}_{\infty 12}\overline{A}_{22} + \overline{Q}_{12} - \epsilon\tilde{Z}_{\infty 11}\overline{S}_{11}\tilde{Z}_{\infty 12} - \epsilon\tilde{Z}_{\infty 12}\overline{S}_{21}\tilde{Z}_{\infty 12} - \tilde{Z}_{\infty 11}\overline{S}_{12}\tilde{Z}_{\infty 22} - \tilde{Z}_{\infty 12}\overline{S}_{22}\tilde{Z}_{\infty 22} = 0 \tag{2.24}$$

$$\epsilon\overline{A}'_{12}\tilde{Z}_{\infty 12} + \overline{A}'_{22}\tilde{Z}_{\infty 22} + \epsilon\tilde{Z}'_{\infty 12}\overline{A}_{12} + \tilde{Z}_{\infty 22}\overline{A}_{22} + \overline{Q}_{22} - \epsilon^2\tilde{Z}'_{\infty 12}\overline{S}_{11}\tilde{Z}_{\infty 12} - \epsilon\tilde{Z}_{\infty 22}\overline{S}_{21}\tilde{Z}_{\infty 12} - \epsilon\tilde{Z}'_{\infty 12}\overline{S}_{12}\tilde{Z}_{\infty 22} - \tilde{Z}_{\infty 22}\overline{S}_{22}\tilde{Z}_{\infty 22} = 0. \tag{2.25}$$

As in the finite-horizon case, by duality arguments, we now substitute the structure (2.15) into (2.22), to arrive at the following coupled algebraic Riccati equations for $\tilde{\Sigma}_{\infty 11}, \tilde{\Sigma}_{\infty 12}$, and $\tilde{\Sigma}_{\infty 22}$.

$$\begin{aligned}\tilde{A}_{11}\tilde{\Sigma}_{\infty 11} + \tilde{A}_{12}\tilde{\Sigma}'_{\infty 12} + \tilde{\Sigma}_{\infty 11}\tilde{A}'_{11} + \tilde{\Sigma}_{\infty 12}\tilde{A}'_{12} + \tilde{M}_{11} \\ - \tilde{\Sigma}_{\infty 11}\tilde{R}_{11}\tilde{\Sigma}_{\infty 11} - \tilde{\Sigma}_{\infty 12}\tilde{R}_{21}\tilde{\Sigma}_{\infty 11} \\ - \tilde{\Sigma}_{\infty 11}\tilde{R}_{12}\tilde{\Sigma}'_{\infty 12} - \tilde{\Sigma}_{\infty 12}\tilde{R}_{22}\tilde{\Sigma}'_{\infty 12} = 0\end{aligned} \tag{2.26}$$

$$\begin{aligned}\epsilon\tilde{A}_{11}\tilde{\Sigma}_{\infty 12} + \tilde{A}_{12}\tilde{\Sigma}_{\infty 22} + \tilde{\Sigma}_{\infty 11}\tilde{A}'_{21} + \tilde{\Sigma}_{\infty 12}\tilde{A}'_{22} + \tilde{M}_{12} \\ - \epsilon\tilde{\Sigma}_{\infty 11}\tilde{R}_{11}\tilde{\Sigma}_{\infty 12} - \epsilon\tilde{\Sigma}_{\infty 12}\tilde{R}_{21}\tilde{\Sigma}_{\infty 12} \\ - \tilde{\Sigma}_{\infty 11}\tilde{R}_{12}\tilde{\Sigma}_{\infty 22} - \tilde{\Sigma}_{\infty 12}\tilde{R}_{22}\tilde{\Sigma}_{\infty 22} = 0\end{aligned} \tag{2.27}$$

$$\begin{aligned}\epsilon\tilde{A}_{21}\tilde{\Sigma}_{\infty 12} + \tilde{A}_{22}\tilde{\Sigma}_{\infty 22} + \epsilon\tilde{\Sigma}'_{\infty 12}\tilde{A}'_{21} + \tilde{\Sigma}_{\infty 22}\tilde{A}'_{22} + \tilde{M}_{22} \\ - \epsilon^2\tilde{\Sigma}'_{\infty 12}\tilde{R}_{11}\tilde{\Sigma}_{\infty 12} - \epsilon\tilde{\Sigma}_{\infty 22}\tilde{R}_{21}\tilde{\Sigma}_{\infty 12} \\ - \epsilon\tilde{\Sigma}'_{\infty 12}\tilde{R}_{12}\tilde{\Sigma}_{\infty 22} - \tilde{\Sigma}_{\infty 22}\tilde{R}_{22}\tilde{\Sigma}_{\infty 22} = 0.\end{aligned} \tag{2.28}$$

Thus completing the analysis of the direct solution to the full-order problem as $\epsilon \to 0$, we now turn, in the next two sections, to the original goal of this paper—which is the derivation of the approximate solution based on a time-scale decomposition. First we identify, in Section III, the slow and fast subsystems associated with the original problem and obtain the solutions of two separate H^∞-optimal control problems, one defined on the slow time scale and the other one on the fast time scale.

III. A Time-Scale Decomposition

The Slow Subsystem and the Associated Soft-Constrained Game

To obtain the slow dynamics associated with (2.1), we let $\epsilon = 0$ and solve for x_2 (to be denoted $\overline{x}_2$) in terms of $x_1 =: x_s, u =: u_s, w =: w_s, y =: y_s$, and under the working Assumption 3:

$$\overline{x}_2 = -A_{22}^{-1}(A_{21}x_s + B_2u_s + D_2w_s). \tag{3.1}$$

Using this in the remaining equations of (2.1), we obtain the reduced-order (slow) dynamics

$$\begin{cases} \dot{x}_s = A_0x_s + B_0u_s + D_0w_s \\ y_s = C_0x_s - C_2A_{22}^{-1}B_2u_s + E_0w_s \end{cases} \tag{3.2}$$

where

$$\begin{aligned} &A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}; \qquad B_0 := B_1 - A_{12}A_{22}^{-1}B_2; \\ &D_0 := D_1 - A_{12}A_{22}^{-1}D_2; \qquad C_0 := C_1 - C_2A_{22}^{-1}A_{21}; \\ &E_0 := E - C_2A_{22}^{-1}D_2. \end{aligned}$$

Using (3.1) also in the cost function (2.3) leads to the reduced (slow) cost (where we have also set $\epsilon = 0$)

$$\begin{aligned} L_s = |x_s(t_f)|^2_{Q_{f11}} + \int_{t_0}^{t_f} (|x_s|^2_{Q_{11}} + 2x_s'Q_{12}\overline{x}_2 \\ + |\overline{x}_2|^2_{Q_{22}} + 2x_s'P_1u_s + 2\overline{x}_2'P_2u_s + |u_s|^2_R)\, dt. \end{aligned} \tag{3.3}$$

The zero-sum differential game associated with this problem has the cost function

$$L_{\gamma s} = L_s - \gamma^2\left(\|w_s\|^2 + |x_s(t_0)|^2_{Q_{011}}\right) \tag{3.4}$$

and we are in fact interested only in the upper value of this game.

As in the perfect state information case covered in [18], under the condition

$$\gamma^2 I - D_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}D_2 > 0 \tag{3.5}$$

we introduce the transformation given below to cancel out the cross terms in x_s, u_s, and w_s:

$$\begin{aligned}\hat{w}_s =& \left(\gamma^2 I - D_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}D_2\right)^{1/2}[w_s + \\ &\left(\gamma^2 I - D_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}D_2\right)^{-1}D_2'A_{22}'^{-1}Q_{22}A_{22}^{-1} \\ &\cdot((A_{22}Q_{22}^{-1}Q_{21} - A_{21})x_s + (A_{22}Q_{22}^{-1}P_2 - B_2)u_s)].\end{aligned} \tag{3.6}$$

Note that the dependence of $\hat{w}_s$ on u_s is allowed here since we are interested in the upper value of the game. Using this transformation in (3.2) and (3.4), we arrive at the following LQ structure:

$$\dot{x}_s = A^\square x_s + B^\square u_s + D^\square \hat{w}_s; \qquad x_s(t_0) = x_{10} \tag{3.7}$$

$$\hat{y}_s := y_s + G^\square u_s = C^\square x_s + E^\square \hat{w}_s \tag{3.8}$$

$$\begin{aligned} L_{\gamma s} = |x_s(t_f)|^2_{Q_{f11}} - \gamma^2|x_s(t_0)|^2_{Q_{011}} \\ + \int_{t_0}^{t_f} \left([x_s' \quad u_s'] \begin{bmatrix} Q^\square & P^\square \\ P^{\square\prime} & R^\square \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} - |\hat{w}_s|^2\right) dt \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} A^\square(\gamma) := A_{11} - A_{12}Q_{22}^{-1}Q_{21} - \left(A_{12}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_1D_2'\right) \\ \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2D_2'\right)^{-1} \\ \cdot (A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \end{aligned} \tag{3.10}$$

$$\begin{aligned} B^\square(\gamma) := B_1 - A_{12}Q_{22}^{-1}P_2 + \left(A_{12}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_1D_2'\right) \\ \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2D_2'\right)^{-1} \\ \cdot (A_{22}Q_{22}^{-1}P_2 - B_2) \end{aligned} \tag{3.11}$$

$$D^\square(\gamma) := D_0\left(\gamma^2 I - D_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}D_2\right)^{-1/2} \tag{3.12}$$

$$\begin{aligned} G^\square(\gamma) := C_2Q_{22}^{-1}P_2 - \left(C_2Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}L_2'\right) \\ \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2D_2'\right)^{-1} \\ \cdot (A_{22}Q_{22}^{-1}P_2 - B_2) \end{aligned} \tag{3.13}$$

$$\begin{aligned} C^\square(\gamma) := C_1 - C_2Q_{22}^{-1}Q_{21} - \left(C_2Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}L_2'\right) \\ \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2}D_2D_2'\right)^{-1} \\ \cdot (A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \end{aligned} \tag{3.14}$$

$$E^\square(\gamma) := E_0\left(\gamma^2 I - D_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} D_2\right)^{-1/2} \tag{3.15}$$

$$\begin{aligned} Q^\square(\gamma) := & Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + (A_{21}' - Q_{12}Q_{22}^{-1}A_{22}') \\ & \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} D_2 D_2'\right)^{-1} \\ & \cdot (A_{21} - A_{22}Q_{22}^{-1}Q_{21}) \end{aligned} \tag{3.16}$$

$$\begin{aligned} P^\square(\gamma) := & P_1 - Q_{12}Q_{22}^{-1}P_2 - (A_{21}' - Q_{12}Q_{22}^{-1}A_{22}') \\ & \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} D_2 D_2'\right)^{-1} \\ & \cdot (A_{22}Q_{22}^{-1}P_2 - B_2) \end{aligned} \tag{3.17}$$

$$\begin{aligned} R^\square(\gamma) := & R - P_2'Q_{22}^{-1}P_2 + (P_2'Q_{22}^{-1}A_{22}' - B_2') \\ & \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} D_2 D_2'\right)^{-1} \\ & \cdot (A_{22}Q_{22}^{-1}P_2 - B_2). \end{aligned} \tag{3.18}$$

Detailed derivations of the above can be found in [23]; see also Appendix B for an outline of the derivation.

We can also compute (see Appendix B and [23] for detailed derivations)

$$\begin{aligned} L^\square := & D^\square E^{\square\prime} \\ = & \frac{1}{\gamma^2} L_1 - A_{12}Q_{22}^{-1}C_2' + \left(A_{12}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} D_1 D_2'\right) \\ & \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} D_1 D_2'\right)^{-1} \\ & \cdot \left(A_{22}Q_{22}^{-1}C_2' - \frac{1}{\gamma^2} L_2\right) \end{aligned} \tag{3.19}$$

$$\begin{aligned} N^\square := & E^\square E^{\square\prime} \\ = & \frac{1}{\gamma^2} N - C_2Q_{22}^{-1}C_2' + \left(C_2Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} L_2'\right) \\ & \cdot \left(A_{22}Q_{22}^{-1}A_{22}' - \frac{1}{\gamma^2} D_2 D_2'\right)^{-1} \\ & \cdot \left(A_{22}Q_{22}^{-1}C_2' - \frac{1}{\gamma^2} L_2\right). \end{aligned} \tag{3.20}$$

The system described by (3.7)–(3.9) is now a standard LQ differential game, but under imperfect state measurements, where the slow control u_s at time t can use only the measurements $\{\hat{y}_s(\tau), \tau \le t\}$. For this system, there will exist a "minimal" $\gamma_I > 0$, such that $\forall\ \gamma > \gamma_I$, the following three conditions are satisfied[1] (see [16, ch. 5, p. 146]).

Condition 1: The following GRDE admits a bounded nonnegative definite solution $Z_{s\gamma}$

$$\dot{\tilde{Z}}_s + \overline{A}^{\square\prime}\tilde{Z}_s + \tilde{Z}_s\overline{A}^\square - \tilde{Z}_s\overline{S}_0\tilde{Z}_s + \overline{Q}^\square = 0; \quad \tilde{Z}_s(t_f) = Q_{f11} \tag{3.21}$$

where

$$\overline{A}^\square := A^\square - B^\square R^{\square-1}P^{\square\prime}; \qquad \overline{Q} := Q^\square - P^\square R^{\square-1}P^{\square\prime}$$
$$\overline{S}_0 := B^\square R^{\square-1}B^{\square\prime} - D^\square D^{\square\prime}.$$

[1] Note that under condition (3.5), $Q^\square(\gamma) \ge 0$.

Condition 2: The following GRDE admits a bounded positive definite solution $\tilde{\Sigma}_s^*$

$$\dot{\tilde{\Sigma}}_s = \tilde{A}^\square\tilde{\Sigma}_s + \tilde{\Sigma}_s\tilde{A}^{\square\prime} - \gamma^2\tilde{\Sigma}_s\tilde{R}_0\tilde{\Sigma}_s + \frac{1}{\gamma^2}\tilde{M}^\square; \quad \tilde{\Sigma}_s(t_0) = \frac{1}{\gamma^2} Q_{011}^{-1} \tag{3.22}$$

where

$$\tilde{A}^\square := A^\square - L^\square N^{\square-1}C^\square; \qquad \gamma^2\tilde{R}_0 := C^{\square\prime}N^{\square-1}C^\square - Q^\square$$
$$\frac{1}{\gamma^2}\tilde{M}^\square := D^\square D^{\square\prime} - L^\square N^{\square-1}L^{\square\prime}.$$

An equivalent condition can be obtained in terms of $\Sigma_{s\gamma} := \gamma^2\tilde{\Sigma}_s^*$, where the counterpart of (3.22) is

$$\dot{\Sigma}_{s\gamma} = \tilde{A}^\square\Sigma_{s\gamma} + \Sigma_{s\gamma}\tilde{A}^{\square\prime} - \Sigma_{s\gamma}\tilde{R}_0\Sigma_{s\gamma} + \tilde{M}^\square; \quad \Sigma_{s\gamma}(t_0) = Q_{011}^{-1}. \tag{3.23}$$

Condition 3: The spectral radius condition

$$\frac{1}{\gamma^2}\Sigma_{s\gamma}^{-1} - Z_{s\gamma} > 0. \tag{3.24}$$

Under these three conditions, the "slow" differential game has a bounded upper value. For $\gamma < \gamma_I$, on the other hand, at least one of the Conditions 1–3 is violated and the game has unbounded upper value.

