Proof of Corollary 2:

a) It is clear that if $p_i^l(s) = p_i^l$ or $e^{j\theta_i^l}$ are constants satisfying the bounding inequalities (4.8, 9), then they lie in the disc $A(\alpha_i)$ of Fig. 3. Applying Lemma 5a and then Theorem 3a gives that the perturbed interconnected system is closed-loop stable.

b) A similar form of argument using Lemma 5b and Theorem 3b applies. □

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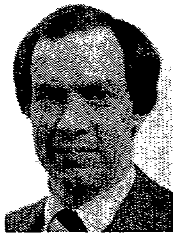

David J. N. Limebeer (M'81) was born in Johannesburg, South Africa, in 1952. He received the B.Sc. degree in electrical engineering from the University of Witwatersrand in 1974, and the M.Sc. and Ph.D. degrees in 1977 and 1980, respectively, from the University of Natal in South Africa.

He was the recipient of CSIR doctoral and post-doctoral fellowships, and was also given an award by the South African Institute of Electrical Engineers for a paper on the subsynchronous resonance of nuclear powered turbo-generators. Since September 1980, he has been a Research Associate in the Control and Management Systems Division of the Department of Engineering, University of Cambridge, England. He was recently appointed a Lecturer in the Department of Electrical Engineering, Imperial College, University of London. His main research interests are multivariable systems theory, computer aided control system design, and the analysis and control of power systems.

Y. S. Hung was born in Hong Kong, in 1952. He received the B.Sc. degrees in electrical engineering and in mathematics from the University of Hong Kong in 1975 and 1977, respectively, and the M.Phil. and Ph.D. degrees in control engineering from the University of Cambridge in 1979 and 1982.

Since 1982, he has been a Research Assistant at Cambridge. Dr. Hung is the author, with A. G. J. MacFarlane, of *Multivariable Feedback: A Quasi-Classical Approach* (Berlin: Springer-Verlag). His current research interests are in multivariable systems theory, large scale interconnected systems, computer aided control system design, and the application of optimization techniques.

Information Induced Multimodel Solutions in Multiple Decisionmaker Problems

VIKRAM R. SAKSENA, MEMBER, IEEE, JOSE B. CRUZ, JR., FELLOW, IEEE, WILLIAM R. PERKINS, FELLOW, IEEE, AND TAMER BASAR, FELLOW, IEEE

Abstract—**This paper is concerned with modeling and control strategy interaction in a multimodel context. The role of the observability structure in multiple decisionmaker (DM) problems is examined. A procedure is developed for generating multimodel solutions based on certain deterministic information patterns. This is achieved by first explicitly identifying the state space structure induced by the observation sets of the DM's, and then overlapping appropriately the input space structure of each DM. Such a representation is used to identify the class of admissible strategies which generate multimodel solutions. Conditions, which depend on the information pattern, are obtained under which these multimodel solutions admit partial noninteraction among the DM's.**

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V. R. Saksena was with the University of Illinois at Urbana. He is now with Bell Laboratories, Holmdel, NJ.

J. B. Cruz, Jr., W. R. Perkins, and T. Basar are with the Department of Electrical Engineering, Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801.

I. Introduction

THE CONCEPT OF multimodel strategies for large scale systems has been introduced in [1]. It accounts for the presence of multiple decisionmakers (DM's) assuming different simplified models of the same system. In [1], [2] multiparameter singular perturbations were employed

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to capture the multimodel nature of interconnected systems with slow and fast dynamics. There was a strong interaction among the DM's through the slow variables, and a weak interaction through the fast subvectors. A multimodel situation resulted when each DM modeled his own fast subsystem dynamics and assumed a certain reduced order equivalent of the rest of the system.

The role of time scales in multimodel strategy design has been studied further in [3]–[5] within the framework of multiparameter singular perturbations. In [3], the implications of relaxing the weak coupling assumption on the fast subsystems were examined. In [4], [5], the effects of random disturbances and nonclassical information patterns were analyzed.

The basic challenge in multimodeling is to identify the "core" where there is a strong interaction among all the DM's, and other low-order subproblems where the interactions are weak. This leads to the possibility of decentralized strategy design by the DM's, using several low-order models of the system. Such a decomposition need not be based on time scale considerations alone.

In large scale systems, the DM's observe, in general, different variables through their individual objective functionals. These observed variables play a crucial role in the solution of the problem. In this paper we shall focus on the role of the observed variables in multimodel strategy design. In fact we shall attempt to identify the core by examining the observability structure induced by the observation sets of the DM's. Firstly, we shall represent the system in the observability decomposition form using the techniques of chained aggregation [6], [7]. Then, by overlapping appropriately the input structure with the observability decomposition, we shall be led to the class of admissible strategies (referred to as Structure Preserving Strategies) for the DM's, which generates multimodel solutions.

The information induced multimodel solutions developed here will be shown to admit partial noninteraction among the DM's under certain conditions which depend on the information pattern. Two types of deterministic information patterns will be considered; namely, Feedback Perfect State (FPS) and Feedback Imperfect State (FIS) (these will be made precise later). The results will be derived primarily for the two DM cases under the noncooperative Nash solution concept; but extensions to many DM problems and Pareto games will also be discussed. In problems where the system is completely observable by each DM, we shall demonstrate how the observability decomposition can be induced using state feedback. The applicability of our approach to the control of large scale interconnected subsystems and the control of multiarea power systems will be examined.

II. Problem Formulation

2.1. The Problem

Consider a linear system controlled by two DM's

$$\dot{x} = Ax + B_1u_1 + B_2u_2, \qquad x(0) = x_0 \tag{1a}$$

$$y_i = C_i x, \qquad i = 1,2 \tag{1b}$$

$$\dim x = n, \quad \dim u_i = m_i, \quad \dim y_i = p_i.$$

The variables y_i will be referred to as the "observation set" of each DM. These are in fact the controlled variables as seen through the performance index of each DM, and may or may not correspond to the actual system outputs available to each DM.

The performance index of each DM is given by

$$J_i(\gamma_1, \gamma_2) = \left\{ \frac{1}{2}\int_0^\infty (y_i'y_i + u_i'R_iu_i)\, dt \,|\, u_i(t) = \gamma_i(\cdot) \right\}, \qquad i = 1,2 \tag{2}$$

where $\gamma_i(\cdot)$ is the admissible strategy of DMi, measurable with respect to the sigma-algebra generated by his information set (to be specified later).

The DM's are to select optimal strategies $\{\gamma_i^* | \gamma_i^* \in \Gamma_i;\ i = 1,2\}$ such that

$$J_i(\gamma_i^*, \gamma_j^*) \leqslant J_i(\gamma_i, \gamma_j^*), \qquad \forall \gamma_i \in \Gamma_i,\ i,j = 1,2,\ i \neq j \tag{3}$$

where $\{\Gamma_i;\ i = 1,2\}$ are some admissible strategy sets for the DM's to be specified later. The pair of inequalities in (3) define the Nash equilibrium point.

In large scale game problems, the "curse of dimensionality" may render any direct approach to the optimal solution computationally intractable. Hence there is a strong motivation for the DM's to look for alternative approaches to the problem which ease the computational difficulties. The approach formulated in the sequel has the desirable feature that it induces a partial noninteraction among the DM's leading to a lower order game. This is done by choosing appropriate admissible strategy sets Γ_i based on a particular structural decomposition of the system.

2.2. Structural Decomposition

The observation sets of the DM's given by (1b) induce a certain observability decomposition on the state space. We propose to exploit this decomposition to obtain multimodel strategies. To do this, we start by exhibiting this observability decomposition explicitly by transforming the state space. This may be done either by performing chained aggregation sequentially with respect to each DM's observation set [6]–[8], [18]; or, equivalently by making a similarity transformation directly following a procedure dual to the one in [9], [17] where a controllability decomposition was achieved.

The transformed system is represented as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_1u_1 + \bar{B}_2u_2, \qquad \bar{x}(0) = \bar{x}_0 \tag{4a}$$

$$y_i = \bar{C}_i\bar{x}, \qquad i = 1,2 \tag{4b}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} & 0 \\ 0 & \bar{A}_{22} & \bar{A}_{23} & 0 \\ 0 & 0 & \bar{A}_{33} & 0 \\ \bar{A}_{41} & \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{44} \end{bmatrix}$$

$$\bar{C}_1 = [\bar{C}_{11} \quad 0 \quad \bar{C}_{13} \quad 0]$$
$$\bar{C}_2 = [0 \quad \bar{C}_{22} \quad \bar{C}_{23} \quad 0]$$
$$\bar{B}_i = \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \\ \bar{B}_{i3} \\ \bar{B}_{i4} \end{bmatrix}, \qquad i=1,2$$

and

$$[\bar{A}_{ii}, \bar{C}_{ii}], \quad \left[\begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} \\ 0 & \bar{A}_{33} \end{bmatrix}, [\bar{C}_{ii} \quad \bar{C}_{i3}]\right], \qquad i=1,2$$

are observable pairs.

The eigenvalues of $\{\bar{A}_{ii};\ i=1,2\}$ represent the modes which are observable only to DMi but not to DMj $(i \neq j)$; the eigenvalues of $\bar{A}_{33}$ represent the modes which are observable to both the DM's; and the eigenvalues of $\bar{A}_{44}$ represent the modes which are unobservable to both the DM's.

For simplicity we shall neglect the jointly unobservable modes. In a well formulated problem these modes are stable and do not contribute anything to the cost. Hence, from now onwards we shall assume the system matrices to have the following form:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} \\ 0 & \bar{A}_{22} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33} \end{bmatrix} \tag{5a}$$

$$\bar{C}_1 = [\bar{C}_{11} \quad 0 \quad \bar{C}_{13}]$$
$$\bar{C}_2 = [0 \quad \bar{C}_{22} \quad \bar{C}_{23}] \tag{5b}$$

$$\bar{B}_i = \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \\ \bar{B}_{i3} \end{bmatrix}, \qquad i=1,2. \tag{5c}$$

The input structure specified by the matrices $\bar{B}_1, \bar{B}_2$ are not in a form suitable for our analysis. We need to make input space transformations in order to appropriately overlap the input structure with the observability decomposition. Assuming that the pairs $\{(\bar{A}, \bar{B}_i), (\bar{A}_{ii}, \bar{B}_{ii});\ i=1,2\}$ are controllable, there exist matrices G_1, G_2 such that the input space transformation $\{u_i = G_i \bar{u}_i;\ i=1,2\}$ gives the new input matrices in the following form [10]:

$$\hat{B}_1 = \bar{B}_1 G_1 = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{14} \\ 0 & \hat{B}_{12} \\ 0 & \hat{B}_{13} \end{bmatrix}$$

$$\hat{B}_2 = \bar{B}_2 G_2 = \begin{bmatrix} 0 & \hat{B}_{21} \\ \hat{B}_{22} & \hat{B}_{24} \\ 0 & \hat{B}_{23} \end{bmatrix} \tag{6}$$

where the pairs $\{(\bar{A}_{ii}, \hat{B}_{ii});\ i=1,2\}$ are controllable.