After some detailed manipulations (see Appendix B, as well as [23]), we arrive at the following forms for $\overline{A}^\square, \overline{S}_0, \overline{Q}^\square, \tilde{A}^\square, \tilde{R}_0$ and $\tilde{M}^\square$.

$$\begin{aligned} \overline{A}^\square = & \overline{A}_{11} - \overline{A}_{12}\overline{Q}_{22}^{-1}\overline{Q}_{21}^{-1} - (\overline{S}_{12} + \overline{A}_{12}\overline{Q}_{22}^{-1}\overline{A}_{22}') \\ & \cdot (\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1}(\overline{A}_{21} - \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}\overline{Q}_{21}) \end{aligned} \tag{3.25}$$

$$\begin{aligned} \overline{Q}^\square = & \overline{Q}_{11} - \overline{Q}_{12}\overline{Q}_{22}^{-1}\overline{Q}_{21} + (\overline{A}_{21}' - \overline{Q}_{12}\overline{Q}_{22}^{-1}\overline{A}_{22}') \\ & \cdot (\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1}(\overline{A}_{21} - \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{Q}_{21}) \end{aligned} \tag{3.26}$$

$$\begin{aligned} \overline{S}_0 = & \overline{S}_{11} + \overline{A}_{12}\overline{Q}_{22}^{-1}\overline{A}_{12}' - (\overline{S}_{12} + \overline{A}_{12}\overline{Q}_{22}^{-1}\overline{A}_{22}') \\ & \cdot (\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1}(\overline{S}_{21} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{12}') \end{aligned} \tag{3.27}$$

$$\begin{aligned} \tilde{A}^\square = & \tilde{A}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{A}_{21} - (\tilde{A}_{12} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{A}_{22}) \\ & \cdot (\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{R}_{21} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{21}) \end{aligned} \tag{3.28}$$

$$\begin{aligned} \tilde{R}_0 = & \tilde{R}_{11} + \tilde{A}_{21}'\tilde{M}_{22}^{-1}\tilde{A}_{21} - (\tilde{R}_{12} + \tilde{A}_{21}'\tilde{M}_{22}^{-1}\tilde{A}_{22}) \\ & \cdot (\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{R}_{21} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{21}) \end{aligned} \tag{3.29}$$

$$\begin{aligned} \tilde{M}^\square = & \tilde{M}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{M}_{21} + (\tilde{A}_{12} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{A}_{22}) \\ & \cdot (\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{A}_{12}' - \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{M}_{21}). \end{aligned} \tag{3.30}$$

In view of this, let us introduce the set

$$\hat{\Gamma}_{Is} := \{\gamma' > 0 : \forall \gamma \geq \gamma', (3.5) \text{ holds and Conditions } 1\text{–}3 \text{ are satisfied}\}.$$

Let us further define

$$\hat{\gamma}_{Is} := \inf\{\gamma \in \hat{\Gamma}_{Is}\}. \tag{3.31}$$

Then, the transformed game with cost function $L_{\gamma s}$ has a bounded upper value if $\gamma > \hat{\gamma}_{Is}$, and only if $\gamma \geq \hat{\gamma}_{Is}$. For $\gamma > \hat{\gamma}_{Is}$, the control policy that delivers the upper value is

$$u_{Is}^* = \hat{\mu}_{Is}^*(t, \hat{y}_{s[t_0,t]}) = -R^{\square-1}(B^{\square\prime} Z_{s\gamma} + P^{\square\prime})\hat{x}_s(t) \tag{3.32}$$

where $\hat{x}_s$ is generated by the following differential equation:

$$\dot{\hat{x}}_s = A^\square \hat{x}_s + B^\square u_{IS}^* + D^\square \hat{w}_s + \left(I - \frac{1}{\gamma^2}\Sigma_{s\gamma} Z_{s\gamma}\right)^{-1} \cdot \left(\frac{1}{\gamma^2}\Sigma_{s\gamma} C^{\square\prime} + L^\square\right) N^{\square-1}(\hat{y}_s - \hat{\hat{y}}_s); \qquad \hat{x}_s(t_0) = 0 \tag{3.33}$$

$$\hat{w}_s = D^{\square\prime} Z_{s\gamma} \hat{x}_s \tag{3.34}$$

$$\hat{\hat{y}}_s = C^\square \hat{x}_s + E^\square \hat{w}_s. \tag{3.35}$$

Substituting (3.34), (3.35), and (3.8) into (3.33), after detailed derivations (see Appendix B and [23] for details), we arrive at

$$u_{Is}^*(t, y_{[t_0,t]}) = -R^{-1}(B_1' Z_{s\gamma} + P_1' - P_2'\overline{U} + B_2'\overline{V})\hat{x}_s(t) \tag{3.36}$$

$$\dot{\hat{x}}_s = (\overline{A}^\square - \overline{S}_0 Z_{s\gamma})\hat{x}_s + \left(I - \frac{1}{\gamma^2}\Sigma_{s\gamma} Z_{s\gamma}\right)^{-1} \cdot (\Sigma_{s\gamma} C_1' + L_1 - \tilde{U}' L_2 + \tilde{V} C_2') N^{-1} \cdot \left(y - \left(C_1 + \frac{1}{\gamma^2} L_1' - C_2\overline{U} + \frac{1}{\gamma^2} L_2'\overline{V}\right)\hat{x}_s\right); \quad \hat{x}_s(t_0) = 0 \tag{3.37}$$

where

$$\overline{U} := \overline{U}_1 Z_{s\gamma} + \overline{U}_2 \tag{3.38}$$

$$\overline{V} := \overline{V}_1 Z_{s\gamma} + \overline{V}_2 \tag{3.39}$$

$$\tilde{U} := \tilde{U}_1 \Sigma_{s\gamma} + \tilde{U}_2 \tag{3.40}$$

$$\tilde{V} := \tilde{V}_1 \Sigma_{s\gamma} + \tilde{V}_2 \tag{3.41}$$

$$\overline{U}_1 := \overline{Q}_{22}^{-1}\overline{A}_{12}' - \overline{Q}_{22}^{-1}\overline{A}_{22}'(\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1} \cdot (\overline{S}_{21} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{12}') \tag{3.42}$$

$$\overline{U}_2 := \overline{Q}_{22}^{-1}\overline{Q}_{21} + \overline{Q}_{22}^{-1}\overline{A}_{22}'(\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1} \cdot (\overline{A}_{21} - \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{Q}_{21}) \tag{3.43}$$

$$\overline{V}_1 := -(\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1}(\overline{S}_{21} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{12}') \tag{3.44}$$

$$\overline{V}_2 := (\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}')^{-1}(\overline{A}_{21} - \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{Q}_{21}) \tag{3.45}$$

$$\tilde{U}_1 := \tilde{M}_{22}^{-1}\tilde{A}_{21} - \tilde{M}_{22}^{-1}\tilde{A}_{22}(\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1} \cdot (\tilde{R}_{21} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{21}) \tag{3.46}$$

$$\tilde{U}_2 := \tilde{M}_{22}^{-1}\tilde{M}_{21} + \tilde{M}_{22}^{-1}\tilde{A}_{22}(\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1} \cdot (\tilde{A}_{12}' - \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{M}_{21}) \tag{3.47}$$

$$\tilde{V}_1 := -(\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{R}_{21} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{21}) \tag{3.48}$$

$$\tilde{V}_2 := (\tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{A}_{12}' - \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{M}_{21}). \tag{3.49}$$

From (3.25)–(3.30), (3.36), and (3.37), we see that the coefficients of the Riccati differential equations (3.21) and (3.22), as well as the $\mu_{Is}^*(t, y_{[t_0,t]})$, do not actually depend on the validity of condition (3.5). This resembles the situation encountered in [18], where we had actually eliminated that condition by introducing a disturbance feedforward into the slow subsystem, which was then realizable using the fast states. Here, however, this does not seem to be possible since the measurements are noisy. Still, motivated by the solution of the full-information problem in [18], we introduce the set

$$\tilde{\Gamma}_{Is} := \{\gamma' > 0 : \ \forall \gamma \geq \gamma', \overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}' > 0, \ \tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22} > 0, (3.21) \text{ has a bounded nonnegative definite solution over } [t_0, t_f], (3.22) \text{ has a bounded positive definite solution over } [t_0, t_f], \text{ and } \gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma} > 0\} \tag{3.50}$$

and further define

$$\gamma_{Is} := \inf\{\gamma \in \tilde{\Gamma}_{Is}\} \tag{3.51}$$

which constitutes a lower bound on the ultimate performance of the full-order system. We will see in the following section that γ_{Is} will indeed play a key role in the solution of the original problem.

The Fast Subsystem and the Associated Soft-Constrained Game

Let $x_f := x_2 - \overline{x}_2, u_f := u - u_s, w_f := w - w_s, y_f := y - y_s$ and $\tau = t' - t/\epsilon$, where we take t to be frozen and let t' vary on the same time scale as t. We define the fast subsystem and the associated cost (as in the standard regulator problem: see [24]) by

$$\frac{d}{d\tau} x_f^t = A_{22}(t) x_f^t + B_2(t) u_f^t + D_2(t) w_f^t \tag{3.52}$$

$$y_f^t = C_2 x_f^t + E w_f^t \tag{3.53}$$

$$L_{\gamma f}^t = \int_{-\infty}^{\infty} \left(|x_f^t|_{Q22(t)}^2 + 2x_f^{t\prime} P_2(t) u_f^t + |u_f^t|_{R(t)}^2 - \gamma^2 |w_f^t|^2\right) d\tau. \tag{3.54}$$

The GARE's associated with this infinite-horizon game, for each t, are

$$\overline{A}_{22}'(t) Z_f + Z_f \overline{A}_{22}(t) + \overline{Q}_{22}(t) - Z_f \overline{S}_{22}(t) Z_f = 0 \tag{3.55}$$

$$\tilde{A}_{22}(t)\Sigma_f + \Sigma_f \tilde{A}_{22}'(t) + \tilde{M}_{22} - \Sigma_f \tilde{R}_{22}(t)\Sigma_f = 0. \tag{3.56}$$

We now let γ_{If}^t denote the minimax disturbance attenuation bound for the H^∞-optimal control problem defined by

(3.52)–(3.54) under imperfect state information,[2] and γ_{of}^{t} denote the same under open-loop information.[3] For every $\gamma > \gamma_{If}^{t}$, let $Z_{f\gamma}(t)$ be the minimal positive definite solution for (3.55) (and, for every $\gamma > \gamma_{of}^{t}$, let $Z_{of\gamma}(t)$ be the solution to the open-loop version of (3.55), with $B_2(t) = 0$) and $\Sigma_{f\gamma}(t)$ be the minimal positive definite solution of (3.56).

We now define

$$\gamma_{If} := \sup_{t\in[t_0,t_f]} \gamma_{If}^{t}\,; \qquad \gamma_{of} := \sup_{t\in[t_0,t_f]} \gamma_{of}^{t}\,.$$

Then, for every $\gamma > \gamma_{If}$, the GARE's (3.55) and (3.56) admit positive definite solutions for all $t \in [t_0, t_f]$. Let

$$\overline{\gamma}_I := \max\{\gamma_{If}, \gamma_{Is}\}\,. \tag{3.57}$$

We will shortly see that this value also plays an important role in the characterization of a reduced-order solution for our problem.

The Infinite-Horizon Case

We now turn to the infinite-horizon case. Let $A, B, D, C, E, P, R,$ and Q be time invariant, Q_f be zero, and $t_0 = -\infty, t_f = \infty$. By following steps similar to those in the finite-horizon case, we first decompose the system into slow and fast subsystems. The slow game is described by

$$\dot{x}_s = A^{\square}x_s + B^{\square}u_s + D^{\square}\hat{w}_s \tag{3.58}$$

$$\hat{y}_s = y + G^{\square}u_s = C^{\square}x_s + E^{\square}\hat{w}_s \tag{3.59}$$

$$L_{\gamma s} = \int_{-\infty}^{\infty}\left(\begin{bmatrix} x_s' & u_s'\end{bmatrix}\begin{bmatrix} Q^{\square} & P^{\square} \\ P^{\square\prime} & R^{\square}\end{bmatrix}\begin{bmatrix} x_s \\ u_s\end{bmatrix} - |\hat{w}_s|^2\right)dt \tag{3.60}$$

where $A^{\square}, B^{\square}, D^{\square}, G^{\square}, C^{\square}, E^{\square}, Q^{\square}, P^{\square}$, and $R^{\square}$ are as defined before, with the only difference being that they are now time-invariant. Similarly we can define $L^{\square}$ and $N^{\square}$ as in (3.19) and (3.20), but now time-invariant.

The associated GARE's are

$$\overline{A}^{\square\prime}\tilde{Z}_{s\infty} + \tilde{Z}_{s\infty}\overline{A}^{\square} - \tilde{Z}_{s\infty}\overline{S}_0\tilde{Z}_{s\infty} + \overline{Q}^{\square} = 0 \tag{3.61}$$

$$\tilde{A}^{\square}\tilde{\Sigma}_{s\infty} + \tilde{\Sigma}_{s\infty}\tilde{A}^{\square\prime} - \gamma^2\tilde{\Sigma}_{s\infty}\tilde{R}_0\tilde{\Sigma}_{s\infty} + \frac{1}{\gamma^2}\tilde{M}^{\square} = 0 \tag{3.62}$$

where $\overline{A}^{\square}, \overline{S}_0, \overline{Q}^{\square}, \tilde{A}^{\square}, \tilde{R}_0$, and $\tilde{M}^{\square}$ are defined as before.