Remarks: Before performing the input space transformation, we might need to do another state space transformation; but this can be done without destroying the observability decomposition. This is to put the system in an appropriate basis such that $\mathcal{X} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \varphi$ where $\mathcal{X}$ is the state space and $\mathcal{R}_i$ is a controllability subspace of DMi. The input space transformations G_i identify explicitly the control channels through which the individually observable modes are completely controllable [10].

2.3. *Structure Preserving Strategies*

The system and the performance indexes, after the observability decomposition and the input space transformation, take the following form:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \hat{B}_1\bar{u}_1 + \hat{B}_2\bar{u}_2, \qquad \bar{x}(0) = \bar{x}_0 \tag{7a}$$

$$y_i = \bar{C}_i \bar{x}, \qquad i=1,2 \tag{7b}$$

$$J_i = \frac{1}{2}\int_0^\infty \left(y_i' y_i + \bar{u}_i' \hat{R}_i \bar{u}_i\right) dt, \qquad i=1,2 \tag{8}$$

where $\hat{R}_i = G_i' R_i G_i$. We shall assume that

$$\hat{R}_i = \begin{bmatrix} \hat{R}_{ii} & 0 \\ 0 & \hat{R}_{ij} \end{bmatrix} > 0, \qquad i,j=1,2,\ i \neq j.$$

The nature of the results obtained here hold for arbitrary positive definite $\hat{R}_i$; but assuming a block diagonal form results in simpler derivations.

Before we obtain the Nash solution, we need to define the set of admissible strategies for each DM. The admissible strategy sets that we are particularly interested here will be referred to as "Structure Preserving" strategies and are defined below.

Definition: A Structure Preserving strategy set is the set of all linear feedback strategies which preserve the observability decomposition (5) of the closed-loop system.

In the single DM case, the three component control of [7], [18] is a Structure Preserving control. After the first component achieves decoupling, the second and third components which control the aggregate and the residual respectively, are Structure Preserving. The design in [7] was purely from a pole-placement point of view without any optimality considerations. Here we shall show that in the multiple DM case, the design of Structure Preserving Nash strategies leads to multimodel solutions.

III. Multimodel Solutions

We shall consider two types of information patterns for the DM's: the Feedback Perfect State (FPS) and the Feedback Imperfect State (FIS). Under the FPS information pattern, each DM knows, at time t, the current state of the system $x(t)$; and under the FIS information pattern, each DM knows, at time t, only the current value of his observation $y(t)$.

3.1. *FPS Information Pattern*

Under the FPS information pattern, the admissible strategy set Γ_i of DMi is the set of linear state feedback strategies which are Structure Preserving. Specifically

$$\Gamma_i = \left\{ \gamma_i | \gamma_i(\bar{x}) = -F_i\bar{x} = -\begin{bmatrix} F_{ii}\delta_{i1} & F_{ii}\delta_{i2} & F_{i3} \\ 0 & 0 & F_{3i} \end{bmatrix}\bar{x} \right\},$$
$$i=1,2 \tag{9}$$

where δ_{ij} is the Kronecker delta.

Now, to find the Nash solution, we need to find a pair $\{\gamma_i^* \in \Gamma_i;\ i=1,2\}$ such that the pair of inequalities (3) are satisfied. Substituting $\bar{u}_i = \gamma_i(\bar{x})$ from (9) in (7) and (8) we get

$$\dot{\bar{x}} = \hat{A}\bar{x}, \qquad \bar{x}(0) = \bar{x}_0 \tag{10a}$$

$$y_i = \bar{C}_i \bar{x} \tag{10b}$$

$$J_i = \frac{1}{2}\int_0^\infty (\bar{x}' Q_i \bar{x})\, dt, \qquad i=1,2 \tag{11}$$

where

$$Q_i = \bar{C}_i'\bar{C}_i + F_i'\hat{R}_i F_i, \qquad i=1,2 \tag{12}$$

and the closed-loop system matrix is

$$\hat{A} = A - \hat{B}_1 F_1 - \hat{B}_2 F_2 = \begin{bmatrix} (\bar{A}_{11} - \hat{B}_{11}F_{11}) & 0 & (\bar{A}_{13} - \hat{B}_{11}F_{13} - \hat{B}_{14}F_{31} - \hat{B}_{21}F_{32}) \\ 0 & (\bar{A}_{22} - \hat{B}_{22}F_{22}) & (\bar{A}_{23} - \hat{B}_{22}F_{23} - \hat{B}_{24}F_{32} - \hat{B}_{12}F_{31}) \\ 0 & 0 & (\bar{A}_{33} - \hat{B}_{13}F_{13} - \hat{B}_{23}F_{32}) \end{bmatrix} \triangleq \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ 0 & 0 & \hat{A}_{33} \end{bmatrix}. \tag{13}$$

The optimal solution $\{F_{ii}^*, F_{i3}^*, F_{3i}^*;\ i=1,2\}$ will depend in general on the initial conditions $\bar{x}_0$ [11]. To remove this dependence, we follow [11] and assume that the initial conditions are random with

$$E[\bar{x}_0\bar{x}_0'] = N > 0 \tag{14}$$

and modify the cost functionals to be

$$J_i = \frac{1}{2}\mathop{E}_{\bar{x}_0}\left\{\int_0^\infty (\bar{x}'Q_i\bar{x})\, dt\right\}, \qquad i=1,2. \tag{15}$$

Introduce $M_1, M_2, L \in \mathbb{R}^{n\times n}$ defined by

$$\frac{1}{2}\bar{x}_0' M_i \bar{x}_0 = \frac{1}{2}\int_0^\infty (\bar{x}'Q_i\bar{x})\, dt; \qquad i=1,2 \tag{16}$$

$$L = \int_0^\infty E[\bar{x}(t)\bar{x}'(t)]\, dt. \tag{17}$$

For any given pair (F_1, F_2) such that $\operatorname{Re}\lambda(\hat{A}) < 0$, $M_i \geqslant 0$ and $L > 0$ satisfy the matrix Lyapunov equations

$$M_i\hat{A} + \hat{A}'M_i + Q_i = 0, \qquad i=1,2 \tag{18}$$

$$\hat{A}L + L\hat{A}' + N = 0. \tag{19}$$

Partition M_i, L, N appropriately

$$M_i = \begin{bmatrix} M_{11}^{(i)} & M_{12}^{(i)} & M_{13}^{(i)} \\ M_{12}^{(i)'} & M_{22}^{(i)} & M_{23}^{(i)} \\ M_{13}^{(i)'} & M_{23}^{(i)'} & M_{33}^{(i)} \end{bmatrix}, \qquad i=1,2 \tag{20a}$$

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12}' & L_{22} & L_{23} \\ L_{13}' & L_{23}' & L_{33} \end{bmatrix} \qquad N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}' & N_{22} & N_{23} \\ N_{13}' & N_{23}' & N_{33} \end{bmatrix}. \tag{20b}$$

Applying the Matrix Minimum Principle [13], the optimal $F_{ii}^*, F_{i3}^*, F_{3i}^*$ for the feedback Nash solution [15] can be shown to satisfy (for $i, j = 1,2;\ i \neq j$)

$$\hat{R}_{ii}F_{ii}^*L_{ii} + \hat{R}_{ii}F_{i3}^*L_{i3}' - \hat{B}_{ii}'M_{ii}^{(i)}L_{ii} - \hat{B}_{ii}'M_{i3}^{(i)}L_{i3}' = 0 \tag{21a}$$

$$\hat{R}_{ii}F_{ii}^*L_{i3} + \hat{R}_{ii}F_{i3}^*L_{33} - \hat{B}_{ii}'M_{i3}^{(i)}L_{33} - \hat{B}_{ii}'M_{ii}^{(i)}L_{i3} = 0 \tag{21b}$$

$$\hat{R}_{ij}F_{3i}^*L_{33} - \hat{B}_{i3}'M_{i3}^{(i)'}L_{i3} - \hat{B}_{i3}'M_{33}^{(i)}L_{33} - \hat{B}_{i4}'M_{ii}^{(i)}L_{i3} - \hat{B}_{i4}'M_{i3}^{(i)}L_{33} = 0 \tag{21c}$$

where

$$M_{ij}^{(i)} = M_{jj}^{(i)} = M_{j3}^{(i)} = 0 \tag{22a}$$

$$M_{ii}^{(i)}\hat{A}_{ii}^* + \hat{A}_{ii}^{*\prime}M_{ii}^{(i)} + \bar{C}_{ii}'\bar{C}_{ii} + F_{ii}^{*\prime}\hat{R}_{ii}F_{ii}^* = 0 \tag{22b}$$

$$M_{ii}^{(i)}\hat{A}_{i3}^* + M_{i3}^{(i)}\hat{A}_{33}^* + \hat{A}_{ii}^{*\prime}M_{i3}^{(i)} + \bar{C}_{ii}'\bar{C}_{i3} + F_{ii}^{*\prime}\hat{R}_{ii}F_{i3}^* = 0 \tag{22c}$$

$$M_{33}^{(i)}\hat{A}_{33}^* + \hat{A}_{33}^{*\prime}M_{33}^{(i)} + M_{i3}^{(i)'}\hat{A}_{i3}^* + \hat{A}_{i3}^{*\prime}M_{i3} + \bar{C}_{i3}'\bar{C}_{i3} + F_{i3}^{*\prime}\hat{R}_{ii}F_{i3}^* + F_{3i}^{*\prime}\hat{R}_{ij}F_{3i}^* = 0 \tag{22d}$$

$$\hat{A}_{ii}^*L_{i3} + \hat{A}_{i3}^*L_{33} + L_{i3}\hat{A}_{33}^{*\prime} + N_{i3} = 0 \tag{23a}$$

$$\hat{A}_{33}^*L_{33} + L_{33}\hat{A}_{33}^{*\prime} + N_{33} = 0 \tag{23b}$$

$\{\hat{A}_{ii}^*, \hat{A}_{i3}^*, \hat{A}_{33}^*;\ i=1,2\}$ are as in (13) with $\{F_{ii} = F_{ii}^*, F_{i3} = F_{i3}^*, F_{3i} = F_{3i}^*;\ i=1,2\}$. Solving (21) we obtain

$$F_{ii}^* = \hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{ii}^{(i)} \tag{24a}$$

$$F_{i3}^* = \hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{i3}^{(i)} \tag{24b}$$

$$F_{3i}^* = \hat{R}_{ij}^{-1}\hat{B}_{i3}'\left[M_{33}^{(i)} + M_{i3}^{(i)'}L_{i3}L_{33}^{-1}\right] + \hat{R}_{ij}^{-1}\hat{B}_{i4}'\left[M_{i3}^{(i)} + M_{ii}^{(i)}L_{i3}L_{33}^{-1}\right]. \tag{24c}$$

Notice that even though (21a) and (21b) are coupled in F_{ii}^* and F_{i3}^*, we are able to solve for them explicitly as in (24a) and (24b). This fact plays a crucial role in showing that the Nash solution admits a partial noninteraction. Substituting (24a) in (22b) we obtain

$$M_{ii}^{(i)}\bar{A}_{ii} + \bar{A}_{ii}'M_{ii}^{(i)} + \bar{C}_{ii}'\bar{C}_{ii} - M_{ii}^{(i)}\hat{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{ii}^{(i)} = 0, \qquad i=1,2. \tag{25}$$

It can be readily seen from (24a) and (25) that F_{ii}^* is the solution of an optimal state regulator problem with parameters $(\bar{A}_{ii}, \hat{B}_{ii}, \bar{C}_{ii}, \hat{R}_{ii})$. The following proposition highlights the multimodel nature of the Nash solution.

Proposition 1

Given the linear system (7) controlled by two DM's, and their performance indices (8), the design of Structure Preserving Feedback Nash strategies under the FPS information pattern, leads to two low-order coupled optimization problems defined by

$$\min_{\bar{u}_i} J_i = \underset{z_{i0}}{E}\left\{ \frac{1}{2}\int_0^\infty \left(y_i' y_i + \bar{u}_i' \hat{R}_i \bar{u}_i \right) dt \right\}$$

subject to

$$\bar{u}_i = \gamma_i(z_i) = -\begin{bmatrix} F_{ii} & F_{i3} \\ 0 & F_{3i} \end{bmatrix} z_i$$

where

$$\dot{z}_i = \begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}F_{3j} \\ 0 & \bar{A}_{33} - \hat{B}_{j3}F_{3j} \end{bmatrix} z_i + \begin{bmatrix} \hat{B}_{ii} & \hat{B}_{i4} \\ 0 & \hat{B}_{i3} \end{bmatrix} \bar{u}_i, \qquad z_i(0) = z_{i0}$$

$$y_i = \begin{bmatrix} \bar{C}_{ii} & \bar{C}_{i3} \end{bmatrix} z_i$$

$$E[z_{i0} \quad z_{i0}'] = \begin{bmatrix} N_{ii} & N_{i3} \\ N_{i3}' & N_{33} \end{bmatrix}, \qquad i, j = 1,2; \; i \neq j.$$

The solution to this pair of coupled optimization problems admits partial noninteraction among the DM's, and is given by the set of equations (22)–(25).

At this point we would like to remark that the controllability–observability of the triple $\{(\bar{A}_{ii}, \hat{B}_{ii}, \bar{C}_{ii});\ i = 1,2\}$ guarantees

$$\operatorname{Re}\lambda(\hat{A}_{ii}^*) < 0, \qquad i = 1,2. \tag{26}$$

For the solution to be well defined we need only to verify that $\operatorname{Re}\lambda(\hat{A}_{33}^*) < 0$.

The coupling between the optimization problems of the two DM's is due to the presence of the control gain F_{3j} of DM j in DM i's low-order model. Partial noninteraction is achieved because each DM can evaluate his control gain F_{ii}^* independently in a decentralized manner by solving (24a) and (25). The control gains $\{F_{i3}^*, F_{3i}^*;\ i = 1,2\}$ are then obtained by solving the coupled set of equations (22c), (22d), (23), (24).

Hence, we have succeeded in identifying the "core" of a high-order game problem where the DM's actually interact, and a pair of low-order control problems, one for each DM. This has been achieved by restricting the admissible strategy sets of the DM's to Structure Preserving strategies under the FPS information pattern; and transforming the state space and input space appropriately.

Notice that F_{ii}^* is independent of the statistics of the initial conditions since it is obtained from (24a) and (25). But $\{F_{i3}^*, F_{3i}^*;\ i = 1,2\}$ do depend, in general, on the statistics of the initial conditions, as they are obtained from the coupled set of equations (22c)–(24), which may be difficult to solve in practice. The gain matrices $\{F_{i3}^*, F_{3i}^*;\ i = 1,2\}$ which result in $\{L_{i3} = 0;\ i = 1,2\}$ are of particular interest, as they are computationally simpler to obtain. Such a set is given by

$$F_{i3}^* = \hat{R}_{ii}^{-1}\hat{B}_{ii}' M_{i3}^{(i)}, \qquad i = 1,2 \tag{27a}$$

$$F_{3i}^* = \hat{R}_{ij}^{-1}\left[\hat{B}_{i3}' M_{33}^{(i)} + \hat{B}_{i4}' M_{i3}^{(i)}\right], \qquad i, j = 1,2,\ i \neq j. \tag{27b}$$

Here $M_{i3}^{(i)}, M_{33}^{(i)}$ satisfy the coupled set of equations (for $i, j = 1,2;\ i \neq j$)

$$\begin{aligned} &M_{ii}^{(i)}\bar{A}_{i3} + M_{i3}^{(i)}\bar{A}_{33} + \bar{A}_{ii}' M_{i3}^{(i)} + \bar{C}_{ii}'\bar{C}_{i3} - M_{ii}^{(i)} S_{ii} M_{i3}^{(i)} \\ &\quad - M_{ii}^{(i)}\hat{S}_i M_{33}^{(i)} - M_{ii}^{(i)} S_{i4} M_{i3}^{(i)} - M_{i3}^{(i)} - M_{ii}^{(i)}\tilde{S}_{ji} M_{33}^{(j)} \\ &\quad - M_{ii}^{(i)}\bar{S}_{ji} M_{j3}^{(j)} - M_{i3}^{(i)} S_{i3} M_{33}^{(i)} - M_{i3}^{(i)}\hat{S}_i' M_{i3}^{(i)} \\ &\quad - M_{i3}^{(i)} S_{j3} M_{33}^{(j)} - M_{i3}^{(i)}\hat{S}_j' M_{j3}^{(j)} = 0 \end{aligned} \tag{28a}$$

$$\begin{aligned} &M_{33}^{(i)}\bar{A}_{33} + \bar{A}_{33}' M_{33}^{(i)} + M_{i3}^{(i)\prime}\bar{A}_{i3} + \bar{A}_{i3}' M_{i3}^{(i)} + \bar{C}_{i3}'\bar{C}_{i3} \\ &\quad - M_{33}^{(i)} S_{i3} M_{33}^{(i)} - M_{33}^{(i)}\hat{S}_i' M_{i3}^{(i)} - M_{i3}^{(i)\prime}\hat{S}_i M_{33}^{(i)} \\ &\quad - M_{33}^{(i)} S_{j3} M_{33}^{(j)} - M_{33}^{(j)} S_{j3} M_{33}^{(i)} - M_{33}^{(i)}\hat{S}_j' M_{j3}^{(j)} \\ &\quad - M_{j3}^{(j)\prime}\hat{S}_j M_{33}^{(i)} - M_{i3}^{(i)\prime} S_{ii} M_{i3}^{(i)} - M_{i3}^{(i)} S_{i4} M_{i3}^{(i)} \\ &\quad - M_{i3}^{(i)\prime}\tilde{S}_{ji} M_{33}^{(j)} - M_{33}^{(j)}\tilde{S}_{ji}' M_{i3}^{(i)} - M_{i3}^{(i)}\bar{S}_{ji} M_{j3}^{(j)} \\ &\quad - M_{j3}^{(j)\prime}\bar{S}_{ji}' M_{i3}^{(i)} = 0 \end{aligned} \tag{28b}$$

where

$$S_{ii} = \hat{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}' \quad S_{i3} = \hat{B}_{i3}\hat{R}_{ij}^{-1}\hat{B}_{i3}' \quad \tilde{S}_{ij} = \hat{B}_{ij}\hat{R}_{ij}^{-1}\hat{B}_{i3}'$$

$$S_{i4} = \hat{B}_{i4}\hat{R}_{ij}^{-1}\hat{B}_{i4}' \quad \bar{S}_{ij} = \hat{B}_{ij}\hat{R}_{ij}^{-1}\hat{B}_{i4}' \quad \hat{S}_i = \hat{B}_{i4}\hat{R}_{ij}^{-1}\hat{B}_{i3}'.$$

Furthermore $\{F_{i3}^*, F_{3i}^*;\ i = 1,2\}$ are such that

$$\hat{A}_{i3}^* L_{33} + N_{i3} = 0, \qquad i = 1,2 \tag{29}$$

where L_{33} is the positive definite solution of (23b).

Notice that if the initial cross covariance $N_{13} = N_{23} = 0$, then (29) is satisfied if and only if $\hat{A}_{13}^* = \hat{A}_{23}^* = 0$; which would be true if the solution of (27) and (28) block diagonalizes the closed-loop system.

3.2. FIS Information Pattern

It can be readily seen that when the output matrices are of the form given by (5b), Structure Preserving strategies involving only static linear output feedback do not exist.

When the output matrices split so that there are separate observation channels for the individually and commonly observable modes, i.e., when

$$\bar{C}_i = \begin{bmatrix} \bar{C}_{ii}\delta_{i1} & \bar{C}_{ii}\delta_{i2} & 0 \\ 0 & 0 & \bar{C}_{i3} \end{bmatrix}, \qquad i = 1,2 \tag{30}$$

linear static output feedback Structure Preserving strategies do exist, and belong to the admissible strategy set $\tilde{\Gamma}_i$ defined by

$$\tilde{\Gamma}_i = \left\{ \tilde{\gamma}_i \,|\, \tilde{\gamma}_i(y_i) = -\tilde{F}_i y_i = -\begin{bmatrix} \tilde{F}_{ii} & \tilde{F}_{i3} \\ 0 & \tilde{F}_{3i} \end{bmatrix} y_i \right\}, \qquad i = 1,2. \tag{31}$$

Substituting $\bar{u}_i = \tilde{\gamma}_i(y_i)$ from (31) in (7) and (8), we get

$$\dot{\bar{x}} = \tilde{A}\bar{x}, \qquad \bar{x}(0) = \bar{x}_0 \tag{32a}$$

$$y_i = \bar{C}_i \bar{x} \tag{32b}$$

$$J_i = \frac{1}{2}\int_0^\infty \left(\bar{x}' \tilde{Q}_i \bar{x} \right) dt, \qquad i = 1,2 \tag{33}$$

where

$$\tilde{Q}_i = \overline{C}_i'\left(I + \tilde{F}_i'\hat{R}_i\tilde{F}_i\right)\overline{C}_i, \qquad i=1,2 \tag{34}$$

and the closed-loop system matrix becomes

$$\tilde{A} = \overline{A} - \hat{B}_1\tilde{F}_1\overline{C}_1 - \hat{B}_2\tilde{F}_2\overline{C}_2$$

$$= \begin{bmatrix} (\overline{A}_{11} - \hat{B}_{11}\tilde{F}_{11}\overline{C}_{11}) & 0 & (\overline{A}_{13} - \hat{B}_{13}\tilde{F}_{13}\overline{C}_{13} - \hat{B}_{14}\tilde{F}_{31}\overline{C}_{13} - \hat{B}_{21}\tilde{F}_{32}\overline{C}_{23}) \\ 0 & (\overline{A}_{22} - \hat{B}_{22}\tilde{F}_{22}\overline{C}_{22}) & (\overline{A}_{23} - \hat{B}_{22}\tilde{F}_{23}\overline{C}_{23} - \hat{B}_{24}\tilde{F}_{32}\overline{C}_{23} - \hat{B}_{12}\tilde{F}_{31}\overline{C}_{13}) \\ 0 & 0 & (\overline{A}_{33} - \hat{B}_{13}\tilde{F}_{31}\overline{C}_{13} - \hat{B}_{23}\tilde{F}_{32}\overline{C}_{23}) \end{bmatrix} \triangleq \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix}. \tag{35}$$