Let us now introduce the following set (as the counterpart of the one in the finite-horizon case).

$$\begin{aligned}\hat{\Gamma}_{Is\infty} := \{\gamma' > 0 :\ & \forall\,\gamma \geq \gamma', (3.5) \text{ holds, } (3.61) \text{ has a minimal positive definite solution } Z_{s\gamma}, \overline{A}^{\square} - \overline{S}_0 Z_{s\gamma} \text{ is Hurwitz, } (3.62) \text{ has a minimal positive} \\ & \text{definite solution, written as } \frac{1}{\gamma^2}\Sigma_{s\gamma}, \tilde{A}^{\square\prime} - \tilde{R}_0\Sigma_{s\gamma} \\ & \text{is Hurwitz, and } \gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma} > 0\}\end{aligned} \tag{3.63}$$

and define[4]

$$\hat{\gamma}_{Is\infty} := \inf\{\gamma \in \hat{\Gamma}_{Is\infty}\}\,. \tag{3.64}$$

For every $\gamma > \hat{\gamma}_{Is\infty}$, let $Z_{s\gamma}$ be the minimal positive definite solution to (3.61), and $\tilde{\Sigma}_{s\infty}^{*}$, which is to be denoted by $(1/\gamma^2)\Sigma_{s\gamma}$, be the minimal positive definite solution to (3.62). Then, for each $\gamma > \hat{\gamma}_{Is\infty}$, the control that attains the optimal upper value is the same as in (3.32)–(3.35), which can further be simplified to (3.36)–(3.37). By an argument similar to that used in the finite-horizon case, we introduce

$$\begin{aligned}\tilde{\Gamma}_{Is\infty} := \{\gamma' > 0 :\ & \forall\,\gamma \geq \gamma', \overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}_{22}' > 0, \\ & \tilde{R}_{22} + \tilde{A}_{22}'\tilde{M}_{22}^{-1}\tilde{A}_{22} > 0, (3.61) \text{ has a minimal} \\ & \text{positive definite solution } Z_{s\gamma}, \overline{A}^{\square} - \overline{S}_0 Z_{s\gamma} \\ & \text{is Hurwitz, } (3.62) \text{ has a minimal positive definite solution, written as } \frac{1}{\gamma^2}\Sigma_{s\gamma}, \tilde{A}^{\square\prime} - \tilde{R}_0\Sigma_{s\gamma} \\ & \text{is Hurwitz, and } \gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma} > 0\}\end{aligned} \tag{3.65}$$

and define

$$\gamma_{Is\infty} := \inf\{\gamma \in \tilde{\Gamma}_{Is\infty}\}\,. \tag{3.66}$$

The fast part of the system is the same as in the finite-horizon case, where the coefficient matrices are now constants. The fast game is described by (3.52)–(3.54), and the GARE's are the same as (3.55)–(3.56). We will use $\gamma_{If\infty}$ to denote the minimax disturbance attenuation bound under imperfect state information pattern (and $\gamma_{of\infty}$ for the open-loop case), let $Z_{f\gamma}$ denote the minimal positive solution to (3.55) (and $Z_{of\gamma}$ for the open-loop case) and $\Sigma_{f\gamma}$ denote the minimal positive solution to (3.56). As the counterpart of (3.35), we define

$$\overline{\gamma}_{I\infty} := \max\{\gamma_{If\infty}, \gamma_{Is\infty}\}\,. \tag{3.67}$$

This quantity will also play an important role in our analysis in the next section.

IV. Optimality of the Slow Controller

We study in this section optimality properties of the slow controller introduced in Section III, when used in the full-order system, for both finite- and infinite-horizon cases.

The Infinite-Horizon Case

To make the $\overline{\gamma}_{I\infty}$ defined by (3.67) remain finite, we make two additional assumptions:

Assumption 4: The pairs $(\overline{A}_{11} - \overline{A}_{12}\overline{A}_{22}^{-1}\overline{A}_{21}, \overline{Q}_{11} - \overline{Q}_{12}\overline{Q}_{22}^{-1}\overline{Q}_{21})$, $(A_{22}C_2)$ and (A_0, C_0) are observable.

Assumption 5: The pairs (A_0, B_0), $(\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}, \tilde{M}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{M}_{21})$ and (A_{22}, B_2) are controllable.

Then we have the following theorem:

[2] To ensure that $\gamma_{If}^{t} < \infty$, it will be sufficient to take the pair $(A_{22}(t), B_2(t))$ to be controllable, and the pair $(A_{22}(t), C_2(t))$ to be observable.

[3] To ensure that $\gamma_{of}^{t} < \infty$, it will be sufficient to take $A_{22}(t)$ to be Hurwitz, for every fixed t.

[4] Again, for $\hat{\gamma}_{Is\infty}$ to be finite, it will be sufficient to have the pair $(A^{\square}, B^{\square})$ controllable and the pair $(A^{\square}, C^{\square})$ observable at $\gamma = \infty$, which is equivalent to having the pair (A_0, B_0) controllable and the pair (A_0, C_0) observable.

Theorem 1: Consider the singularly perturbed system (2.1)–(2.6), with $t_f = \infty, t_0 = -\infty, Q_f = 0, Q_0 = \infty$ and A, B, D, C, E, P, R, Q time-variant. If assumptions 1–5 hold, then

1) $\lim_{\epsilon \to 0^+} \gamma_I^*(\epsilon) = \overline{\gamma}_{I\infty}$, where $\overline{\gamma}_{I\infty}$, as defined in (3.67), is finite.

2) $\forall\ \gamma > \overline{\gamma}_{I\infty}, \exists\ \epsilon_\gamma > 0$ such that $\forall\ \epsilon \in [0, \epsilon_\gamma)$, the GARE (2.21) admits a positive definite solution, where the minimal such solution can be approximated by

$$\tilde{Z}(\epsilon) = \begin{bmatrix} Z_{s\gamma} + O(\epsilon) & \epsilon\big(Z_{f\gamma}\overline{U} + \overline{V}\big)' + O(\epsilon^2) \\ \epsilon\big(Z_{f\gamma}\overline{U} + \overline{V}\big) + O(\epsilon^2) & \epsilon Z_{f\gamma} + O(\epsilon^2) \end{bmatrix}. \tag{4.1}$$

Similarly, the GARE (2.22) admits a positive definite solution, where the minimal such solution can be approximated by

$$\tilde{\Sigma}(\epsilon) = \begin{bmatrix} \Sigma_{s\gamma} + O(\epsilon) & (\Sigma_{f\gamma}\tilde{U} + \tilde{V})' + O(\epsilon) \\ \Sigma_{f\gamma}\tilde{U} + \tilde{V} + O(\epsilon) & \frac{1}{\epsilon}\left(\Sigma_{f\gamma} + O(\epsilon)\right) \end{bmatrix}. \tag{4.2}$$

Furthermore, $\left(1/\gamma^2\right)\tilde{\Sigma}^{-1}(\epsilon) - \tilde{Z}(\epsilon) > 0$. Consequently, the game has a finite upper value.

3) $\forall\ \gamma > \max\{\gamma_{Is\infty}, \gamma_{of\infty}\} = \max\{\hat{\gamma}_{Is\infty}, \gamma_{of\infty}\}$, if we apply to the system the "slow" controller μ_{Is}^* defined by (3.36)–(3.37), then $\exists\ \epsilon'_\gamma > 0$ such that, $\forall\ \epsilon \in [0, \epsilon'_\gamma)$, the disturbance attenuation level γ is attained for the full-order system.

Proof: We first note that under Assumptions 1–5, both slow and fast games admit saddle-point solutions for sufficiently large values of γ, since at $\gamma = \infty$ both problems become "regular" LQR problems admitting stabilizing optimal controllers. This shows that both $\gamma_{Is\infty}$ and $\gamma_{If\infty}$ are finite, implying that $\overline{\gamma}_{I\infty}$ is also finite. Fix $\gamma > \overline{\gamma}_{I\infty}$. By an argument similar to that used in the proof of Theorem 1 of [18] and duality, we conclude that the pair (A_ϵ, Q) is observable and the pair (A_ϵ, D_ϵ) is controllable. We know that GARE (3.61) admits a minimal positive definite solution $Z_{s\gamma}$ and the GARE (3.55) (with time-invariant coefficients) admits a minimal positive definite solution $Z_{f\gamma}$. By the definition of $\gamma_{Is\infty}$, $\overline{A}^{\square} - \overline{S}_0 Z_{s\gamma}$ is Hurwitz. It is well-known that, for $\gamma > \gamma_{If\infty}, \overline{A}_{22} - \overline{S}_{22} Z_{f\gamma}$ is Hurwitz. Thus, we can conclude, by applying the method of proof of Theorem 1 in [18], that GARE (2.21) admits a positive definite solution for sufficiently small positive ϵ, which can further be approximated by (4.1).

On the other hand, GARE (3.62) admits a minimal positive definite solution $\left(1/\gamma^2\right)\Sigma_{s\gamma}$, with $\tilde{A}^{\square\prime} - \tilde{R}_0\Sigma_{s\gamma}$ Hurwitz, and furthermore GARE (3.56) (with time-invariant coefficients) admits a minimal positive solution $\Sigma_{f\gamma}$, with $\tilde{A}'_{22} - \tilde{R}_{22}\Sigma_{f\gamma}$ Hurwitz. Thus, by applying the method of proof of Theorem 1 in [18], (2.26)–(2.28) admit a solution and hence GARE (2.22) admits a solution, which can further be approximated by (4.2). Then if follows that, for sufficiently small positive ϵ, the solution in (4.2) is positive definite.

Now, we can easily evaluate

$$I - \frac{1}{\gamma^2}\tilde{\Sigma}\tilde{Z} = \begin{bmatrix} I - \frac{1}{\gamma^2}\Sigma_{s\gamma}Z_{s\gamma} + O(\epsilon) & O(\epsilon) \\ * & I - \frac{1}{\gamma^2}\Sigma_{f\gamma}Z_{f\gamma} + O(\epsilon) \end{bmatrix}$$

where $*$ stands for any constant terms of $O(1)$. This shows that for ϵ sufficiently small, the matrix $I - \left(1/\gamma^2\right)\tilde{\Sigma}\tilde{Z}$ can have only positive eigenvalues. Thus, 1) is proved.

To prove part 1), note that $\forall \gamma < \overline{\gamma}_{I\infty}$, either at least one of the GARE's (3.61), (3.55), (3.62), and (3.56) does not admit any positive definite solution, or one of the matrices $\gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma}$ and $\gamma^2\Sigma_{f\gamma}^{-1} - Z_{f\gamma}$has at least one negative eigenvalue. The former implies that one of the GARE's (2.21) and (2.22) does not admit any positive definite solution for sufficiently small ϵ, by the result of [18], which further implies that $\gamma < \gamma_I^*(\epsilon)$.The latter implies that the matrix $I - \frac{1}{\gamma^2}\tilde{\Sigma}\tilde{Z}$ has at least one negative eigenvalue for sufficiently small ϵ, which again implies that $\gamma < \gamma_I^*(\epsilon)$. Hence, $\gamma < \gamma_I^*(\epsilon)$ for sufficiently small ϵ, $\forall \gamma < \overline{\gamma}_{I\infty}$. Thus, 1) is proved.

For 3), we first note that for $\gamma > \gamma_{of\infty}$, condition (3.5) holds. Thus we can write the closed-loop system, using the parameter matrices of the slow subsystem, as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\hat{x}}_s \\ \epsilon\dot{x}_2 \end{bmatrix} = F^e \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix} + D^e w \tag{4.3}$$

where

$$F^e := \begin{bmatrix} F_{11}^e & F_{12}^e \\ F_{21}^e & F_{22}^e \end{bmatrix}; \qquad F_{11}^e := \begin{bmatrix} F_{11}^{e11} & F_{12}^{e11} \\ F_{21}^{e11} & F_{22}^{e11} \end{bmatrix}$$

$$F_{11}^{e11} := A_{11}; \qquad F_{12}^{e11} := -B_1 R^{\square-1}(B^{\square\prime} Z_{s\gamma} + P^{\square\prime})$$

$$F_{21}^{e11} := \left(I - \frac{1}{\gamma^2}\Sigma_{s\gamma}Z_{s\gamma}\right)^{-1}\left(\frac{1}{\gamma^2}\Sigma_{s\gamma}C^{\square\prime} + L^{\square}\right)N^{\square-1}C_1$$

$$\begin{aligned} F_{22}^{e11} := \overline{A}^{\square} - \overline{S}_0 Z_{s\gamma} - \left(I - \frac{1}{\gamma^2}\Sigma_{s\gamma}Z_{s\gamma}\right)^{-1} \\ \cdot \left(\frac{1}{\gamma^2}\Sigma_{s\gamma}C^{\square\prime} + L^{\square}\right)N^{\square-1} \\ \cdot \left(G^{\square}R^{\square-1}(B^{\square\prime}Z_{s\gamma} + P^{\square\prime}) + C^{\square} + L^{\square\prime}Z_{s\gamma}\right) \end{aligned}$$

$$F_{12}^e := \begin{bmatrix} A_{12} \\ \left(I - \frac{1}{\gamma^2}\Sigma_{s\gamma}Z_{s\gamma}\right)^{-1}\left(\frac{1}{\gamma^2}\Sigma_{s\gamma}C^{\square\prime} + L^{\square}\right)N^{\square-1}C_2 \end{bmatrix}$$

$$F_{21}^e := \left[A_{21} \vdots - B_2 R^{\square-1}(B^{\square\prime}Z_{s\gamma} + P^{\square\prime})\right];$$

$$F_{22}^e := A_{22}; \qquad D_2^e := D_2$$

$$D^e := \begin{bmatrix} D_1^e \\ D_2^e \end{bmatrix}; \qquad D_1^e := \begin{bmatrix} D_1 \\ \hdashline \left(I - \frac{1}{\gamma^2}\Sigma_{s\gamma}Z_{s\gamma}\right)^{-1} \\ \cdot \left(\frac{1}{\gamma^2}\Sigma_{s\gamma}\, C^{\square\prime} + L^{\square}\right) \\ \cdot N^{\square-1}E \end{bmatrix}$$

where we have used the slow controller in the form of (3.32)–(3.35) and replaced $\hat{y}_s$ by its expression given by (3.8). We can also rewrite the cost function as

$$J = \int_{-\infty}^{\infty} \left\{ \left\| \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix} \right\|^2_{Q^e} - \gamma^2 |w|^2 \right\} dt \qquad (4.4)$$

where

$$Q^e := \begin{bmatrix} Q^e_{11} & Q^e_{12} \\ Q^e_{21} & Q^e_{21} \end{bmatrix}; \qquad Q^e_{11} := \begin{bmatrix} Q^{e11}_{11} & Q^{e11}_{12} \\ Q^{e11}_{21} & Q^{e11}_{21} \end{bmatrix};$$

$$Q^{e11}_{11} := Q_{11}$$

$$Q^{e11}_{12} := -P_1 R^{\square -1}(B^{\square\prime} Z_{s\gamma} + P^{\square\prime}); \qquad Q^{e11}_{21} := Q^{e11\prime}_{12}$$

$$Q^{e11}_{22} := (Z_{s\gamma} B^{\square} + P^{\square}) R^{\square -1} R R^{\square -1} (B^{\square\prime} Z_{s\gamma} + P^{\square\prime})$$

$$Q^e_{12} := \begin{bmatrix} Q_{12} \\ -(Z_{s\gamma} B^{\square} + P^{\square}) R^{\square -1} P_2' \end{bmatrix}; \qquad Q^e_{21} := Q^{e\prime}_{12};$$

$$Q^e_{22} := Q_{22}.$$

This is an infinite-horizon, one-person maximization problem, with the initial state at $t = -\infty$ being 0. It has a finite maximum (which will be zero since the initial state is zero) if the following ARE admits a positive definite solution:

$$F^{e\prime}_\epsilon \tilde{\Xi} + \tilde{\Xi} F^e_\epsilon + \tilde{\Xi} \frac{1}{\gamma^2} D^e_\epsilon D^e_\epsilon \tilde{\Xi} + Q^e = 0 \qquad (4.5)$$

where

$$F^e_\epsilon := \begin{bmatrix} F^e_{11} & F^e_{12} \\ \frac{1}{\epsilon} F^e_{21} & \frac{1}{\epsilon} F^e_{22} \end{bmatrix}; \qquad D^e_\epsilon := \begin{bmatrix} D^e_1 \\ \frac{1}{\epsilon} D^e_2 \end{bmatrix}.$$