Define M_1, M_2, and L as in (16), (17), and partition them as in (20). For any given pair $(\tilde{F}_1, \tilde{F}_2)$, such that $\mathrm{Re}\,\lambda(\tilde{A}) < 0$, $M_i \geq 0$, and $L > 0$ satisfy the matrix Lyapunov equations

$$M_i\tilde{A} + \tilde{A}'M_i + \tilde{Q}_i = 0; \qquad i=1,2 \tag{36}$$

$$\tilde{A}L + L\tilde{A}' + N = 0. \tag{37}$$

Applying the Matrix Minimum Principle, the optimal $\tilde{F}_{ii}^*, \tilde{F}_{i3}^*, \tilde{F}_{3i}^*$ for the Feedback Nash solution can be shown to satisfy (for $i, j=1,2;\ i \neq j$)

$$\tilde{F}_{ii}^*\overline{C}_{ii}L_{ii}\overline{C}_{ii}' + \tilde{F}_{i3}^*\overline{C}_{i3}L_{i3}'\overline{C}_{ii}'$$
$$= \hat{R}_{ii}^{-1}\hat{B}_{ii}'\left[M_{ii}^{(i)}L_{ii}\overline{C}_{ii}' + M_{i3}^{(i)}L_{i3}'\overline{C}_{ii}'\right] \tag{38a}$$

$$\tilde{F}_{ii}^*\overline{C}_{ii}L_{i3}\overline{C}_{i3}' + \tilde{F}_{i3}^*\overline{C}_{i3}L_{33}\overline{C}_{i3}'$$
$$= \hat{R}_{ii}^{-1}\hat{B}_{ii}'\left[M_{ii}^{(i)}L_{i3}\overline{C}_{i3}' + M_{i3}^{(i)} + L_{33}\overline{C}_{i3}'\right] \tag{38b}$$

$$\tilde{F}_{3i}^* = \hat{R}_{ij}^{-1}\left[\hat{B}_{i3}'M_{33}^{(i)}L_{33}\overline{C}_{i3}' + \hat{B}_{i3}'M_{i3}^{(i)'}L_{i3}\overline{C}_{i3}'\right.$$
$$\left. + \hat{B}_{i4}'M_{ii}^{(i)}L_{i3}\overline{C}_{i3}' + \hat{B}_{i4}'M_{i3}^{(i)}L_{33}\overline{C}_{i3}'\right] \times \left[\overline{C}_{i3}L_{33}\overline{C}_{i3}'\right]^{-1} \tag{38c}$$

where

$$M_{ij}^{(i)} = M_{jj}^{(i)} = M_{j3}^{(i)} = 0 \tag{39a}$$

$$M_{ii}^{(i)}\tilde{A}_{ii}^* + \tilde{A}_{ii}^{*\prime}M_{ii}^{(i)} + \overline{C}_{ii}'\left(I + \tilde{F}_{ii}^{*\prime}\hat{R}_{ii}\tilde{F}_{ii}^*\right)\overline{C}_{ii} = 0 \tag{39b}$$

$$M_{ii}^{(i)}\tilde{A}_{i3}^* + M_{i3}^{(i)}\tilde{A}_{33}^* + \tilde{A}_{ii}^{*\prime}M_{i3}^{(i)}$$
$$+ \overline{C}_{ii}'\left(I + \tilde{F}_{ii}^{*\prime}\hat{R}_{ii}\tilde{F}_{i3}^*\right)\overline{C}_{i3} = 0 \tag{39c}$$

$$M_{33}^{(i)}\tilde{A}_{33}^* + \tilde{A}^{*\prime}{}_{33}M_{33}^{(i)} + M_{i3}^{(i)\prime}\tilde{A}_{i3}^* + \tilde{A}_{i3}^{*\prime}M_{i3}^{(i)}$$
$$+ \overline{C}_{i3}'\left(I + \tilde{F}_{i3}^{*\prime}\hat{R}_{ii}\tilde{F}_{i3}^* + \tilde{F}_{3i}^{*\prime}\hat{R}_{ij}\tilde{F}_{3i}^*\right)\overline{C}_{i3} = 0 \tag{39d}$$

$$\tilde{A}_{ii}^*L_{ii} + L_{ii}\tilde{A}_{ii}^{*\prime} + \tilde{A}_{i3}^*L_{i3}' + L_{i3}\tilde{A}_{i3}^{*\prime} + N_{ii} = 0 \tag{40a}$$

$$\tilde{A}_{ii}^*L_{i3} + \tilde{A}_{i3}^*L_{33} + L_{i3}\tilde{A}_{33}^{*\prime} + N_{i3} = 0 \tag{40b}$$

$$\tilde{A}_{33}^*L_{33} + L_{33}\tilde{A}_{33}^{*\prime} + N_{33} = 0. \tag{40c}$$

$\{\tilde{A}_{ii}^*, \tilde{A}_{i3}^*, \tilde{A}_{33}^*;\ i=1,2\}$ are as in (35) with $\{\tilde{F}_{ii} = \tilde{F}_{ii}^*, = \tilde{F}_{i3} = \tilde{F}_{i3}^*,\ \tilde{F}_{3i} = \tilde{F}_{3i}^*;\ i=1,2\}$. The following proposition highlights the multimodel nature of the Nash solution.

Proposition 2

Given the linear system (7a) controlled by two DM's, their observation sets (30) and their performance indexes (8), the design of Structure Preserving Feedback Nash strategies under the FIS information pattern leads to two low-order coupled optimization problems defined by

$$\min_{\overline{u}_i} J_i = \mathop{E}_{z_{i0}}\left\{\frac{1}{2}\int_0^\infty \left(y_i'y_i + \overline{u}_i'\hat{R}_i\overline{u}_i\right)dt\right\}$$

subject to

$$\overline{u}_i = \overline{\gamma}_i(y_i) = -\begin{bmatrix} \tilde{F}_{ii} & \tilde{F}_{i3} \\ 0 & \tilde{F}_{3i} \end{bmatrix} y_i$$

where

$$y_i = \begin{bmatrix} \overline{C}_{ii} & 0 \\ 0 & \overline{C}_{i3} \end{bmatrix} z_i$$

$$\dot{z}_i = \begin{bmatrix} \overline{A}_{ii} & \overline{A}_{i3} - \hat{B}_{ji}\tilde{F}_{3j}\overline{C}_{j3} \\ 0 & \overline{A}_{33} - \hat{B}_{j3}\tilde{F}_{3j}\overline{C}_{j3} \end{bmatrix} z_i + \begin{bmatrix} \hat{B}_{ii} & \hat{B}_{i4} \\ 0 & \hat{B}_{i3} \end{bmatrix} \overline{u}_i, \qquad z_i(0) = z_{i0}$$

$$E[z_{i0}z_{i0}'] = \begin{bmatrix} N_{ii} & N_{i3} \\ N_{i3}' & N_{33} \end{bmatrix}, \qquad i, j=1,2;\ i \neq j.$$

The solution to this pair of coupled optimization problems is given by the set of equations (38)–(40).

Now, unlike the earlier problem of Section III-3.1, we need to verify that

$$\mathrm{Re}\,\lambda\left(\hat{A}_{ii}^*\right) < 0, \qquad i=1,2,3$$

for the solution to be well defined. Also unlike the earlier case, the Structure Preserving Feedback Nash solution of Proposition 2 is completely interacting. This is essentially due to the fact that (38a) and (38b) cannot be solved explicitly for $\tilde{F}_{ii}^*$ and $\tilde{F}_{i3}^*$. Another significant difference is that now all the optimal gains $\{F_{ii}^*, F_{i3}^*, F_{3i}^*;\ i=1,2\}$ depend on the statistics of the initial conditions.

Partial noninteraction results when $\{L_{i3} = 0;\ i=1,2\}$. In this case the optimal solution is given by $(i, j=1,2;\ i \neq j)$

$$\tilde{F}_{ii}^* = \hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{ii}^{(i)}L_{ii}\overline{C}_{ii}'\left(\overline{C}_{ii}L_{ii}\overline{C}_{ii}'\right)^{-1} \tag{41a}$$

$$\tilde{F}_{i3}^* = \hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{i3}^{(i)}L_{33}\overline{C}_{i3}'\left(\overline{C}_{i3}L_{33}\overline{C}_{i3}'\right)^{-1} \tag{41b}$$

$$\tilde{F}_{3i}^* = \hat{R}_{ij}^{-1}\left[\hat{B}_{i3}'M_{33}^{(i)}L_{33}\overline{C}_{i3}' + \hat{B}_{i4}'M_{i3}^{(i)}L_{33}\overline{C}_{i3}'\right]$$
$$\cdot\left(\overline{C}_{i3}L_{33}\overline{C}_{i3}'\right)^{-1}. \tag{41c}$$

$M_{ii}^{(i)}, M_{i3}^{(i)}, M_{33}^{(i)}$ are obtained from (39) with the control gains given by (41). L_{33} is obtained from (40c), and L_{ii} is the positive definite solution of

$$\tilde{A}_{ii}^*L_{ii} + L_{ii}\tilde{A}_{ii}^{*\prime} + N_{ii} = 0. \tag{42}$$

Furthermore $\{\tilde{F}_{i3}^*, \tilde{F}_{3i}^*;\ i=1,2\}$ are such that

$$\tilde{A}_{i3}^*L_{33} + N_{i3} = 0, \qquad i=1,2. \tag{43}$$

Now $\tilde{F}_{ii}^*$ is first obtained by each DM independently on solving (39b), (41a), and (42). This is the optimal solution of an output regulator problem with parameters $(\overline{A}_{ii}, \hat{B}_{ii}, \overline{C}_{ii}, \hat{R}_{ii}, N_{ii})$ [11].

In cases when the output matrices do not split as in (30), the FPS Structure Preserving Nash strategies of Proposition 1 can be synthesized as feedback strategies using dynamic observers.