We now first decompose (4.5) into slow and fast parts, each of which is an ARE independent of $\epsilon > 0$. Then we observe that the solution to the fast ARE is $Z_{of\gamma}$, and that (see Appendix C) a solution to the slow ARE is

$$\begin{bmatrix} \gamma^2 \Sigma^{-1}_{s\gamma} & -(\gamma^2 \Sigma^{-1}_{s\gamma} - Z_{s\gamma}) \\ -(\gamma^2 \Sigma^{-1}_{s\gamma} - Z_{s\gamma}) & \gamma^2 \Sigma^{-1}_{s\gamma} - Z_{s\gamma} \end{bmatrix} := \Xi_{s\gamma} \qquad (4.6)$$

which is clearly positive definite, since $\gamma^2 \Sigma^{-1}_{s\gamma} - Z_{s\gamma} > 0$ from the definition of $\hat{\Gamma}_{Is\infty}$. Then, there exists a minimal positive definite solution $\overline{\Xi}_{s\gamma}$ to the slow ARE.

Now using an analysis similar to that used in the proof of Theorem 1 of Part I (which involved the use of the implicit function theorem) we can show that existence of positive definite solutions to the slow and fast parts of (4.5) implies existence of a positive definite solution to the full-order equation (4.5) for sufficiently small $\epsilon > 0$. This completes the proof of part 3) of the theorem. □

The Finite-Horizon Case

Theorem 2: For the singularly perturbed system (2.1)–(2.6), let Assumptions 1–3 be satisfied, the pair $(A_{22}(t), B_2(t))$ be controllable, the pair $(A_{22}(t), C_2(t))$ be observable for each $t \in [t_0, t_f]$, and the following conditions hold:

$Q_{f22} < Z_{f\gamma}(t_f)$, where $Z_{f\gamma}(t_f)$ is the solution to (3.55) at $t = t_f$ with γ fixed.

$Q_{022} > \Sigma^{-1}_{f\gamma}(t_0)$, where $\Sigma_{f\gamma}(t_0)$ is the solution to (3.56) at $t = t_0$ with γ fixed.

Then,

1) $\gamma^*_I(\epsilon) \le \overline{\gamma}_I$, asymptotically as $\epsilon \to 0$, where $\overline{\gamma}_I$ as defined in (3.57), is finite.

2) $\forall\ \gamma > \overline{\gamma}_I, \exists\ \epsilon_\gamma > 0$ such that $\forall\ \epsilon \in [0, \epsilon_\gamma)$, the GRDE (2.9) admits a nonnegative definite solution, which can be approximated by (4.7), given below, for all $t \in [t_0, t_f]$, where $Z_{cb}(\tau)$ and $Z_{cb}(\tau)$ are boundary layer correction terms at $t = t_f$, and as $\tau \to -\infty$ they converge to 0 exponentially in the τ time scale. Also the GRDE (2.10) admits a positive definite solution, which can be approximated by (4.8) given below, for all $t \in [t_0, t_f]$, where $\Sigma_{cb}(\tau)$ and $\Sigma_{fb}(\tau)$ are boundary layer correction terms at $t = t_0$, and as $\tau \to \infty$ they converge to 0 exponentially in the τ time scale.

Furthermore, $(1/\gamma^2)\tilde{\Sigma}^{-1} - \tilde{Z} > 0$. Consequently, the game (2.1)–(2.6) has a finite value.

3) With γ fixed, let $\gamma^2 Q_{022} > Z_{of\gamma}(t_0)$. If furthermore $\gamma > \max\{\gamma_{Is}, \gamma_{of}\} = \max\{\hat{\gamma}_{Is}, \gamma_{of}\}$, and we apply to the system the "slow" controller μ^*_{Is} defined by (3.36)–(3.37), then $\exists\ \epsilon'_\gamma > 0$ such that, $\forall\ \epsilon \in [0, \epsilon'_\gamma)$, the disturbance attenuation level γ is attained for the full-order system

Proof: By a reasoning similar to that used in Theorem 1, we can apply Theorem 2 (and its method of proof) in [18] to verify (4.7) and (4.8). Hence, we only need to show the validity of the spectral radius condition for the full-order system to prove 1) and 2). We can simply evaluate $I - (1/\gamma^2)\tilde{\Sigma}\tilde{Z}$ as in (o), found on the bottom of the next page, where $*$ stands for any constant term of $O(1)$. Since $\gamma > \gamma_{Is}, I - (1/\gamma^2)\Sigma_{s\gamma}(t) Z_{s\gamma}(t)$ has only positive eigenvalues. Furthermore, since $\gamma > \gamma_{If}$, $I - (1/\gamma^2)\Sigma_{f\gamma}(t) Z_{f\gamma}(t)$ has only positive eigenvalues. For the boundary layer terms, we first consider $t \approx t_f$. By choosing ϵ sufficiently small, $\Sigma_{fb}(\tau)$ can be made arbitrary small. We can see from the differential equation for $Z_{fb}(\tau)$ (see (2.21) and (2.23) in [18]) that $Z_{f\gamma}(t_f) + Z_{fb}(\tau) \le Z_{f\gamma}(t_f) < \gamma^2 \Sigma_{f\gamma}(t_f)\ \forall\ \tau$. Thus, we can conclude that, for ϵ sufficiently small, the 22-block of the matrix (o) can be made to have only positive eigenvalues at $t \approx t_f$, due to the continuity of $Z_{f\gamma}(t)$. The same holds for

$$\tilde{Z} = \begin{bmatrix} Z_{s\gamma} + O(\epsilon) & \epsilon(Z_{f\gamma}\overline{U} + \overline{V} + Z_{cb}(\tau))' + O(\epsilon^2) \\ \epsilon(Z_{f\gamma}\overline{U} + \overline{V} + Z_{cb}(\tau)) + O(\epsilon^2) & \epsilon(Z_{f\gamma} + Z_{fb}(\tau)) + O(\epsilon^2) \end{bmatrix} \qquad (4.7)$$

$$\tilde{\Sigma} = \begin{bmatrix} \Sigma_{s\gamma} + O(\epsilon) & (\Sigma_{f\gamma}\tilde{U} + \tilde{V} + \Sigma_{cb}(\tau))' + O(\epsilon) \\ \Sigma_{f\gamma}\tilde{U} + \tilde{V} + \Sigma_{cb}(\tau) + O(\epsilon) & \frac{1}{\epsilon}(\Sigma_{f\gamma} + \Sigma_{fb}(\tau) + O(\epsilon)) \end{bmatrix} \qquad (4.8)$$

$t \approx t_0$. For $t \in (t_0, t_f)$, choose ϵ sufficiently small to make the two-boundary-layer terms arbitrary small. Then, by choosing ϵ sufficiently small, the 22-block of the above matrix can be made to have only positive eigenvalues $\forall\, t \in [t_0, t_f]$. Thus, we have established 1) and 2).

For 3), we follow a reasoning similar to that used in the proof of Theorem 1. First we rewrite the system equation as in (4.3) and the cost function in a form similar to (4.4):

$$
\begin{aligned}
J = &\int_{t_0}^{t_f} \left\{ \left\| \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix} \right\|^2_{Q^e} - \gamma^2 |w|^2 \right\} dt + \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix}' \\
&\cdot \begin{bmatrix} Q_{f11} & 0 & \epsilon Q_{f12} \\ 0 & 0 & 0 \\ \epsilon Q_{f12} & 0 & \epsilon Q_{f22} \end{bmatrix} \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix} \Bigg|_{t=t_f} - \gamma^2 \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix}' \\
&\cdot \begin{bmatrix} Q_{011} & 0 & \epsilon Q_{012} \\ 0 & mI & 0 \\ \epsilon Q_{012} & 0 & \epsilon Q_{022} \end{bmatrix} \begin{bmatrix} x_1 \\ \hat{x}_s \\ x_2 \end{bmatrix} \Bigg|_{t=t_0} \\
\equiv &\|x^e\|^2_{Q^e} + |x^e(t_f)|^2_{Q^e_f} - \gamma^2 |x^e(t_0)|^2_{Q^e_0} \qquad (4.9)
\end{aligned}
$$

where m is any sufficiently large positive number (we can do this since $\hat{x}_s(t_0) = 0$), and the coefficient matrices F^e and Q^e are defined as before. Let $\tilde{\Xi}$ be the bounded positive definite solution to the following RDE (the existence of $\tilde{\Xi}$ will be verified shortly)

$$
\begin{aligned}
&\dot{\tilde{\Xi}} + F_\epsilon^{e\prime} \tilde{\Xi} + \tilde{\Xi} F_\epsilon^e + \tilde{\Xi} \frac{1}{\gamma^2} D_\epsilon^e D_\epsilon^e \tilde{\Xi} + Q^e = 0; \\
&\tilde{\Xi}(t_f) = Q_f^e \qquad t \in [t_0, t_f]. \qquad (4.10)
\end{aligned}
$$

Then, the maximal cost can be bounded by

$$J \leq |x^e(t_0)|^2_{-\gamma^2 Q_0^e + \tilde{\Xi}(t_0)}.$$

Let us assume for the moment that the RDE (4.10) admits a decomposition with well-defined slow and fast parts. Then, its solution can be approximated by

$$\tilde{\Xi} = \begin{bmatrix} \tilde{\Xi}_{s\gamma}(t) + O(\epsilon) & O(\epsilon) \\ O(\epsilon) & \epsilon(Z_{of\gamma}(t) + \tilde{\Xi}_{fb}(\tau) + O(\epsilon)) \end{bmatrix} \quad (4.11)$$

where $\tilde{\Xi}_{s\gamma}$ is the solution to the slow sub-RDE, and $\tilde{\Xi}_{fb}(\tau)$ is the boundary layer term at $t = t_f$. Now, since $\Xi_{s\gamma}$ defined in (4.6) is also a solution to the same RDE as $\Xi_{s\gamma}$ (see Appendix C), and

$$\Xi_{s\gamma}(t_f) > \begin{bmatrix} Q_{f11} & 0 \\ 0 & 0 \end{bmatrix}$$

it follows that $\tilde{\Xi}_{s\gamma}(t)$ exists and $\Xi_{s\gamma}(t_0) > \tilde{\Xi}_{s\gamma}(t_0)$, which in turn guarantees the existence of $\tilde{\Xi}$ by using the generalized implicit function theorem given in [25, theorem L]. Since $\Sigma_{s\gamma}(t_0) = Q_{011}^{-1}$, we have that $-\gamma^2 Q_0^e + \tilde{\Xi}(t_0) < 0$ for small enough ϵ and large enough m. This then establishes 3). □

V. A Sequential Design of the Composite Controller

We see from the analysis of the previous section that for the finite-horizon case a "slow" controller can achieve a performance level $\gamma > \max\{\gamma_{Is}, \gamma_{of}\}$ (but not necessarily $\max\{\gamma_{Is}, \gamma_{If}\} < \gamma < \max\{\gamma_{Is}, \gamma_{0f}\}$). We may find in many situations that the fast subsystem is more sensitive to noise, which means that $\gamma_{of} \gg \gamma_{If}$ and $\gamma_{of} \gg \gamma_{Is}$. In such situations, it may be possible to obtain significantly better performance by designing a composite controller. Toward a study of this possibility, we first make the following simplifying assumption.[5]

Assumption 6: $R = I, L = 0$, and $P = 0$.

Note that this means that the cost function has no cross terms between the state and the control, has unity weighting on control, and the system and measurement noises are independent.

We will discuss the design of the composite controller first for the finite-horizon case.

Fast Subsystem Revisited

For the fast subsystem described by (3.52)–(3.53), we cannot design a dynamic output feedback controller since it is not a feasible design when the true value of ϵ is not known. This then leaves, as the only possibility, the use of static output feedback controllers.

First we state a lemma regarding the static output feedback control design for a general system, for which a proof can be found in Appendix D.

Lemma 1: Consider the system

$$\dot{x} = Ax + Bu + Dw \quad (5.1)$$

and the measurement equation

$$y = Cx + Ew \quad (5.2)$$

along with the cost function

$$J = \int_{-\infty}^{\infty} (|x|_Q^2 + |u|^2)\, dt; \qquad Q \geq 0 \quad (5.3)$$

where $DE' = 0$ and $N := EE'$ is invertible. Let $x \to 0$ as $t \to -\infty$, and assume that the pair (A, D) is controllable and the pair (A, Q) is observable.