We let

$$\bar{u}_i = \bar{\gamma}_i(\hat{z}_i) = -\begin{bmatrix} F_{ii}^* & F_{i3}^* \\ 0 & F_{3i}^* \end{bmatrix} \hat{z}_i, \qquad i=1,2 \tag{44}$$

where

$$\dot{\hat{z}}_i = \begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}F_{3j}^* \\ 0 & \bar{A}_{33} - \hat{B}_{j3}F_{3j}^* \end{bmatrix} z_i + \begin{bmatrix} \hat{B}_{ii} & \hat{B}_{i4} \\ 0 & \hat{B}_{i3} \end{bmatrix} \bar{u}_i + K_i\{y_i - [\bar{C}_{ii} \quad \bar{C}_{i3}]\hat{z}_i\}, \qquad i,j=1,2;\ i \neq j \tag{45}$$

$\{F_{ii}^*, F_{i3}^*, F_{3i}^*;\ i=1,2\}$ are given by (24), and K_i is the observer gain to be chosen by each DM.

Notice that the dimension of the observer of each DM is equal to the dimension of his own observable eigenspace, which is all he needs to reconstruct in order to implement the FPS Structure Preserving strategy.

Defining $e_i = z_i - \hat{z}_i$, we get

$$\dot{e}_i = \begin{bmatrix} \bar{A}_{ii} - K_{i1}\bar{C}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}F_{3j}^* - K_{i1}\bar{C}_{i3} \\ -K_{i2}\bar{C}_{ii} & \bar{A}_{33} - \hat{B}_{j3}F_{3j}^* - K_{i2}\bar{C}_{i3} \end{bmatrix} e_i \quad K_i = \begin{bmatrix} K_{i1} \\ K_{i2} \end{bmatrix}$$
$$= \hat{A}_i e_i; \qquad i,j=1,2;\ i \neq j. \tag{46}$$

If we choose K_i such that

$$\operatorname{Re}\lambda(\hat{A}_i) < -\frac{1}{\mu_i}, \qquad \mu_i > 0,\ i=1,2 \tag{47}$$

then we can write

$$\mu_i \dot{e}_i = \tilde{A}_i e_i$$
$$\operatorname{Re}\lambda(\tilde{A}_i) < -1, \qquad i=1,2. \tag{48}$$

Hence, by making the observer dynamics arbitrarily fast, we can represent the error system as a stable singularly perturbed system, i.e., $e_i \to 0$ as $\mu_i \to 0$. Rewriting the composite system and the feedback strategies as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \hat{B}_1\bar{u}_1 + \hat{B}_2\bar{u}_2$$
$$\mu_i\dot{e}_i = \tilde{A}_i e_i, \qquad i=1,2 \tag{49}$$

$$\bar{u}_i = \gamma_i(\bar{x}, e_i) = -\begin{bmatrix} F_{ii}^*\delta_{i1} & F_{ii}^*\delta_{i2} & F_{i3}^* \\ 0 & 0 & F_{3i}^* \end{bmatrix} \bar{x} + \begin{bmatrix} F_{ii}^* & F_{i3}^* \\ 0 & F_{3i}^* \end{bmatrix} e_i, \qquad i=1,2. \tag{50}$$

Since $e_i \to 0$ as $\mu_i \to 0$, $\gamma_i(\bar{x}, e_i)$ converges in open loop to a policy having a unique feedback representation, which we denote by $\bar{\gamma}_i^f(\bar{x})$, and

$$\bar{\gamma}_i^f(\bar{x}) = \gamma_i^*(\bar{x}), \qquad i=1,2 \tag{51}$$

where $\{\gamma_i^*(\bar{x});\ i=1,2\}$ is the FPS Structure Preserving Feedback Nash strategy of Proposition 1.

Due to (51) and the results of [12], we have

$$\lim_{\|\mu\| \to 0} J_i(\gamma_1, \gamma_2) = J_i(\gamma_1^*, \gamma_2^*), \qquad i=1,2. \tag{52}$$

It is to be noted that (50) is not the Feedback Nash strategy for the system (49) and the performance indices (8) within the class of admissible strategies $\hat{\Gamma}_i$ defined by

$$\Gamma_i\left\{\gamma_i | \gamma_i(\bar{x}, e_i) = -\begin{bmatrix} F_{ii}\delta_{i1} & F_{ii}\delta_{i2} & F_{i3} \\ 0 & 0 & F_{3i} \end{bmatrix}\bar{x} + \begin{bmatrix} F_{ii} & F_{i3} \\ 0 & F_{3i} \end{bmatrix} e_i = -\begin{bmatrix} F_{ii} & F_{i3} \\ 0 & F_{3i} \end{bmatrix}\hat{z}_i;\ K_i \text{ fixed}\right\}, \qquad i=1,2. \tag{53}$$

The Feedback Nash strategy $\hat{\gamma}_i^* \in \hat{\Gamma}_i$ will in general depend on the choice of the observer gains K_i. We do not compute $\hat{\gamma}_i^*$ because the strategy $\bar{\gamma}_i$ given by (44), or equivalently by (50), has the property of being Near-Equilibrium and Asymptotic Nash [16] as established by the following proposition.

Proposition 3

The strategy $\bar{\gamma}_i(\hat{z}_i) = \gamma_i(\bar{x}, e_i)$ given by (44) (or (50)) is Near-Equilibrium and Asymptotic Nash within the class $\hat{\Gamma}_i$ defined by (53). That is

$$\lim_{\|\mu\| \to 0} \{J_i(\bar{\gamma}_i, \bar{\gamma}_j) - J_i(\hat{\gamma}_i, \bar{\gamma}_j)\} = 0, \qquad \forall \hat{\gamma}_i \in \hat{\Gamma}_i,\ i,j=1,2,\ i \neq j$$

and

$$\lim_{\|\mu\| \to 0} \{J_i(\bar{\gamma}_i, \hat{\gamma}_j) - J_i(\bar{\gamma}_i, \bar{\gamma}_j)\} = 0\cdot$$

$\forall \hat{\gamma}_j \in \hat{\Gamma}_j$ such that $J_j(\bar{\gamma}_i, \hat{\gamma}_j) \leqslant (\bar{\gamma}_i, \bar{\gamma}_j)$; $i,j=1,2$; $i \neq j$.

The proof of the above Proposition follows readily from the results of [3].

IV. Decoupling of Completely Observable Systems

In situations when the whole system is completely observable through the observation set of each DM, the "core" is the full problem itself. But in some such cases, if the DM's have access to all the states then the observability decomposition can be induced by using state feedback. The role of the decoupling control in reduced order modeling has been studied in detail in [18]. Here we shall outline the procedure for multiple DM problems.

Suppose after appropriate state space, input and output space transformations, the system can be put in the following form [18]:

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix}\bar{x} + \begin{bmatrix} \bar{B}_{11} & \bar{B}_{14} \\ 0 & \bar{B}_{12} \\ 0 & \bar{B}_{13} \end{bmatrix}\bar{u}_1 + \begin{bmatrix} 0 & \bar{B}_{21} \\ \bar{B}_{22} & \bar{B}_{24} \\ 0 & \bar{B}_{23} \end{bmatrix}\bar{u}_2 \tag{54a}$$

$$\bar{y}_1 = [\bar{C}_{11} \quad 0 \quad \bar{C}_{13}]\bar{x}$$
$$\bar{y}_2 = [0 \quad \bar{C}_{22} \quad \bar{C}_{23}]\bar{x} \tag{54b}$$

with $\{\bar{B}_{ii}, \bar{B}_{i3}, i=1,2\}$ being square and nonsingular.

Now if the DM's use the following strategies

$$\bar{u}_1 = \begin{bmatrix} 0 & -\bar{B}_{11}^{-1}(\bar{A}_{12} - \bar{B}_{21}\bar{B}_{23}^{-1}\bar{A}_{32}) & 0 \\ -\bar{B}_{13}^{-1}\bar{A}_{31} & 0 & 0 \end{bmatrix} \bar{x} + \hat{u}_1 \tag{55a}$$

$$\bar{u}_2 = \begin{bmatrix} -\bar{B}_{22}^{-1}(\bar{A}_{21} - \bar{B}_{12}\bar{B}_{13}^{-1}\bar{A}_{31}) & 0 & 0 \\ 0 & -\bar{B}_{23}^{-1}\bar{A}_{32} & 0 \end{bmatrix} \bar{x} + \hat{u}_2 \tag{55b}$$

then the resulting partially closed-loop system has the form

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{11} - \bar{B}_{14}\bar{B}_{13}^{-1}\bar{A}_{31} & 0 & \bar{A}_{13} \\ 0 & \bar{A}_{22} - \bar{B}_{24}\bar{B}_{23}^{-1}\bar{A}_{32} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33} \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{B}_{11} & \bar{B}_{14} \\ 0 & \bar{B}_{12} \\ 0 & \bar{B}_{13} \end{bmatrix} \hat{u}_1 + \begin{bmatrix} 0 & \bar{B}_{21} \\ \bar{B}_{22} & \bar{B}_{24} \\ 0 & \bar{B}_{23} \end{bmatrix} \hat{u}_2. \tag{56}$$

It can be readily seen that the system (56), (54b) has the desired form of (7). Under appropriate assumptions, Proposition 1 can be applied to design $\hat{u}_1, \hat{u}_2$ as FPS Structure Preserving strategies.

It is significant to note that making the dimension of $\bar{B}_{ii}$ as large as possible results in a "maximally decoupled" system, i.e., a system in which the decentralized control problems are of the highest possible dimension, and consequently the "core" problem is of lowest possible dimension [18].

The use of decoupling control introduces a degree of suboptimality if the performance indices are chosen *a priori*. This is because the decoupling control is chosen from a purely algebraic point of view without any optimality considerations.

We would like to remark that the use of decoupling control requires a degree of mutual cooperation among the DM's. This cannot be guaranteed under the noncooperative Nash concept in general, unless, the resulting advantages constitute a strong enough incentive for the DM's to compensate for the performance loss resulting from the use of decoupling control. But, within a cooperative framework, the use of decoupling control can be readily ensured.

In problems where there is a need for the DM's to use the decoupling control, it will be more appropriate for them to choose their performance indexes with respect to the strategies $\hat{u}_i$ after the decoupling has been achieved. Again, this is easier to ensure in a cooperative framework than in a noncooperative framework.

Hence, in situations when the decoupling control has to be used, a semicooperative or cooperative framework is desirable for the application of our techniques.

V. Applications

Now we shall examine the applicability of our design methodologies to the control of large scale interconnected subsystems and multiarea power systems.

5.1. *Large Scale Interconnected Subsystems*

Consider the large scale system wherein each subsystem is controlled by one DM having his own performance objective. The system considered is of the form

$$\dot{x}_i = A_{ii}x_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} A_{ij}y_j + B_{ii}u_i \tag{57a}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \cdots, N \tag{57b}$$

where the output variables y_i are the interconnection variables. The above problem has been considered in [7], [8] as a single DM problem. We shall demonstrate that when viewed as a multiple DM problem, the techniques developed in this paper can be applied for optimal strategy design.

For simplicity we shall consider the two subsystem case ($N = 2$). As in [7] suppose that each subsystem is transformed with respect to its own output. The transformed system can be represented as

$$\left[\begin{array}{c} \dot{y}_1 \\ \dot{x}_{1r} \\ \hline \dot{y}_2 \\ \dot{x}_{2r} \end{array}\right] = \left[\begin{array}{cc|cc} F_{11}^{(1)} & F_{12}^{(1)} & F^{12} & 0 \\ F_{21}^{(1)} & F_{22}^{(2)} & F_r^{12} & 0 \\ \hline F^{21} & 0 & F_{11}^{(2)} & F_{12}^{(2)} \\ F_r^{21} & 0 & F_{21}^{(2)} & F_{22}^{(2)} \end{array}\right] \left[\begin{array}{c} y_1 \\ x_{1r} \\ \hline y_2 \\ x_{2r} \end{array}\right] + \left[\begin{array}{c} G_{11} \\ G_{12} \\ \hline 0 \\ 0 \end{array}\right] u_1 + \left[\begin{array}{c} 0 \\ 0 \\ \hline G_{22} \\ G_{21} \end{array}\right] u_2. \tag{58}$$

By a simple reordering of variables (58) can be written as

$$\left[\begin{array}{c} \dot{x}_{1r} \\ \hline \dot{x}_{2r} \\ \hline \dot{y}_1 \\ \dot{y}_2 \end{array}\right] = \left[\begin{array}{c|c|cc} F_{22}^{(1)} & 0 & F_{21}^{(1)} & F_r^{12} \\ \hline 0 & F_{22}^{(2)} & F_r^{21} & R_{21}^{(2)} \\ \hline F_{12}^{(1)} & 0 & F_{11}^{(1)} & F^{12} \\ 0 & F_{12}^{(2)} & F^{21} & F_{11}^{(2)} \end{array}\right] \left[\begin{array}{c} x_{1r} \\ \hline x_{2r} \\ \hline y_1 \\ y_2 \end{array}\right] + \left[\begin{array}{c} G_{12} \\ 0 \\ G_{11} \\ 0 \end{array}\right] u_1 + \left[\begin{array}{c} 0 \\ \hline G_{21} \\ \hline 0 \\ G_{22} \end{array}\right] u_2. \tag{59}$$

Now, by making an appropriate input space transformation [7] and letting DMi use his own residual state feedback to cancel the terms $F_{12}^{(i)}x_{ir}$, we obtain a system which is in the familiar observability decomposition form. The interconnection variables y_i represent the variables observable by both the DM's, and the residual variables x_{ir} represent the variables observable by DMi alone.

Suppose DMi chooses his performance index as

$$J_i = \frac{1}{2}\int_0^\infty (y_i'y_i + x_{ir}'Q_i x_{ir} + \hat{u}_i'R_i\hat{u}_i)\,dt, \qquad i=1,2 \tag{60}$$

where u_i = decoupling control + $\hat{u}_i$, then assuming that each DM has access to all the interconnection variables and his own subsystem variables, Structure Preserving Linear Feedback Nash strategies $\hat{u}_i$ can be generated from multimodel solutions of Proposition 1.

5.2. Two Area Power System

This example has been considered in [7] in the single DM context. Here we assume that each area is under a different control authority. We shall demonstrate how the system can be transformed into our desired form given by (7) without using numerical values for the system parameters. Once this has been accomplished, it is a routine numerical exercise to compute the optimal strategies for a given set of system parameters.

A two area power system with each area containing two thermal plants is constructed from [14]. The system is modeled by

$$\begin{aligned} \dot{x} &= Ax + B_1u_1 + B_2u_2 \\ y_i &= C_i x, \qquad i=1,2 \end{aligned} \tag{61}$$

where $x \in R^{19}$, $u_1 \in R^2$, $u_2 \in R^2$, $y_1 \in R^2$, $y_2 \in R^2$. The state, control, and output variables are defined in Appendix A. The system matrices are given by

$$A = \begin{bmatrix} A_{11}^{(1)} & 0 & A_{13}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & 0 & 0 & 0 & 0 \\ A_{31}^{(1)} & A_{32}^{(1)} & a_{99}^{(1)} & h_{12} & 0 & 0 & 0 \\ 0 & 0 & h_{21} & 0 & h_{23} & 0 & 0 \\ 0 & 0 & 0 & h_{32} & a_{99}^{(2)} & A_{31}^{(2)} & A_{32}^{(2)} \\ 0 & 0 & 0 & 0 & A_{13}^{(2)} & A_{11}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & A_{23}^{(2)} & 0 & A_{22}^{(2)} \end{bmatrix}$$

$$A_{11}^{(i)} = \begin{bmatrix} a_{11}^{(i)} & 0 & 0 & 0 \\ a_{21}^{(i)} & a_{22}^{(i)} & 0 & 0 \\ 0 & a_{32}^{(i)} & a_{33}^{(i)} & 0 \\ 0 & 0 & a_{43}^{(i)} & a_{44}^{(i)} \end{bmatrix}$$

$$A_{22}^{(i)} = \begin{bmatrix} a_{55}^{(i)} & 0 & 0 & 0 \\ a_{65}^{(i)} & a_{66}^{(i)} & 0 & 0 \\ 0 & a_{76}^{(i)} & a_{77}^{(i)} & 0 \\ 0 & 0 & a_{87}^{(i)} & a_{88}^{(i)} \end{bmatrix}$$

$$A_{13}^{(i)} = \begin{bmatrix} a_{19}^{(i)} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A_{23}^{(i)} = \begin{bmatrix} a_{59}^{(i)} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_{31}^{(i)} = \begin{bmatrix} 0 & a_{92}^{(i)} & a_{93}^{(i)} & a_{94}^{(i)} \end{bmatrix}$$

$$A_{32}^{(i)} = \begin{bmatrix} 0 & a_{96}^{(i)} & a_{97}^{(i)} & a_{98}^{(i)} \end{bmatrix}$$

$$B_1 = \left[\begin{array}{cc} B_{11}^{(1)} & 0 \\ 0 & B_{22}^{(1)} \\ \hline \multicolumn{2}{c}{0_{11\times 2}} \end{array}\right], \quad B_2 = \left[\begin{array}{cc} \multicolumn{2}{c}{0_{11\times 2}} \\ \hline B_{11}^{(2)} & 0 \\ 0 & B_{22}^{(2)} \end{array}\right]$$

$$B_{11}^{(i)} = \begin{bmatrix} b_{11}^{(i)} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{22}^{(i)} = \begin{bmatrix} b_{52}^{(i)} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \left[\begin{array}{c|cc|c} 0_{2\times 8} & 1 & 0 & 0_{2\times 9} \\ & 0 & 1 & \end{array}\right], \quad C_2 = \left[\begin{array}{c|cc|c} 0_{2\times 9} & 1 & 0 & 0_{2\times 8} \\ & 0 & 1 & \end{array}\right].$$

The parameters appearing in the system matrices are defined in Appendix B.

After two steps of chained aggregation [6], [7] and one input space transformation, we obtain the following representation:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} F_{11}^{(1)} & 0 & F_{13} \\ 0 & F_{11}^{(2)} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} G_{11}^{(1)} & G_{12}^{(1)} \\ 0 & 0 \\ 0 & G_{31} \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(1)} \\ \bar{u}_2^{(1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_{11}^{(2)} & G_{12}^{(2)} \\ 0 & G_{32} \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(2)} \\ \bar{u}_2^{(2)} \end{bmatrix},$$

$$\bar{x}_1 \in R^6; \quad \bar{x}_2 \in R^6; \quad \bar{x}_3 \in R^7 \tag{62}$$

where

$$F_{11}^{(i)} = \begin{bmatrix} f_{33}^{(i)} & f_{34}^{(i)} & 0 & 0 & 0 & 0 \\ 0 & f_{44}^{(i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{55}^{(i)} & f_{56}^{(i)} & 0 & 0 \\ 0 & 0 & 0 & f_{66}^{(i)} & f_{67}^{(i)} & 0 \\ 0 & 0 & 0 & 0 & f_{77}^{(i)} & f_{78}^{(i)} \\ 0 & 0 & 0 & 0 & 0 & f_{88}^{(i)} \end{bmatrix}$$

$$F_{33} = \begin{bmatrix} a_{99}^{(1)} & h_{12} & 0 & 1 & 0 & 0 & 0 \\ h_{21} & 0 & h_{23} & 0 & 0 & 0 & 0 \\ 0 & h_{32} & a_{99}^{(2)} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & f_{11}^{(1)} & 1 & 0 & 0 \\ d_{21}^{(1)} & 0 & 0 & 0 & f_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_{11}^{(2)} & 1 \\ 0 & 0 & d_{21}^{(2)} & 0 & 0 & 0 & f_{22}^{(2)} \end{bmatrix}$$

$$F_{31} = \left[\begin{array}{c} 0_{4\times 6} \\ \hline F_2^{(1)} \\ \hline 0_{2\times 6} \end{array}\right], \quad F_{32} = \left[\begin{array}{c} 0_{6\times 6} \\ \hline F_2^{(2)} \end{array}\right]$$

$$F_2^{(i)} = \begin{bmatrix} f_{23}^{(i)} & f_{24}^{(i)} & f_{25}^{(i)} & f_{26}^{(i)} & f_{27}^{(i)} & f_{28}^{(i)} \end{bmatrix}$$

$$F_{13} = \left[\begin{array}{c|c|c} 0_{6\times 3} & D & 0_{6\times 3} \end{array}\right], \quad F_{23} = \left[\begin{array}{c|c|c} 0_{6\times 5} & D & 0_{6\times 1} \end{array}\right]$$

$$D=[0 \quad d_{41} \quad 0 \quad 0 \quad 0 \quad d_{81}]'$$
$$G_{11}^{(i)}=[0 \quad g_{42}^{(i)} \quad 0 \quad 0 \quad 0 \quad g_{52}^{(i)}]'$$
$$G_{12}^{(i)}=[0 \quad g_{41}^{(i)} \quad 0 \quad 0 \quad 0 \quad 0]'$$
$$G_{31}=[0 \quad 0 \quad 0 \quad 0 \quad g_{21}^{(1)} \quad 0 \quad 0]'$$
$$G_{32}=[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad g_{21}^{(2)}]'.$$

The parameters appearing in the transformed system (62) are given in terms of the parameters of original system in Appendix C.