Then, for any $\gamma > 0$, there exists a control law in the form $\mu(y) = K(\gamma)y$, for some constant matrix $K(\gamma)$, such that the disturbance attenuation level for the system is less than or equal to γ if

$$I - \frac{1}{\gamma^2} KN(t)K' > 0 \quad (5.4)$$

[5] This is done to simplify the ensuing analysis and not to burden it with unnecessary notational complexity. The assumption does not bring in any loss of generality of conceptual nature.

$$I - \frac{1}{\gamma^2} \tilde{\Sigma} \tilde{Z} = \begin{bmatrix} I - \frac{1}{\gamma^2} \Sigma_{s\gamma} Z_{s\gamma} + O(\epsilon) & O(\epsilon) \\ * & I - \frac{1}{\gamma^2} (\Sigma_{f\gamma} + \Sigma_{fb}(\tau))(Z_{f\gamma} + Z_{fb}(\tau)) + O(\epsilon) \end{bmatrix} \quad (o)$$

and under the matrix $K(\gamma)$, the following GARE admits a positive definite solution.

$$A'Z + ZA - Z(BB' - \frac{1}{\gamma^2}DD')Z + Q + (C'K' + ZB) \cdot (I - \frac{1}{\gamma^2}KNK')^{-1}(B'Z + KC) = 0. \quad (5.5)$$

□

In view of the above lemma, the GARE associated with the fast subsystem is

$$A_{22}'(t)Z + ZA_{22}(t) - ZS_{22}(t)Z + Q_{22}(t) + (C_2'(t)K' + ZB_2(t))\Big(I - \frac{1}{\gamma^2}KN(t)K'\Big)^{-1} \cdot (B_2'(t)Z + KC_2(t)) = 0. \quad (5.6)$$

Define

$$\chi(t,\gamma) := I - \frac{1}{\gamma^2}KN(t)K' \quad (5.7)$$

to simplify the notation. Introduce the set

$$\Gamma_{Ifs}^t := \{\gamma' \ge 0 : \forall\, \gamma > \gamma', \text{there exists a } K(t,\gamma) \text{ of appropriate dimensions such that } \chi > 0, \text{ GARE (5.6) admits a minimal positive definite solution } Z_{fs\gamma} \text{ and the matrix } A_{22} - B_2\chi^{-1}KC_2 - (S_{22} - B_2\chi^{-1}B_2')Z_{fs\gamma} \text{ is Hurwitz}\} \quad (5.8)$$

and further define

$$\gamma_{Ifs}^t := \inf\{\gamma \in \Gamma_{Ifs}^t\}; \qquad \gamma_{Ifs} := \sup_{t\in[t_0,t_f]} \gamma_{Ifs}^t.$$

For each $\gamma > \gamma_{Ifs}$, let the matrix $K(t,\gamma)$ be the matrix that is characterized in Lemma 1 for each fixed t in $[t_0, t_f]$. Then the control law that delivers the γ performance bound for each t is

$$u_{Ifs}^{t*}(\tau) = \mu_{Ifs}^{t*}(y_f^t(\tau)) = K(t,\gamma)y_f^t(\tau). \quad (5.9)$$

This control law, when transformed to the t time scale, yields

$$u_{Ifs}^*(t) = \mu_{Ifs}^*(t, y(t)) = \mu_{Ifs}^{t*}(y_f^t(0)) = K(t,\gamma)y(t). \quad (5.10)$$

Composite Design

Let $\gamma > \gamma_{Ifs}$, and introduce the composite control

$$u = u_s + u_{Ifs}^*(t) \quad (5.11)$$

for the full-order system, where $u_{Ifs}^*(t)$ is defined in (5.10). Then, substituting the above into (2.1) and (2.3), we have

$$\begin{cases} \dot{x}_1 = (A_{11} + B_1KC_1)x_1 + (A_{12} + B_1KC_2)x_2 \\ \quad + B_1u_s + (D_1 + B_1KE)w \\ \epsilon\dot{x}_2 = (A_{21} + B_2KC_1)x_1 + (A_{22} + B_2KC_2)x_2 \\ \quad + B_2u_s + (D_2 + B_2KE)w \\ y = C_1x_1 + C_2x_2 + Ew \end{cases} \quad (5.12)$$

with initial condition $x(t_0) = x_0$, where x_0 is unknown as in Section II, and

$$L(u_s w, x_0) = \int_{t_0}^{t_f} (|x(t)|_{Q(t)}^2 + |u_s(t) + K(t)y(t)|^2)\, dt + |x(t_f)|_{Q_f}^2. \quad (5.13)$$

The soft-constrained zero-sum differential game associated with this problem has the cost function

$$L_\gamma(u_s, w, x_0) = L(u_s, w, x_0) - \gamma^2\Big\{\|w\|^2 + |x_0|_{Q_0}^2\Big\}. \quad (5.14)$$

Since we are interested only in the upper value of the game, we introduce the following transformation to cancel out cross terms in x, u_s and w.[6]

$$\breve{w} = (\gamma^2 I - E'K'KE)^{1/2} \cdot (w - (\gamma^2 I - E'K'KE)^{-1} \cdot (E'K'u_s + [E'K'KC_1 \quad E'K'KC_2\,]x)). \quad (5.15)$$

Under this transformation, the system can be represented by (see Appendix E for the derivation)

$$\begin{cases} \dot{x}_1 = (A_{11} + B_1\chi^{-1}KC_1)x_1 + (A_{12} + B_1\chi^{-1}KC_2)x_2 \\ \quad + B_1\chi^{-1}u_s + (D_1 + B_1KE)(\gamma^2 I - E'K'KE)^{-1/2}\breve{w}; \\ \quad x_1(t_0) = x_{10} \\ \epsilon\dot{x}_2 = (A_{21} + B_2\chi^{-1}KC_1)x_1 + (A_{22} + B_2\chi^{-1}KC_2)x_2 \\ \quad + B_2\chi^{-1}u_s + (D_2 + B_2KE)(\gamma^2 I - E'K'KE)^{-1/2}\breve{w}; \\ \quad x_2(t_0) = x_{20} \end{cases} \quad (5.16)$$

where χ was defined by (5.7). The measurement equation can be rewritten as (see Appendix E)

$$\begin{aligned} \breve{y} :&= y - E(\gamma^2 I - E'K'KE)^{-1}E'K'u_s \\ &= \gamma^2 E(\gamma^2 I - E'K'KE)^{-1}E'N^{-1}(C_1x_1 + C_2x_2) \\ &\quad + E(\gamma^2 I - E'K'KE)^{-1/2}\breve{w}. \end{aligned} \quad (5.17)$$

Also the cost function in (5.14) can be rewritten as (see again Appendix E)

$$L_\gamma(u_s\breve{w}, x_0) = |x(t_f)|_{Q_f}^2 - \gamma^2|x_0|_{Q_0}^2 + \int_{t_0}^{t_f} (|x|_{\breve{Q}}^2 + 2x'C'K'\mathcal{X}^{-1}u_s + |u_s|_{\mathcal{X}^{-1}}^2 - |\breve{w}|^2)\, dt. \quad (5.18)$$

where $\breve{Q}$ is defined as

$$\breve{Q} := \begin{bmatrix} Q_{11} + C_1'K'\chi^{-1}KC_1 & Q_{12} + C_1'K'\chi^{-1}KC_2 \\ Q_{21} + C_2'K'\chi^{-1}KC_1 & Q_{22} + C_2'K'\chi^{-1}KC_2 \end{bmatrix}.$$

Now, we can apply the results in Sections II–IV to the transformed game (5.16)–(5.18), which leads to the following correspondences:

$$A_{ij} \leftarrow A_{ij} + B_i\chi^{-1}KC_j;$$
$$D_i \leftarrow (D_i + B_iKE)(\gamma^2 I - E'K'KE)^{-1/2}$$

[6] For $\gamma > \gamma_{Ifs}$, we have $\chi > 0$, which is equivalent to $\gamma^2 I - E'K'KE > 0$.

$$C_i \leftarrow \gamma^2 E(\gamma^2 I - E'K'KE)^{-1} EN^{-1} C_i;$$
$$E \leftarrow E(\gamma^2 I - E'K'KE)^{-1/2}$$
$$B_i \leftarrow B_i \mathcal{X}^{-1}; \qquad y \leftarrow \breve{y};$$
$$Q_{ij} \leftarrow Q_{ij} + C_i' K' \chi^{-1} K C_j; \qquad \gamma \leftarrow 1$$
$$P_i \leftarrow C_i' K' \chi^{-1}; \qquad R \leftarrow \chi^{-1};$$
$$Q_{fij} \leftarrow Q_{fij}; \qquad Q_{0ij} \leftarrow \gamma^2 Q_{0ij}$$

where $i = 1, 2$ $j = 1, 2$. We can further obtain the following:

$$N \leftarrow E(\gamma^2 I - E'K'KE)^{-1} E'; \qquad L_i \leftarrow \frac{1}{\gamma^2} B_i \chi^{-1} KN$$
$$\overline{A}_{ij} \leftarrow A_{ij} = \overline{A}_{ij}; \qquad \overline{S}_{ij} \leftarrow \overline{S}_{ij}; \qquad \overline{Q}_{ij} \leftarrow Q_{ij} = \overline{Q}_{ij}$$
$$\tilde{A}_{ij} \leftarrow A_{ij} = \tilde{A}_{ij}; \qquad \tilde{R}_{ij} \leftarrow \gamma^2 \tilde{R}_{ij};$$
$$\tilde{M}_{ij} \leftarrow \frac{1}{\gamma^2} D_i D_j' = \frac{1}{\gamma^2} \tilde{M}_{ij}$$

where $i = 1, 2$ $j = 1, 2$. From the above, we can deduce that, under the conditions $\overline{S}_{22} + \overline{A}_{22} \overline{Q}_{22}^{-1} \overline{A}_{22}' > 0$ and $\tilde{R}_{22} + \tilde{A}_{22}' \tilde{M}_{22}^{-1} \tilde{A}_{22} > 0$,

$$\overline{A}^{\square} \leftarrow \overline{A}^{\square}; \qquad \overline{S}_0 \leftarrow \overline{S}_0; \qquad \overline{Q}^{\square} \leftarrow \overline{Q}^{\square}$$
$$\tilde{A}^{\square} \leftarrow \tilde{A}^{\square}; \qquad \tilde{R}_0 \leftarrow \gamma^2 \tilde{R}_0; \qquad \tilde{M}^{\square} \leftarrow \frac{1}{\gamma^2} \tilde{M}^{\square}.$$

Then, for $\gamma > \gamma_{Is}$, where γ_{Is} is defined in (3.51) for system (2.1)–(2.6), we have the following direct consequences:

$$Z_{s\gamma} \leftarrow Z_{s\gamma}; \qquad \Sigma_{s\gamma} \leftarrow \frac{1}{\gamma^2} \Sigma_{s\gamma}$$

and the conditions specified in (3.50) are satisfied for the game (5.16)–(5.18). Thus the slow controller for problem (5.16)–(5.18) is

$$\begin{aligned} u_s^*(t) &= \breve{\mu}_{Is}^*(t, \breve{y}(t)) \\ &= -(B_1' Z_{s\gamma} + KC_1 + B_2' \overline{V} - KC_2 \overline{U}) \hat{x}_s \end{aligned} \tag{5.19}$$

$$\begin{aligned} \dot{\hat{x}}_s = &\left(\overline{A}^{\square} - \overline{S}_0 Z_{s\gamma}\right) \hat{x}_s + \left(I - \frac{1}{\gamma^2} \Sigma_{s\gamma} Z_{s\gamma}\right)^{-1} \\ &\cdot (\Sigma_{s\gamma} C_1' N^{-1} + B_1 K - \tilde{U}' B_2 K + \tilde{V} C_2' N^{-1}) \\ &\cdot \left(\breve{y} - E(\gamma^2 I - E'K'KE)^{-1} E'\right. \\ &\cdot \left.(\gamma^2 N^{-1} C_1 + K' B_1' - \gamma^2 N^{-1} C_2 \overline{U} + K' B_2' \overline{V}) \hat{x}_s\right); \\ &\hat{x}_s(t_0) = 0. \end{aligned} \tag{5.20}$$

Verification of the above is straightforward if we observe the correspondences

$$\overline{U}_1 \leftarrow \overline{U}_1; \qquad \overline{U}_2 \leftarrow \overline{U}_2; \qquad \overline{V}_1 \leftarrow \overline{V}_1; \qquad \overline{V}_2 \leftarrow \overline{V}_2$$
$$\tilde{U}_1 \leftarrow \gamma^2 \tilde{U}_1; \qquad \tilde{U}_2 \leftarrow \tilde{U}_2; \qquad \tilde{V}_1 \leftarrow \tilde{V}_1; \qquad \tilde{V}_2 \leftarrow \frac{1}{\gamma^2} \tilde{V}_2.$$

Now, we replace $\breve{y}$ by its expression given by (5.17) and obtain the composite control law (see Appendix E for detailed derivations)

$$\begin{aligned} \mu_{Ic}^*(t, y_{[t_0,t]}) &= \mu_{Ifs}^*(t, y(t)) + \breve{\mu}_{Is}^*(t, \breve{y}_{[t_0,t]}) \\ &= Ky - (B_1' Z_{s\gamma} + KC_1 + B_2' \overline{V} - KC_2 \overline{U}) \hat{x}_c \end{aligned} \tag{5.21}$$

$$\begin{aligned} \dot{\hat{x}}_c = &\left(\overline{A}^{\square} - \overline{S}_0 Z_{s\gamma}\right) \hat{x}_c + \left(I - \frac{1}{\gamma^2} \Sigma_{s\gamma} Z_{s\gamma}\right)^{-1} \\ &\cdot (\Sigma_{s\gamma} C_1' N^{-1} + B_1 K - \tilde{U}' B_2 K + \tilde{V} C_2' N^{-1}) \\ &\cdot \left(y - (C_1 - C_2 \overline{U}) \hat{x}_c\right); \qquad \hat{x}_c(t_0) = 0. \end{aligned} \tag{5.22}$$

This leads to the following theorem

Theorem 3: For the singularly perturbed system (2.1)–(2.6), let Assumptions 1–3 and 6 be satisfied, the pair $(A_{22}(t), B_2(t))$ be controllable, the pair $(A_{22}(t), C_2(t))$ be observable for each $t \in [t_0, t_f]$, and the following conditions hold:

$Q_{f22} < Z_{f\gamma}(t_f)$, where $Z_{f\gamma}(t_f)$ is the solution to (3.55) at $t = t_f$ with γ fixed.

$\gamma^2 Q_{022} > Z_{fs\gamma}(t_0)$, where $Z_{fs\gamma}(t_0)$ is the solution to (5.6) at $t = t_0$ with γ fixed.

Then, $\forall\, \gamma > \max\{\gamma_{Is}, \gamma_{Ifs}\}$, if we apply to the system the composite controller μ_{Ic}^* defined in (5.21) and (5.22), $\exists\, \epsilon_\gamma' > 0$ such that, $\forall\, \epsilon \in [0, \epsilon_\gamma']$, the disturbance attenuation level γ is attained for the full-order system.

Proof: Fix $\gamma > \max\{\gamma_{Is}, \gamma_{Ifs}\}$, and consider the differential game described by (5.16)–(5.18). The fast subsystem is

$$\begin{aligned} \dot{x}_f^t = &(A_{22} + B_2 \chi^{-1} K C_2) x_f^t + B_2 \chi^{-1} u_{sf}^t \\ &+ (D_2 + B_2 KE)(\gamma^2 I - E'K'KE)^{-1/2} \breve{w}_f^t \\ \breve{y}_f^t := &\gamma^2 E(\gamma^2 I - E'K'KE)^{-1} E' N^{-1} C_2 x_f^t \\ &+ E(\gamma^2 I - E'K'KE)^{-1/2} \breve{w}_f^t \end{aligned}$$

with the associated cost function being

$$\begin{aligned} \breve{L}_{\gamma f}^t(u_{sf}^t, \breve{w}_f^t) = \int_{-\infty}^{\infty} &\left(|x_f^t|_{\tilde{Q}_{22}}^2 + 2 x_f^{t\prime} C_2' K' \chi^{-1} u_{sf}^t \right. \\ &\left. + |u_{sf}^t|_{\chi^{-1}}^2 - |\breve{w}_f^t|^2 \right) dt. \end{aligned}$$

The open-loop GARE of the above game is the same as (5.6), which admits a minimal positive definite solution $Z_{fs\gamma}$. Then both $A_{22} + B_2 \chi^{-1} K C_2$ and $A_{22} - B_2 \chi^{-1} K C_2 - (S_{22} - B_2 \chi^{-1} B_2') Z_{fs\gamma}$ are Hurwitz. We can also see from earlier correspondences that the slow GRDE's for the transformed game (5.16)–(5.18) are the same as (3.21) and (3.22). Thus, by the proof of 3) of Theorem 2, we deduce that when the control law described by (5.19) and (5.20) is applied to the game (5.16)–(5.18), the maximal cost with respect to the disturbance $\breve{w}$ is bounded above by zero. Hence, when the composite controller is applied to the game (2.1)–(2.6), the maximal cost with respect to the disturbance w is less than or equal to zero, which means that the disturbance attenuation level γ is achieved for the system (2.1)–(2.6). □

The Infinite-Horizon Case

We now turn to the infinite-horizon case. Let A, B, C, D, E, and Q be time-invariant, Q_f be zero, and $t_0 = -\infty, t_f = \infty$. Similar to the finite-horizon case, we revisit the fast subsystem and introduce the GARE (5.6), where the coefficient matrices are now independent of t. We define χ as in (5.7) and introduce the set $\Gamma_{Ifs\infty}$ as in (5.8) (we drop the superscript t since now the set is t invariant). We define $\gamma_{Ifs\infty}$ to be the infimum of $\gamma_{Ifs\infty}$. Then the optimal static feedback control law for the fast subsystem is the same as (5.10).

Now, let the fast control law be active in the full system by defining u to be as in (5.11). Also, introduce the transformation (5.15). The transformed game is then described by

$$\begin{cases} \dot{x}_1 = (A_{11} + B_1\chi^{-1}KC_1)x_1 + (A_{12} + B_1\chi^{-1}KC_2)x_2 \\ \quad + B_1\chi^{-1}u_s + (D_1 + B_1KE)(\gamma^2 I - E'K'KE)^{-1/2}\breve{w} \\ \epsilon\dot{x}_2 = (A_{21} + B_2\chi^{-1}KC_1)x_1 + (A_{22} + B_2\chi^{-1}KC_2)x_2 \\ \quad + B_2\chi^{-1}u_s + (D_2 + B_2KE)(\gamma^2 I - E'K'KE)^{-1/2}\breve{w} \end{cases} \tag{5.23}$$

$$\begin{aligned} \breve{y} &:= y - E(\gamma^2 I - E'K'KE)^{-1}E'K'u_s \\ &= \gamma^2 E(\gamma^2 I - E'K'KE)^{-1}E'N^{-1}(C_1x_1 + C_2x_2) \\ &\quad + E(\gamma^2 I - E'K'KE)^{-1/2}\breve{w} \end{aligned} \tag{5.24}$$

$$\begin{aligned} L_\gamma(u_s, \breve{w}) = \int_{-\infty}^{\infty} (|x|^2_{\breve{Q}} &+ 2x'C'K'\chi^{-1}u_s \\ &+ |u_s|^2_{\chi^{-1}} - |\breve{w}|^2)\, dt \end{aligned} \tag{5.25}$$

where $u_s(t)$ is now a function of $\breve{y}(\tau)$, $\tau \le t$, and $\chi, \breve{Q}$ are as defined before. Now, several substitutions similar to those in the finite-horizon case apply here, leading to a slow controller, which is the same as in (5.19) and (5.20). Finally, the composite control law μ^*_{Ic} is the same as (5.21) and (5.22). These results are summarized in the following theorem.

Theorem 4: Consider the singularity perturbed system (2.1)–(2.6), with $t_f = \infty, t_0 = -\infty, Q_f = 0, Q_0 = \infty$ and A, B, D, C, E, Q time-invariant. If assumptions 1–6 hold, $\forall\ \gamma > \max\{\gamma_{Is\infty}, \gamma_{Ifs\infty}\}$, if we apply to the system the composite controller μ^*_{Ic} defined in (5.21) and (5.22), then $\exists\ \epsilon'_\gamma > 0$ such that, $\forall\ \epsilon \in [0, \epsilon'_\gamma)$, the disturbance attenuation level γ is attained for the full-order system.

Proof: Fix $\gamma > \max\{\gamma_{Is\infty}, \gamma_{Ifs\infty}\}$, and consider the differential game described by (5.23)–(5.25). The fast subsystem is

$$\begin{aligned} \dot{x}_f &= (A_{22} + B_2\chi^{-1}KC_2)x_f + B_2\chi^{-1}u_{sf} \\ &\quad + (D_2 + B_2KE)(\gamma^2 I - E'K'KE)^{-1/2}\breve{w}_f \\ \breve{y}_f &:= \gamma^2 E(\gamma^2 I - E'K'KE)^{-1}E'N^{-1}C_2x_f \\ &\quad + E(\gamma^2 I - E'K'KE)^{-1/2}\breve{w}_f \end{aligned}$$

with the associated cost function being

$$\begin{aligned} \breve{L}_{\gamma f}(u_{sf}, \breve{w}_f) = \int_{-\infty}^{\infty} (|x_f|^2_{\breve{Q}_{22}} &+ 2x'_fC'_2K'\chi^{-1}u_{sf} \\ &+ |u_{sf}|^2_{\chi^{-1}} - |\breve{w}_f|^2)\, dt. \end{aligned}$$

The open-loop GARE of the above game is the same as (5.6), which admits a minimal positive definite solution $Z_{fs\gamma}$. Then both $A_{22} + B_2\chi^{-1}KC_2$ and $A_{22} - B_2\chi^{-1}KC_2 - (S_{22} - B_2\chi^{-1}B'_2)Z_{fs\gamma}$ are Hurwitz. We can also see from the earlier correspondences that the slow GARE's for the transformed game (5.16)–(5.18) are the same as (3.61) and (3.62). Then, by the proof of part 3) of Theorem 1, we deduce that when the control law described in (5.19) and (5.20) is applied to the game (5.23)–(5.25), the maximal cost with respect to the disturbance $\breve{w}$ is bounded by zero. Hence, when the composite controller is applied to the game (2.1)–(2.6), the maximal cost with respect to the disturbance w is less than or equal to zero, which means that the disturbance attenuation level γ is achieved for the system (2.1)–(2.6). □

TABLE I

ϵ	1	0.1	0.01	0.001	0.000
$\gamma^*_{I\infty}(\epsilon)$	55.1993	17.3221	9.9331	9.1197	9.0308

VI. Examples

We present here some numerical results for the infinite horizon case. As stressed earlier, the four quantities $\gamma_{Is\infty}, \gamma_{If\infty}, \gamma_{Ifs\infty}$, and $\gamma_{of\infty}$ play important roles in the computation of an approximate value for $\gamma^*_\infty(\epsilon)$. We already know the relationship among the latter three quantities, namely, $\gamma_{If\infty} \le \gamma_{Ifs\infty} \le \gamma_{of\infty}$, but we do not know how $\gamma_{Is\infty}$ is related to them. In the following examples, we study this relationship numerically, as well as the behavior of the composite and slow controllers. We will also see the effectiveness of the approximate controller on the original system for nonzero values of $\epsilon > 0$.[7]

Example 1

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}u + \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}w \tag{6.1}$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \end{bmatrix}w \tag{6.2}$$

$$L_\gamma = \int_{-\infty}^{\infty} (2x_1^2 + 2x_1x_2 + 3x_2^2 + |u|^2 - \gamma^2|w|^2)\, dt \tag{6.3}$$

where the fast subsystem is open-loop unstable. By using a particular search algorithm, we can compute the four basic quantities:

$$\gamma_{Is\infty} = 6.1160; \qquad \gamma_{If\infty} = 9.0208;$$
$$\gamma_{Ifs\infty} = 9.0208; \qquad \gamma_{of\infty} = \infty.$$

In Table I we compute the minimax disturbance attenuation level $\gamma^*_{I\infty}(\epsilon)$ of the system (6.1)–(6.3) for different fixed values of ϵ.

Note that as $\epsilon \to 0, \gamma^*_{I\infty}(\epsilon) \to \max\{\gamma_{Is\infty}, \gamma_{If\infty}\}$. Now, we choose $\gamma = 9.1 > \max\{\gamma_{Is\infty}, \gamma_{Ifs\infty}\}$ and design the suboptimal controllers for the system based on this value of γ

$$Z_{s\gamma} = 5.9052; \qquad \Sigma_{s\gamma} = 5.8966;$$
$$\mu^*_{Is} = -2.3463\hat{x}_s; \qquad \mu^*_{Ic} = 12.0913\hat{x}_c - 2.9945y$$

where

$$\dot{\hat{x}}_s = -3.1178\hat{x}_s + 2.6166(y - 4.8214\hat{x}_s);$$
$$\dot{\hat{x}}_c = -3.1178\hat{x}_c - 14.8382(y - 4.8214\hat{x}_c).$$

[7] In these examples, the relative accuracy is 0.002.

TABLE II

ϵ	1	0.1	0.01	0.001	0.0001
γ_s^*	∞	∞	∞	∞	∞
γ_c^*	∞	∞	10.0070	9.1265	9.0976

TABLE III

ϵ	1	0.1	0.01	0.001	0.0001
$\gamma_{I\infty}^*(\epsilon)$	1.4058	1.5071	1.5345	1.5378	1.5382

Then, we use μ_{Is}^*, μ_{Ic}^* in system (6.1)–(6.3) and obtain the corresponding disturbance attenuation bounds γ_s^* and γ_c^*, which are in Table II.

We see that only the composite controller achieves the desired performance bound for small values of $\epsilon > 0$. The slow controller, designed based on the slow subsystem only, leads to an infinite attenuation level.

Example 2

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} w \tag{6.4}$$

$$y = [3 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0 \quad 1] w \tag{6.5}$$

$$L_\gamma = \int_{-\infty}^{\infty} (2x_1^2 + 2x_1x_2 + 3x_2^2 + |u|^2 - \gamma^2 |w|^2)\, dt \tag{6.6}$$

where now the fast subsystem is open-loop stable. The four performance levels for this system are

$$\gamma_{Is\infty} = 1.5382\,; \qquad \gamma_{If\infty} = 1.5000\,;$$
$$\gamma_{Ifs\infty} = 1.5000\,; \qquad \gamma_{of\infty} = 2.5992\,.$$

Note that here we have the relationship $\gamma_{of\infty} > \gamma_{Is\infty} > \gamma_{Ifs\infty} = \gamma_{If\infty}$.

We can also compute the minimax disturbance attenuation level $\gamma_{I\infty}^*(\epsilon)$ of the system (6.4)–(6.6) for different values of ϵ, tabulated in Table III.

Note again that as $\epsilon \to 0, \gamma_{I\infty}^*(\epsilon) \to \max\{\gamma_{Is\infty}, \gamma_{If\infty}\}$. Now, we choose $\gamma = 2.65 > \max\{\gamma_{Is\infty}, \gamma_{of\infty}\}$ and design the slow and composite controllers based on this value of γ:

$$Z_{s\gamma} = 2.1786\,; \qquad \Sigma_{s\gamma} = 0.5290\,;$$
$$\mu_{Is}^* = -1.0674\hat{x}_s\,; \qquad \mu_{Ic}^* = -0.14451\hat{x}_c - 1.0936y$$

where

$$\dot{\hat{x}}_s = -2.6834\hat{x}_s + 1.9426(y - 0.8439\hat{x}_s)\,;$$
$$\dot{\hat{x}}_c = -2.6834\hat{x}_c + 0.023011(y - 0.8439\hat{x}_c)\,.$$

Then, we use μ_{Is}^* and μ_{Ic}^* in system (6.4)–(6.6) and obtain the corresponding disturbance attenuation bounds γ_s^* and γ_c^*, which are tabulated in Table IV.

Note that when we choose a γ larger than the maximum of $\gamma_{Is\infty}$ and $\gamma_{of\infty}$, both controllers achieve the desired performance level for the full order system. But, the composite controller achieves a much lower attenuation level than the slow controller. Now, suppose we choose $\gamma = 1.6$, which is larger than the $\max\{\gamma_{Is\infty}, \gamma_{If\infty}\}$ but smaller than $\max\{\gamma_{Is\infty}, \gamma_{of\infty}\}$, and design the controllers based on this value of γ:

TABLE IV

ϵ	1	0.1	0.01	0.001	0.0001
γ_s^*	3.3600	2.8121	2.6231	2.5993	2.5993
γ_c^*	1.4798	1.5100	1.5507	1.5716	1.5716

TABLE V

ϵ	1	0.1	0.01	0.001	0.0001
γ_s^*	∞	∞	∞	∞	∞
γ_c^*	1.5556	1.5556	1.5556	1.5556	1.5556

$$Z_{s\gamma} = 3.3788\,; \qquad \Sigma_{s\gamma} = 0.56402\,;$$
$$\mu_{Is}^* = -1.6447\hat{x}_s\,; \qquad \mu_{Ic}^* = -0.099718\hat{x}_c - 1.3840y$$

where

$$\dot{\hat{x}}_s = -2.5744\hat{x}_s + 7.6196(y + 1.2640\hat{x}_s)\,;$$
$$\dot{\hat{x}}_c = -2.5744\hat{x}_c - 0.89407(y + 1.2640\hat{x}_c)\,.$$

Then, if we use μ_{Is}^* and μ_{Ic}^* in system (6.4)–(6.6), we obtain the disturbance attenuation bounds γ_s^* and γ_c^*, as shown in Table V.