Now we need to apply a decoupling control to cancel out the terms F_{31} and F_{32} in (62). The decoupling control is chosen to be

$$\bar{u}_2^{(i)}=[k_1^{(i)} \quad k_2^{(i)} \quad k_3^{(i)} \quad k_4^{(i)} \quad k_5^{(i)} \quad k_6^{(i)}]\bar{x}_i+\hat{u}_2^{(i)}$$
$$k_j^{(i)}=-\frac{f_{2,j+2}^{(i)}}{g_{21}^{(i)}}, \qquad j=1,2,\cdots,6, \ i=1,2. \tag{63}$$

Substituting (63) we obtain

$$\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\\ \dot{\bar{x}}_3\end{bmatrix}=\begin{bmatrix}\bar{F}_{11}^{(1)} & 0 & F_{13}\\ 0 & \bar{F}_{11}^{(2)} & F_{23}\\ 0 & 0 & F_{33}\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}$$
$$+\begin{bmatrix}G_{11}^{(1)} & G_{12}^{(1)}\\ 0 & 0\\ 0 & G_{31}\end{bmatrix}\begin{bmatrix}\bar{u}_1^{(1)}\\ \hat{u}_2^{(1)}\end{bmatrix}+\begin{bmatrix}0 & 0\\ G_{11}^{(2)} & G_{12}^{(2)}\\ 0 & G_{32}\end{bmatrix}\begin{bmatrix}\bar{u}_1^{(2)}\\ \hat{u}_2^{(2)}\end{bmatrix} \tag{64}$$

where the values of $\bar{F}_{11}^{(i)}$ have been altered by the decoupling controls.

Now the system is precisely in a form suitable for our design techniques. The frequency deviations in the two areas and the tie line power flow comprise part of the core variables $\bar{x}_3$. The variables $\bar{x}_1$, $\bar{x}_2$ are the residual variables associated with each area.

The nineteenth-order game in its original form (61) may be computationally intractable. But in the form (64), and allowing only FPS Structure Preserving strategies, we need only to solve two sixth-order optimal control problems, and one seventh-order problem where the two DM's interact.

VI. Extensions

In this section we shall discuss briefly extensions of our ideas to many DM problems and cooperative Pareto games.

6.1. Many-DM Problems

In situations with more than two DM's there is more than one way to approach the problem; each approach resulting in a different order of simplification. Ideally one would like to identify the individually observable modes, the pairwise observable modes, and so on; and overlap appropriately the input structure of each DM with this observability decomposition. The design of structure preserving Nash strategies would then lead to the solution of low-order control problems, problems where two DM's interact, problems where three DM's interact and so on up to the core problem where all the DM's interact.

In the three-DM case the (A, C_1, C_2, C_3) matrices in the observability decomposition form will look like

$$A=\begin{bmatrix}A_{11} & 0 & 0 & A_{14} & 0 & A_{16} & A_{17}\\ 0 & A_{22} & 0 & A_{24} & A_{25} & 0 & A_{27}\\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} & A_{37}\\ 0 & 0 & 0 & A_{44} & 0 & 0 & A_{47}\\ 0 & 0 & 0 & 0 & A_{55} & 0 & A_{57}\\ 0 & 0 & 0 & 0 & 0 & A_{66} & A_{67}\\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77}\end{bmatrix}$$
$$C_1=[C_{11} \quad 0 \quad 0 \quad C_{14} \quad 0 \quad C_{16} \quad C_{17}]$$
$$C_2=[0 \quad C_{22} \quad 0 \quad C_{24} \quad C_{25} \quad 0 \quad C_{27}]$$
$$C_3=[0 \quad 0 \quad C_{33} \quad 0 \quad C_{35} \quad C_{36} \quad C_{37}].$$

It can be readily seen that the number of blocks to be identified in the system matrices grows exponentially as the number of DM's increase. Hence for a large number of DM's such a decomposition may be difficult to achieve in practice. The other extreme would be to identify only the modes observable by each DM alone, and consider the rest as commonly observable modes. This will result in only a first order of simplification because the core problem will be of a higher dimension. Of course in practice, depending on the problem, any approach in between these two extremes may be adopted, resulting in different orders of simplification.

6.2. Pareto Games

Multimodel solutions to cooperative Pareto games based on the structural decompositions of Section II can be obtained in a straightforward manner. To illustrate this point we shall give below the Structure Preserving Pareto strategies under the FPS information pattern.

Define the overall system cost as

$$J=\sum_{i=1}^{2}\alpha_i J_i, \qquad 0 \leqslant \alpha_i \leqslant 1, \quad \alpha_1+\alpha_2=1. \tag{65}$$

Applying the Matrix Minimum Principle, the FPS structure preserving Pareto strategy $\gamma_i^*(\cdot)\in\Gamma_i$ (defined by (9)) for the system (7) and performance index (65) is obtained as (for $i=1,2$)

$$F_{ii}^*=\hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{ii} \tag{66a}$$
$$F_{i3}^*=\hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{i3} \tag{66b}$$
$$F_{3i}^*=\hat{R}_{ij}^{-1}\hat{B}_{i3}'\left[\frac{1}{\alpha_i}M_{33}+M_{i3}'L_{i3}L_{33}^{-1}\right]$$
$$+\hat{R}_{ij}^{-1}\hat{B}_{i4}'\left[M_{i3}+M_{ii}L_{i3}L_{33}^{-1}\right] \tag{66c}$$

where

$$M_{ii}\bar{A}_{ii}'+\bar{A}_{ii}M_{ii}+\bar{C}_{ii}'\bar{C}_{ii}-M_{ii}\hat{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{ii}=0 \tag{67a}$$
$$M_{ii}\hat{A}_{i3}^*+M_{i3}\hat{A}_{33}^*+\hat{A}_{ii}^{*\prime}M_{i3}+\bar{C}_{ii}'\bar{C}_{i3}+F_{ii}^{*\prime}\hat{R}_{ii}F_{i3}^*=0 \tag{67b}$$
$$M_{33}\hat{A}_{33}^*+\hat{A}_{33}^*M_{33}+\sum_{i=1}^{2}\alpha_i\left(M_{i3}'\hat{A}_{i3}^*+\hat{A}_{i3}^{*\prime}M_{i3}+\bar{C}_{i3}'\bar{C}_{i3}\right.$$
$$\left.+F_{i3}^{*\prime}\hat{R}_{ii}F_{i3}^*\right)+\alpha_1F_{31}^{*\prime}\hat{R}_{12}F_{31}^*+\alpha_2F_{32}^{*\prime}\hat{R}_{21}F_{32}^*=0 \tag{67c}$$

$$\hat{A}_{ii}^{*} L_{i3} + \hat{A}_{i3}^{*} L_{33} + L_{i3} \hat{A}_{33}^{*\prime} + N_{i3} = 0 \tag{68a}$$

$$\hat{A}_{33}^{*} L_{33} + L_{33} \hat{A}_{33}^{*\prime} + N_{33} = 0. \tag{68b}$$

The controllability–observability of the triple $\{(\overline{A}_{ii}, \hat{B}_{ii}, \overline{C}_{ii}); \; i = 1,2\}$ guarantees $\{\operatorname{Re}\lambda(\hat{A}_{ii}^{*}) < 0; \; i = 1,2\}$. For the solution to be well defined we need only to verify that $\operatorname{Re}\lambda(\hat{A}_{33}^{*}) < 0$. The solution given by (66)–(68) has features similar to the Nash problem of Section III-3.1 (like partial noninteraction, etc).

The Structure Preserving Pareto strategies under the FIS information can be obtained in a similar manner. The solution will have features similar to the Nash problem of Section III-3.2.

VII. Conclusions

This paper has dealt with modeling and control strategy interaction in a multimodel context. The role of the observability structure in multiple DM problems has been examined. By identifying explicitly the observability decomposition induced by the observation sets of the DM's, and by overlapping appropriately the input structure of each DM, we have shown that the design of Structure Preserving Feedback Nash strategies leads to multimodel solutions. Under the FPS information pattern, the multimodel solutions are shown to admit partial noninteraction among the DM's. Under the FIS information pattern, Structure Preserving strategies involving only linear static output feedback do not exist in general. When the output matrices split so that there are separate observation channels for the individually and commonly observable modes, Structure Preserving strategies do exist and are again generated from multimodel solutions. But in this case, the solution is completely interacting unless certain conditions on the statistics of the state variables are satisfied. When the output matrices do not split, the FPS Structure Preserving strategies can be synthesized using observers with arbitrarily fast dynamics. This strategy has the property of being near-equilibrium and asymptotic Nash. When the system is completely observable by each DM, the observability decomposition can be induced by using the decoupling controls. But in such situations, a semicooperative or cooperative framework is desirable.

Applications to the control of large scale interconnected subsystems and control of multiarea power systems have been examined; and extensions to many DM problems and cooperative Pareto games have been discussed.

In practice, it may be possible to combine the multitime scale approach of [1]–[5] and the information based approach developed here. Based on the slow observation sets of the DM's, we could isolate a core of lower dimension within the slow core.

Appendix A

The state, control, and output variables have the following physical meaning:

- x_1, x_{12} valve position displacement in first thermal unit of area 1 and 2;
- x_2, x_{13} power output displacement of HP turbine in first thermal unit of area 1 and 2;
- x_3, x_{14} power output displacement of IP turbine in first thermal unit of area 1 and 2;
- x_4, x_{15} power output displacement of LP turbine in first thermal unit of area 1 and 2;
- x_5, x_{16} valve position displacement in second thermal unit of area 1 and 2;
- x_6, x_{17} power output displacement of HP turbine in second thermal unit of area 1 and 2;
- x_7, x_{18} power output displacement of IP turbine in second thermal unit of area 1 and 2;
- x_8, x_{19} power output displacement of LP turbine in second thermal unit of area 1 and 2;
- x_9, x_{11} frequency deviation of area 1 and 2;
- x_{10} tie-line power flow connecting area 1 and 2;
- $u_1^{(1)}, u_1^{(2)}$ set point adjustment of first thermal unit in area 1 and 2;
- $u_2^{(1)}, u_2^{(2)}$ set point adjustment of second thermal unit in area 1 and 2;
- $y_1^{(1)}, y_2^{(2)}$ frequency deviation of area 1 and 2;
- $y_2^{(1)}, y_1^{(2)}$ tie-line power flow of area 1 and 2.

Appendix B

The physical parameters for each area are as follows:

- r permanent speed drop;
- T_s time constant of the system pilot valve-servomotor turbine gates;
- T_u time constant of the turbine (characterizes the delay between control valve action and turbine nozzle action);
- T_r time constant characterizing the time delay in the HP turbine reheater and reheat piping;
- T_n time constant characterizing the time delay in the IP turbine and crossover piping;
- C_v fraction of total power generated by HP turbine;
- C_s fraction of total power generated by IP turbine;
- e_T coefficient characterizing the influence of frequency variation on turbine output variation (turbine self-regulation);
- K_t proportionality factor connecting the control valves position variation and HP turbine output variation in the steady state (K_t for IP and LP turbines are very close to 1 since power variations of these turbines in the steady state are equal);
- e_i participation of the unit in total system output;
- e_p^t load turbine and system self-regulation coefficient not including the participation of the unit under consideration (lead characteristic); $e = e_p^t - \Sigma_{i=1}^2 e_i T_{r_i}$;
- T system acceleration time constant $= T_p + \Sigma_{i=1}^2 e_i T_{G_i}$;
- T_p time constant due to the mechanical inertia of the rotating masses in the load;
- T_{G_i} unit acceleration time constant, $T_{G_i} = 2H_i$;
- H inertia time constant.

The nonzero elements of the matrices A, B_1, B_2, C_1, C_2 expressed in terms of the physical parameters are:

$$a_{11} = -\frac{r_1}{T_{s_1}}; \quad a_{19} = -\frac{1}{T_{s_1}}; \quad a_{21} = \frac{K_{t_1}}{T_{u_1}}; \quad a_{22} = -\frac{1}{T_{u_1}};$$

$$a_{32} = \frac{1}{T_{r_1}}; \quad a_{33} = -\frac{1}{T_{r_1}}; \quad a_{43} = \frac{1}{T_{n_1}}; \quad a_{44} = -\frac{1}{T_{n_1}};$$

$$a_{55}=-\frac{r_2}{T_{s_2}};\quad a_{59}=-\frac{1}{T_{s_2}};\quad a_{65}=\frac{K_{t_2}}{T_{u_2}};\quad a_{66}=-\frac{1}{T_{u_2}};$$

$$a_{76}=\frac{1}{T_{r_2}};\quad a_{77}=-\frac{1}{T_{r_2}};\quad a_{87}=\frac{1}{T_{n_2}};\quad a_{88}=-\frac{1}{T_{n_2}};$$

$$a_{92}=\frac{e_1C_{v_1}}{T};\quad a_{93}=\frac{e_1C_{s_1}(1-C_{v_1})}{T};$$

$$a_{94}=\frac{e_1(1-C_{s_1})(1-C_{v_1})}{T};$$

$$a_{96}=\frac{e_2C_{v_2}}{T};\quad a_{97}=\frac{e_2C_{s_2}(1-C_{v_2})}{T};$$

$$a_{98}=\frac{e_2(1-C_{s_2})(1-C_{v_2})}{T};$$

$$a_{99}=-\frac{e}{T};\quad b_{11}=\frac{1}{T_{s_1}};\quad b_{52}=\frac{1}{T_{s_2}}.$$

The parameters h_{12}, h_{21}, h_{23}, and h_{32} describe the tie-line dynamics and interconnection with the two areas.

Appendix C

The parameters for each area, appearing in the transformed system (62), are obtained as follows:

$$f_{11}=a_{44};\quad f_{22}=a_{33};$$

$$f_{23}=(a_{43}a_{94}-a_{94}a_{93}+a_{93}+a_{23})a_{32}+(a_{22}a_{92}+a_{32}a_{93}-a_{44}a_{92})(a_{22}-a_{23});$$

$$f_{24}=(a_{22}a_{92}+a_{32}a_{93}-a_{44}a_{92})a_{21}+a_{91}a_{21}(a_{11}-a_{33});$$

$$f_{25}=(a_{88}-a_{33})(a_{88}a_{98}-a_{44}a_{98});$$

$$f_{26}=(a_{88}a_{98}-a_{44}a_{98})a_{87}+(a_{77}-a_{33})(a_{77}a_{97}+a_{88}a_{98}-a_{44}a_{97});$$

$$f_{27}=(a_{77}a_{97}+a_{88}a_{98}-a_{44}a_{97})a_{76}+(a_{66}-a_{33})(a_{96}a_{66}+a_{97}a_{76}-a_{44}a_{96});$$

$$f_{28}=(a_{96}a_{66}+a_{97}a_{76}-a_{44}a_{96})a_{65}+(a_{55}-a_{33})a_{65}a_{96};$$

$$f_{33}=a_{22};\quad f_{34}=a_{21};\quad f_{44}=a_{11};$$

$$f_{55}=a_{88};\quad f_{56}=a_{87};\quad f_{66}=a_{77};$$

$$f_{67}=a_{78};\quad f_{77}=a_{66};\quad f_{78}=a_{65};$$

$$f_{88}=a_{55};\quad g_{21}=a_{92}a_{21}b_{14};$$

$$g_{22}=a_{65}a_{96}b_{52};\quad g_{41}=b_{14};$$

$$g_{52}=b_{52};\quad g_{42}=-\frac{g_{41}g_{22}}{g_{21}};$$

$$d_{21}=a_{92}a_{21}a_{19}+a_{65}a_{96}a_{59};$$

$$d_{41}=a_{19};\quad d_{81}=a_{59}.$$

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Vikram R. Saksena (M'82) was born in Hyderabad, India, on March 21, 1956. He received the B.Tech. degree in electrical engineering from the Indian Institute of Technology, Kharagpur, in 1978, where he was awarded the President of India Gold Medal for academic excellence, and the M.S. and Ph.D. degrees in electrical engineering from the University of Illinois at Urbana-Champaign, in 1980 and 1982, respectively.

From 1978 to 1982, he was a Research Assistant in the Decision and Control Laboratory of the Coordinated Science Laboratory, where he worked on problems in multimodeling of large scale systems. He is currently a member of the Trunk Traffic Engineering Department of Bell Laboratories at Holmdel, NJ, where he is involved in Packet Network Design problems.

Dr. Saksena is a member of Tau Beta Pi, Eta Kappa Nu, and Phi Kappa Phi.

Jose B. Cruz, Jr. (S'56–M'57–SM'61–F'68) was born in the Philippines on September 17, 1932. He received the B.S.E.E. degree (summa cum laude) from the University of the Philippines in 1952, and taught there for a year as an Instructor in Electrical Engineering. He was a Research Assistant at the Research Laboratory of Electronics of the Massachusetts Institute of Technology, Cambridge, MA, from 1954 to 1956, and in 1956 he received the S.M. degree in electrical engineering. He then joined the faculty of the

University of Illinois, Urbana, in 1956, as an Instructor. After completing the Ph.D. degree in electrical engineering at Illinois in 1959, he was promoted to Assistant Professor in 1959, Associate Professor in 1961, and Professor in 1965. In 1964 he was a Visiting Associate Professor at the University of California, Berkeley; in 1967 he was an Associate Member of the Center for Advanced Study at Illinois; and in the fall of 1973 he was a Visiting Professor at M.I.T. and at Harvard University, Cambridge, MA. He currently is Professor of Electrical Engineering and Research Professor in the Coordinated Science Laboratory, University of Illinois. He is also President of Dynamics Systems, a research and development company. His areas of research are modeling and control of systems with multiple decisionmakers, decentralized control of large-scale systems, sensitivity analysis, and stochastic control of systems with uncertain parameters. He is the coauthor of three books, editor and coauthor of two books, and he is author or coauthor of more than 140 papers.

Dr. Cruz has been active in the IEEE. He served as President for 1979 of the IEEE Control Systems Society, and previously served the Society in various capacities including Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and General Chairman of the 1975 IEEE Conference on Decision and Control. He served the IEEE Circuits and Systems Society as Associate Editor for Systems in 1962–1964. At the Institute level, he served as a member of the IEEE Fellow Committee, member of the IEEE Educational Activities Board, member of the IEEE Education Medal Committee, member of the IEEE Technical Activities Board, member of the IEEE United States Activities Board, and a member of the Board of Directors of IEEE as Director of Division I. Presently, he is the IEEE Vice-President for Technical Activities and Chairman of the IEEE Technical Activities Board.

For several years he was on selection panels for National Science Foundation Research initiation grants, and he served as NSF Consultant for Engineering Education in India through the Agency for International Development. He has served as a Consultant to the U.S. Army Frankford Arsenal, Indiana University, Bendix Corporation, Components Corporation, Central University of Venezuela, and several book publishers. He is the Chairman of the Theory Committee of the International Federation on Automatic Control Council. In 1963 he cofounded the Annual Allerton Conference on Circuit and System Theory, now called the Allerton Conference on Communication, Control, and Computing. He is an Associated Editor of the *Journal of the Franklin Institute*, and the *Journal of Optimization Theory and Applications*.

In 1972 he received the Curtis W. McGraw Research Award of the American Society for the Engineering Education, and in 1981 he received the Halliburton Engineering Education Leadership Award. In 1968 he was elected a Fellow of the IEEE for "significant contributions in circuit theory and the sensitivity analysis of control systems," and in 1980 was elected a member of the National Academy of Engineering for "contributions to the control of large-scale systems with multiple goals and to sensitivity analysis," He is a Founding Member of the Phil-American Academy of Science and Engineering. He has been elected to membership in Phi Kappa Phi, Sigma Xi, and Eta Kappa Nu. He is listed in *American Men and Women of Science*, *Who's Who in America*, *Who's Who in Engineering*, and *Who's Who in Technology Today*. He is a Registered Professional Engineer in Illinois. He is a member of the American Society for Engineering Education and the National Society of Professional Engineers.

William R. Perkins (S'55–M'61–F'72) received the A.B. degree (cum laude) in engineering and applied physics from Harvard University, Cambridge, MA, in 1956, and the M.S. and Ph.D. degrees in electrical engineering from Stanford University, Stanford, CA, in 1957 and 1961, respectively.

Since February 1961 he has been at the University of Illinois, Urbana, where he is currently Professor of Electrical Engineering and Research Professor in the Coordinated Science Laboratory. He is also Director of the Decision and Control Laboratory. In 1971–1972 he was an Associate of the Center for Advanced Study, University of Illinois.

Dr. Perkins is Vice President, Financial and Administrative Activities of the IEEE Control Systems Society. He has served on many committees of the IEEE Control Systems Society, including current membership on the Administrative Committee and recent service as an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He was General Chairman of the 1981 IEEE Conference on Decision and Control. He is a member of the IFAC Committee on Theory, and is Cochairman of the National Organizing Committee for the 4th IFAC/IFORS Conference on the Modeling and Control of National Economies.

Tamer Basar (S'71–M'73–SM'79–F'83) was born in Istanbul, Turkey, in 1946. He received the B.S.E.E. degree from Robert College, Istanbul, in 1969, and the M.S., M.Phil., and Ph.D. degrees from Yale University, New Haven, CT, in 1970, 1971, and 1972, respectively.

From March 1972 until August 1973 he was a Research Fellow at Harvard University; from August 1973 until December 1980 he was a Senior Research Scientist at Marmara Research Institute, Gebze, Kocaeli, Turkey, and was also affiliated with Bogazici University, Istanbul; and since January 1981 he has been on the faculty of the Electrical Engineering Department and the Coordinated Science Laboratory, University of Illinois. During 1978–1979 he spent a sabbatical year at Twente University of Technology, Enschede, The Netherlands. He is currently an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, of the IFAC Journal *Automatica*, and of *Large Scale Systems*. He is Chairman of the Technical Committee on "Large Scale Systems and Differential Games," and a member of the Administrative Committee of the IEEE Control Systems Society. He is the recipient of the 1976 TUBITAK (The Scientific and Technical Research Council of Turkey) Young Scientist Award in Applied Mathematics and of the 1979 Sedat Simavi Foundation Award in Mathematical Sciences. He has authored or coauthored more than sixty papers in the general areas of optimal control, dynamic games, stochastic control, estimation, stochastic processes, information theory, and mathematical economics, and is the coauthor of a recently published book on *Dynamic Noncooperative Game Theory* (Academic Press, 1982). Dr. Basar is a member of Sigma Xi, and a member of the New York Academy of Sciences.