This time, only the composite controller μ_{Ic}^* achieves the desired performance level. The slow controller μ_{Is}^* does not yield a finite performance, despite the fact that the fast subsystem is open-loop stable. □

Example 3

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} w \tag{6.7}$$

$$y = [2 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0 \quad 1] w \tag{6.8}$$

$$L_\gamma = \int_{-\infty}^{\infty} (2x_1^2 + 2x_1x_2 + 2x_2^2 + |u|^2 - \gamma^2 |w|^2)\, dt\,. \tag{6.9}$$

We again compute the four performance levels:

$$\gamma_{Is\infty} = 2.1596\,; \qquad \gamma_{If\infty} = 0.63234\,;$$
$$\gamma_{Ifs\infty} = 0.63246\,; \qquad \gamma_{of\infty} = 0.70708\,.$$

Note that here we have the relationship $\gamma_{Is\infty} > \gamma_{of\infty} > \gamma_{Ifs\infty} = \gamma_{If\infty}$.

TABLE VI

ϵ	1	0.1	0.01	0.001	0.0001
$\gamma^*_{I\infty}(\epsilon)$	1.8396	2.1009	2.1533	2.1590	2.1595

TABLE VII

ϵ	1	0.1	0.01	0.001	0.0001
γ^*_s	8.1079	3.8172	2.1997	2.1997	2.1997
γ^*_c	6.0486	2.8533	2.1994	2.1994	2.1994

We can compute the minimax disturbance attenuation level $\gamma^*_{I\infty}(\epsilon)$ of the system (6.7)–(6.9) for different values of ϵ as shown in Table VI.

Note again that as $\epsilon \to 0, \gamma^*_{I\infty}(\epsilon) \to \max\{\gamma_{Is\infty}, \gamma_{If\infty}\}$. Now, we choose $\gamma = 2.2 > \max\{\gamma_{Is\infty}, \gamma_{of\infty}\}$, and design controllers for the subsystems based on this value of γ

$$Z_{s\gamma} = 1.2442; \qquad \Sigma_{s\gamma} = 3.5928;$$
$$\mu^*_{Is} = -1.7770\hat{x}_s; \qquad \mu^*_{Ic} = -1.6632\hat{x}_c - 0.68303y$$

where

$$\dot{\hat{x}}_s = -3.2774\hat{x}_s + 43.1995(y - 0.16654\hat{x}_s)$$
$$\dot{\hat{x}}_c = -3.2774\hat{x}_c + 39.4653(y - 0.16654\hat{x}_c).$$

Then, we use μ^*_{Is} and μ^*_{Ic} in system (6.7)–(6.9) and obtain the corresponding disturbance attenuation bounds γ^*_s and γ^*_c, which are tabulated in Table VII. Now, both controllers achieve the desired performance level as $\epsilon \to 0$. This is to be expected, since $\gamma_{Is\infty} > \gamma_{of\infty}$.

VII. Conclusion

In this paper, we have provided a complete analysis of the singularly perturbed H^∞-optimal control problem with imperfect state measurements, in both finite and infinite horizons, by relating it to a class of singularly perturbed differential games. A major contribution of the paper is the proof of existence and construction of a composite controller, independent of the singular perturbation parameter, under which the associated differential game has a bounded upper value, and the full-order system meets a desired H^∞-performance bound. Such a composite controller is designed sequentially, based on saddle-point solutions of parameterized fast and slow subgames. The design procedure is first to obtain a static output feedback controller using the fast dynamics, to be followed by a dynamic output feedback which uses the dynamics of the "reduced" slow game. As discussed in the paper, such a composite controller may not be able to push the performance of the full-order plant to the ultimate limit $\max\{\gamma_{Is}, \gamma_{If}\}$, no matter how small $\epsilon > 0$ is, whereas in the perfect state measurements case studied in [18] this was possible; this indicates a loss of *robustness* due to the imperfect nature of the measurements. The paper has also presented conditions under which a controller designed based on the slow subsystem only can achieve a desired (albeit, inferior) performance level.

The composite controller design methodology presented in this paper admits two significant interpretations, with important implications for practical controller design applications. In the case that the full-order system is completely known, our composite controller can be tuned precisely to guarantee a certain desired performance bound and also to exhibit additional robustness with respect to changes in the value of the singular perturbation parameter ϵ. If, on the other hand, only a reduced-order model is available, which is a common situation in most real world applications, our results show that a composite controller (PI controller) is generally more robust than a slow controller (I controller). In this case, one can only expect a performance level up to $\hat{\gamma}_{Is}$ ($\hat{\gamma}_{Is\infty}$ in the infinite-horizon case) and only for the slow subsystem. As shown in the paper, such a controller design, which yields essentially an I controller, does not guarantee any performance for the full-order system. Our results show that, with a PI controller design, one can achieve a performance level up to $\max\{\gamma_{Is}, \gamma_{Ifs}\}$ ($\max\{\gamma_{Is\infty}, \gamma_{Ifs\infty}\}$ in the infinite-horizon case) "safely" if the controller is designed properly, and the attained performance level may in fact be well below the level $\hat{\gamma}_{Is}$ ($\hat{\gamma}_{Is\infty}$ in the infinite-horizon case) if the fast subsystem is "benign." Thus, our results provide a theoretical basis for the common observation that a properly tuned PI controller often performs remarkably well in real world applications. Also, our results suggest two possible general purpose schemes to tune the desired PI controller. One scheme is to design a P controller first to cope satisfactorily with the fast modes of the system, and then, with the P controller being active in the system, to reevaluate the reduced-order model and design an I controller for it. For the other scheme, we first form the slow controller, then design the P controller, and finally adjust the slow controller and observer gains according to (5.21) and (5.22). In both cases, the P and I controllers should be designed based on a common performance level γ.

One immediate, but not trivial, extension of these results is to the sampled-data measurement case, so as to obtain the counterparts of the results of [14] and [16] in the singularly perturbed case. This work has already been completed for the (sampled) perfect state measurements case, and the results have been presented in [26]. The derivation of the counterpart of this in the imperfect state measurements case also seems to be within reach. Another extension would be to the problem with digital control action, i.e., when the control action remains constant over each sampling interval, in the perfect state or imperfect state measurements cases [15], [27], [28]. Other possible extensions would be to the multitime scale H^∞-optimal control problem and also to singularly perturbed nonlinear control systems, which are topics currently under study.

Appendix A

A Set of Useful Identities

We first observe the following useful relationships between $\overline{U}_1, \overline{U}_2, \overline{V}_1, \overline{V}_2, \tilde{U}_1, \tilde{U}_2, \tilde{V}_1, \tilde{V}_2, \overline{U}, \overline{V}, \tilde{U}$ and $\tilde{V}$, which were defined in (3.42)–(3.49) and (3.38)–(3.41) (see [18, Appendix B] for a proof).

Set of Identities 1:

1) $\overline{A}_{22}\overline{U}_1 + \overline{S}_{22}\overline{V}_1 = -\overline{S}_{21}$
2) $\overline{A}_{22}\overline{U}_2 + \overline{S}_{22}\overline{V}_2 = \overline{A}_{21}$
3) $\overline{Q}_{22}\overline{U}_2 - \overline{A}'_{22}\overline{V}_2 = \overline{Q}_{21}$
4) $\overline{Q}_{22}\overline{U}_1 - \overline{A}'_{22}\overline{V}_1 = \overline{A}'_{12}$
5) $\tilde{A}'_{22}\tilde{U}_1 + \tilde{R}_{22}\tilde{V}_1 = -\tilde{R}_{21}$
6) $\tilde{A}'_{22}\tilde{U}_2 + \tilde{R}_{22}\tilde{V}_2 = \tilde{A}'_{12}$
7) $\tilde{M}_{22}\tilde{U}_2 - \tilde{A}_{22}\tilde{V}_2 = \tilde{M}_{21}$
8) $\tilde{M}_{22}\tilde{U}_1 - \tilde{A}_{22}\tilde{V}_1 = \tilde{A}_{21}$
9) $\overline{A}_{22}\overline{U} + \overline{S}_{22}\overline{V} = \overline{A}_{21} - \overline{S}_{21}Z_{s\gamma}$
10) $\overline{Q}_{22}\overline{U} - \overline{A}'_{22}\overline{V} = \overline{Q}_{21} + \overline{A}'_{12}Z_{s\gamma}$
11) $\tilde{A}'_{22}\tilde{U} + \tilde{R}_{22}\tilde{V} = \tilde{A}'_{12} - \tilde{R}_{21}\sum_{s\gamma}$
12) $\tilde{M}_{22}\tilde{U} - \tilde{A}_{22}\tilde{V} = \tilde{M}_{21} + \tilde{A}_{21}\sum_{s\gamma}$ $\square$

We will use the above relationships throughout the derivations in the appendixes to follow.

Appendix B

Simplification of Parameter Matrices of the Slow Subsystem

First, we introduce some matrices to simplify the derivations in the sequel.

$$\pi := A_{22}Q_{22}^{-1}A'_{22}; \qquad \rho := \gamma^2 I - D'_2 A'^{-1}_{22} Q_{22} A_{22}^{-1} D_2; \qquad \sigma := A'_{22}(D_2D'_2)^{-1}A_{22}.$$

In terms of these matrices, we can obtain the following expressions using some simple matrix operations and matrix inversion identities:

$$\rho = \gamma^2 I - D'_2\pi^{-1}D_2 \tag{B.1}$$

$$\rho^{-1} = \frac{1}{\gamma^2}\left(I + \frac{1}{\gamma^2}D'_2\left(\pi - \frac{1}{\gamma^2}D_2D'_2\right)^{-1}D_2\right) \tag{B.2}$$

$$\rho^{-1}D'_2\pi^{-1} = \frac{1}{\gamma^2}D'_2\left(\pi - \frac{1}{\gamma^2}D_2D'_2\right)^{-1} \tag{B.3}$$

$$\left(\pi - \frac{1}{\gamma^2}D_2D'_2\right)^{-1}A_{22}Q_{22}^{-1} = (D_2D'_2)^{-1} \cdot A_{22}\left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1} \tag{B.4}$$

$$\left(\pi - \frac{1}{\gamma^2}D_2D'_2\right)^{-1} = \gamma^2(D_2D'_2)^{-1}A_{22}\left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1} \cdot A'_{22}(D_2D'_2)^{-1} - \gamma^2(D_2D'_2)^{-1}. \tag{B.5}$$

By using the identities (B.1)–(B.5), the verification of (3.10)–(3.20) is straightforward but lengthy; see [23] for details.

In the verification of (3.25)–(3.27), on the other hand, a different set of matrix identities is used, which are given below. The details are again lengthy, but brute force; they can be found in [23].

Set of Identities 2:

1) $\left(R - P'_2Q_{22}^{-1}P_2\right)^{-1} = R^{-1} + R^{-1}P'_2\overline{Q}_{22}^{-1}P_2R^{-1}$
2) $\left(R - P'_2Q_{22}^{-1}P_2\right)^{-1}P'_2Q_{22}^{-1} = R^{-1}P'_2\overline{Q}_{22}^{-1}$
3) $R^{\square-1} = R^{-1} + R^{-1}P'_2\overline{Q}_{22}^{-1}P_2R^{-1} - R^{-1}(P'_2\overline{Q}_{22}^{-1}\overline{A}'_{22} - B'_2)(\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}'_{22})^{-1}(\overline{A}_{22}\overline{Q}_{22}^{-1}P_2 - B_2)R^{-1}$
4) $R^{\square-1}\left(A_{22}Q_{22}^{-1}P_2 - B_2\right)'(\pi - \frac{1}{\gamma^2}D_2D'_2)^{-1} = R^{-1}(P'_2\overline{Q}_{22}^{-1}\overline{A}'_{22} - B'_2)(\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}'_{22})^{-1}$
5) $\left(B_2 - A_{22}Q_{22}^{-1}P_2\right)R^{-1}(P'_2\overline{Q}_{22}^{-1}\overline{A}'_{22} - B'_2) = \pi - \frac{1}{\gamma^2}D_2D'_2 - \left(\overline{S}_{22} + \overline{A}_{22}\overline{Q}_{22}^{-1}\overline{A}'_{22}\right)$
6) $\left(B_1 - A_{12}Q_{22}^{-1}P_2\right)R^{-1}(P'_2\overline{Q}_{22}^{-1}\overline{A}'_{22} - B'_2) - A_{12}Q_{22}^{-1}A'_{22} + \frac{1}{\gamma^2}D_1D'_2 = -\overline{S}_{12} - \overline{A}_{12}\overline{Q}_{22}^{-1}\overline{A}'_{22}$
7) $(\overline{A}_{22}\overline{Q}_{22}^{-1}P_2 - B_2)R^{-1}(P'_1 - P'_2Q_{22}^{-1}Q_{21}) - (A_{22}Q_{22}^{-1}Q_{21} - A_{21}) = -\overline{A}_{22}\overline{Q}_{22}^{-1}\overline{Q}_{21} + \overline{A}_{21}.$ $\square$

To prove (3.28)–(3.30), we rewrite matrices $A^\square, L^\square, N^\square, C^\square, Q^\square$, and $D^\square D^{\square\prime}$ in different forms, by repeated application of the relationships (B.4) and (B.5)

$$A^\square = A_{11} - (D_1D'_2)(D_2D'_2)^{-1}A_{21} + ((D_1D'_2) \cdot (D_2D'_2)^{-1}A_{22} - A_{12})\left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1} \cdot \left(A'_{22}(D_2D'_2)^{-1}A_{21} - \frac{1}{\gamma^2}Q_{21}\right)$$

$$L^\square = \frac{1}{\gamma^2}(L_1 - (D_1D'_2)(D_2D'_2)^{-1}L_2 + ((D_1D'_2)(D_2D'_2)^{-1}A_{22} - A_{12}) \cdot \left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1}(A'_{22}(D_2D'_2)^{-1}L_2 - C'_2))$$

$$N^\square = \frac{1}{\gamma^2}(N - L'_2(D_2D'_2)^{-1}L_2 + (L'_2(D_2D'_2)^{-1}A_{22} - C_2) \cdot \left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1}(A'_{22}(D_2D'_2)^{-1}L_2 - C'_2)$$

$$C^\square = C_1 - L'_2(D_2D'_2)^{-1}A_{21} + (L'_2(D_2D'_2)^{-1}A_{22} - C_2) \cdot \left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1}\left(A'_{22}(D_2D'_2)^{-1}A_{21} - \frac{1}{\gamma^2}Q_{21}\right)$$

$$Q^\square = Q_{11} - \gamma^2A'_{21}(D_2D'_2)^{-1}A_{21} + \gamma^2\left(A'_{21}(D_2D'_2)^{-1}A_{22} - \frac{1}{\gamma^2}Q_{12}\right) \cdot \left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1}\left(A'_{22}(D_2D'_2)^{-1}A_{21} - \frac{1}{\gamma^2}Q_{21}\right)$$

$$D^\square D^{\square\prime} = \frac{1}{\gamma^2}(D_1D'_1 - (D_1D'_2)(D_2D'_2)^{-1}(D_2D'_1) + ((D_1D'_2)(D_2D'_2)^{-1}A_{22} - A_{12}) \cdot \left(\sigma - \frac{1}{\gamma^2}Q_{22}\right)^{-1} \cdot (A'_{22}(D_2D'_2)^{-1}(D_2D'_1) - \acute{A}_{12})).$$

Next, we introduce the following mapping rules (correspondences):

$$
\begin{array}{lll}
A_{ij} \rightsquigarrow A'_{ji}; & B_i \rightsquigarrow C'_i; & N \rightsquigarrow R \\
P_i \rightsquigarrow L_i; & D_i D'_j \rightsquigarrow Q_{ij}; & Q_{ij} \rightsquigarrow D_i D'_j
\end{array}
$$

where $i = 1, 2$ $j = 1, 2$. Then, we have the following:

$$
\begin{aligned}
\overline{A}^{\Box} &\rightsquigarrow \tilde{A}'_{11} - \tilde{A}'_{21}\tilde{M}_{22}^{-1}\tilde{M}_{21} - (\tilde{R}_{12} + \tilde{A}'_{21}\tilde{M}_{22}^{-1}\tilde{A}_{22}) \\
&\quad \cdot (\tilde{R}_{22} + \tilde{A}'_{22}\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{A}'_{12} - \tilde{A}'_{22}\tilde{M}_{22}^{-1}\tilde{M}_{21}) \\
\overline{S}_0 &\rightsquigarrow \tilde{R}_{11} + \tilde{A}'_{21}\tilde{M}_{22}^{-1}\tilde{A}_{21} - (\tilde{R}_{12} + \tilde{A}'_{21}\tilde{M}_{22}^{-1}\tilde{A}_{22}) \\
&\quad \cdot (\tilde{R}_{22} + \tilde{A}'_{22}\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{R}_{21} + \tilde{A}'_{22}\tilde{M}_{22}^{-1}\tilde{A}_{21}) \\
\overline{Q}^{\Box} &\rightsquigarrow \tilde{M}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{M}_{21} + (\tilde{A}_{12} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{A}_{22}) \\
&\quad \cdot (\tilde{R}_{22} + \tilde{A}'_{22}\tilde{M}_{22}^{-1}\tilde{A}_{22})^{-1}(\tilde{A}'_{12} - \tilde{A}'_{22}\tilde{M}_{22}^{-1}\tilde{M}_{21}).
\end{aligned}
$$

It is also to see that

$$
\begin{array}{lll}
A^{\Box} \rightsquigarrow A^{\Box\prime}; & B^{\Box} \rightsquigarrow C^{\Box\prime}; & P^{\Box} \rightsquigarrow \gamma^2 L^{\Box} \\
R^{\Box} \rightsquigarrow \gamma^2 N^{\Box}; & Q^{\Box} \rightsquigarrow \gamma^2 D^{\Box} D^{\Box\prime}; & \\
& D^{\Box} D^{\Box\prime} \rightsquigarrow \dfrac{1}{\gamma^2} Q^{\Box}. &
\end{array}
$$

This completes the verification of (3.28)–(3.30).

Verification of (3.36) and (3.37)

We only need to establish the following three identities:

Set of Identities 3:

1) $R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) = R^{-1}(B'_1 Z_{s\gamma} + P'_1 - P'_2 \overline{U} + B'_2 \overline{V})$
2) $N^{\Box-1}(C^{\Box} \frac{1}{\gamma^2} \sum_{s\gamma} + L^{\Box\prime}) = N^{-1}(C_1 \sum_{s\gamma} + L'_1 - L'_2 \tilde{U} + C_2 \tilde{V})$
3) $G^{\Box} R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) + C^{\Box} + L^{\Box\prime} Z_{s\gamma} = C_1 + \frac{1}{\gamma^2} L'_1 - C_2 \overline{U} + \frac{1}{\gamma^2} L'_2 \overline{V}$.

Proof: The first one can be proven using the earlier sets of identities 1 and 2 (see [23] for details).

For the second identity, first we note that

$$
\overline{U} \rightsquigarrow \tilde{U}; \qquad \overline{V} \rightsquigarrow \tilde{V}
$$

where we have introduced the additional correspondence

$$
Z_{s\gamma} \rightsquigarrow \Sigma_{s\gamma}.
$$

Then

$$
R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) \rightsquigarrow N^{\Box-1}\left(C^{\Box}\frac{1}{\gamma^2}\Sigma_{s\gamma} + L^{\Box\prime}\right)
$$

$$
R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) \rightsquigarrow N^{-1}\left(C_1 \Sigma_{s\gamma} + L'_1 + C_2 \tilde{V} - L'_2 \tilde{U}\right).
$$

Thus, the second identity is true.

For the third identity, we first substitute on the left-hand side the first identity and the expressions for $G^{\Box}, C^{\Box}$, and $L^{\Box}$; then we apply the Set of Identities 1 to arrive at the right-hand side; see [23] for details.

APPENDIX C

VERIFICATION: (4.31) IS THE SOLUTION TO THE SLOW SUB RDE (RESPECTIVELY, SUB-ARE) OF (4.35) [RESPECTIVELY, (4.30)]

First, we evaluate the parameter matrices for the slow sub-RDE (respectively sub-ARE) according to formulas (3.15), (3.18), and (3.20) in [18] (see also equations at the bottom of the page).

$$
\begin{aligned}
\Pi := \overline{A}^{\Box} - \overline{S}_0 Z_{s\gamma} - \left(\gamma^2 \Sigma_{s\gamma}^{-1} - Z_{s\gamma}\right)^{-1} \\
\cdot \left(C^{\Box\prime} + \gamma^2 \Sigma_{s\gamma}^{-1} L^{\Box}\right) N^{\Box-1}(C^{\Box} + L^{\Box\prime} Z_{s\gamma})
\end{aligned}
$$

$$
S_0^e = \begin{bmatrix} -\tilde{D}_0 \tilde{D}'_0 & \Omega \\ \Omega' & \Theta \end{bmatrix};
$$

$$
\Omega := -L^{\Box} N^{\Box-1}\left(C^{\Box} + L^{\Box\prime}\gamma^2\Sigma_{s\gamma}^{-1}\right)\left(\gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma}\right)^{-1}
$$

$$
\begin{aligned}
\Theta := -\left(\gamma^2 \Sigma_{s\gamma}^{-1} - Z_{s\gamma}\right)^{-1}\left(C^{\Box\prime} + \gamma^2 \Sigma_{s\gamma}^{-1} L^{\Box}\right) \\
\cdot N^{\Box-1}\left(C^{\Box} + L^{\Box\prime}\gamma^2\Sigma_{s\gamma}^{-1}\right)\left(\gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma}\right)^{-1}.
\end{aligned}
$$

Next, we verify that $\Xi_{s\gamma}$ is the solution to the following RDE:

$$
\dot{\Xi}_{s\gamma} + \tilde{F}_0^{e\prime}\Xi_{s\gamma} + \Xi_{s\gamma}\tilde{F}_0^e - \Xi_{s\gamma} S_0^e \Xi_{s\gamma} + \tilde{Q}^e = 0.
$$

We can simply show that the 11-block, 12-block, and 22-block of the left-hand side are equal to zero. The verification is straightforward but tedious, and it can be found in [23].

Similarly, we can show that $\Xi_{s\gamma}$ is the solution to the slow sub-ARE in the infinite-horizon case. Then, by Theorem 5 of [29], there exists a minimal solution $\Xi_{s\gamma}$ to the slow sub-ARE such that $\tilde{F}_0^e - S_0^e \overline{\Xi}_{s\gamma}$ is Hurwitz.

$$
\tilde{F}_0^e = \begin{bmatrix} A^{\Box} & -B^{\Box} R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) \\ \left(\gamma^2\Sigma_{s\gamma}^{-1} - Z_{s\gamma}\right)^{-1}\left(C^{\Box\prime} + \gamma^2\Sigma_{s\gamma}^{-1} L^{\Box}\right) N^{\Box-1} C^{\Box} & \Pi \end{bmatrix}
$$

$$
\tilde{Q}^e = \begin{bmatrix} Q^{\Box} & -P^{\Box} R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) \\ -(Z_{s\gamma} B^{\Box} + P^{\Box}) R^{\Box-1} P^{\Box\prime} & (Z_{s\gamma} B^{\Box} + P^{\Box}) R^{\Box-1}(B^{\Box\prime} Z_{s\gamma} + P^{\Box\prime}) \end{bmatrix}
$$

Appendix D

Proof of Lemma 1

The differential game associated with this problem has the cost function

$$L_\gamma = J - \gamma^2\|w\|^2. \tag{D.1}$$

Substituting the control law $u = K(\gamma)y$ into (5.1) and (D.1), we arrive at

$$\begin{aligned}\dot{x} &= (A+BKC)x + (D+BKE)w\\ L_\gamma &= \int_{-\infty}^{\infty}(|x|^2_{Q+C'K'KC} + 2x'C'K'KEw\\ &\quad - |w|^2_{\gamma^2 I - E'K'KE})\,dt.\end{aligned}$$

We need to show that, under the condition specified in the lemma, the maximum cost (with respect to w) to the one-person maximization problem above is zero. By Fact 1 in [18], the observability of (A, Q) implies that $(A+BKC, Q+C'K'KC)$ is observable. Under the condition of (5.4), the maximum value is bounded if the following ARE admits a nonnegative definite solution:

$$A_e'Z + ZA_e + ZM_eZ + Q_e = 0$$

where

$$\begin{aligned}A_e &:= A + BKC + (D+BKE)\\ &\quad\cdot(\gamma^2 I - E'K'KE)^{-1}E'K'KC\\ M_e &:= (D+BKE)(\gamma^2 I - E'K'KE)^{-1}\\ &\quad\cdot(D' + E'K'B')\\ Q_e &:= Q + C'K'KC + C'K'KE\\ &\quad\cdot(\gamma^2 I - E'K'KE)^{-1}E'K'KC.\end{aligned}$$

Now, we observe the following simple relationships:

$$\begin{aligned}(\gamma^2 I - E'K'KE)^{-1} &= \frac{1}{\gamma^2}\Big(I + \frac{1}{\gamma^2}E'K'\\ &\quad\cdot\Big(I - \frac{1}{\gamma^2}KNK'\Big)^{-1}KE\Big)\\ (\gamma^2 I - E'K'KE)^{-1}E'K' &= \frac{1}{\gamma^2}E'K'\Big(I - \frac{1}{\gamma^2}KNK'\Big)^{-1}\end{aligned}$$

which can be proved by simple matrix inversion identities. Then, we can rewrite A_e, M_e, and Q_e in the following form:

$$\begin{aligned}A_e &= A + B\Big(I - \frac{1}{\gamma^2}KNK'\Big)^{-1}KC\\ M_e &= -BB' + \frac{1}{\gamma^2}DD' + B\Big(I - \frac{1}{\gamma^2}KNK'\Big)^{-1}B'\\ Q_e &= Q + C'K'\Big(I - \frac{1}{\gamma^2}KNK'\Big)^{-1}KC.\end{aligned}$$

Hence, the ARE can be rewritten as (5.5). This completes the proof.

Appendix E

Derivations for the Composite Controller

Verification of (5.16)–(5.18)

We first note the following set of identities, which can be proved by simple matrix inversion formulas:

Set of Identities 4:

1) $(\gamma^2 I - E'K'KE)^{-1} = (1/\gamma^2)(I + (1/\gamma^2)E'K'\mathcal{X}^{-1}KE)$
2) $(\gamma^2 I - E'K'KE)^{-1}E'K' = (1/\gamma^2)E'K'\mathcal{X}^{-1}$
3) $(\gamma^2 N^{-1} - K'K)^{-1} = E(\gamma^2 I - E'K'KE)^{-1}E'$. □

In view of these identities, it is a simple matter to establish the validity of (5.16)–(5.18).

Verification of (5.22)

We only need to show that

$$\begin{aligned}&-E(\gamma^2 I - E'K'KE)^{-1}E'K'\,(B_1'Z_{s\gamma} + KC_1 + B_2'\overline{V}\\ &-KC_2\overline{U}) + E(\gamma^2 I - E'K'KE)^{-1}\,E'(\gamma^2 N^{-1}C_1\\ &+K'B_1' - \gamma^2 N^{-1}C_2\overline{U} + K'B_2'\overline{V}) = C_1 - C_2\overline{U}\end{aligned}$$

which is true since the left-hand side of the above can be written as

$$\begin{aligned}&-E(\gamma^2 I - E'K'KE)^{-1}E'K'\,(B_1'Z_{s\gamma} + KC_1\\ &+B_2'\overline{V} - KC_2\overline{U}) + E(\gamma^2 I - E'K'KE)^{-1}\\ &\cdot E'(\gamma^2 N^{-1}C_1 + K'B_1' - \gamma^2 N^{-1}C_2\overline{U} + K'B_2'\overline{V})\\ &= E(\gamma^2 I - E'K'KE)^{-1}E'(\gamma^2 N^{-1} - K'K)\\ &\qquad\cdot(C_1 - C_2\overline{U}) = C_1 - C_2\overline{U}.\end{aligned}$$

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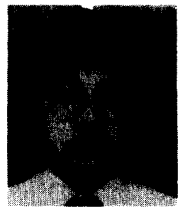

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