Optimal Nonlinear Pricing for a Monopolistic Network Service Provider with Complete and Incomplete Information

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***Abstract*— In the communication network pricing literature, it is the linear pricing schemes that have been largely adopted as the means of controlling network usage or generating profits for network service providers. This paper extends the framework to nonlinear pricing and investigates optimal nonlinear pricing policy design for a monopolistic service provider. The problem is formulated as an incentive-design problem, and incentive (pricing) policies are obtained for a many-users regime, which enable the service provider to approach arbitrarily close to Pareto-optimal solutions. Under the assumption that the service provider knows the true user types, analytical and numerical results indicate a profit improvement exceeding 38% over linear pricing by the introduction of nonlinear pricing. We also consider the scenario where the service provider has incomplete information on user types. A comparative study of the results for complete information and incomplete information is carried out as well, with numerical results pointing to 25%–40% loss of profit by the service provider due to incompleteness of information on the user types.**

***Index Terms*— Nonlinear pricing, incomplete information, price discrimination, network externality, incentives.**

I. Introduction

IN THIS paper, we study the problem of optimal nonlinear pricing policy design for a monopolistic network service provider in the face of multiple user types, when he has complete or incomplete information on the user types. The goal of the service provider is to maximize his profit. [1] We had analyzed earlier in [1] a special single user case, and here we extend the analysis to the multiple user case, and especially to the asymptotic case with respect to the number of users, since our focus is on communication networks with a large number of users.

Most prior work on network pricing for a monopolistic network service provider has dealt with linear pricing. We call a pricing scheme *linear* if it is usage-independent, that is, the unit price to a given user is fixed, such that the charge to this user is a linear function of his usage. In this case, there is no price discrimination via quantity discounts. However, there may or may not exist price discrimination with respect to different users, depending on whether the service provider has complete or incomplete information of user types. For a service provider with incomplete information, he knows only the distribution of a user's type (which we assume to be identical and independent across different users), but not the true type of an individual user. In this case, the optimal linear pricing scheme should have no price discrimination with respect to different users. It is this type of pricing scheme that has been largely adopted in the pricing of service in communication networks, such as in [2], [3], [4], [5], [6], [7].[2] On the other hand, a service provider with complete information observes the users' true types, and thus can charge different users different unit prices. This kind of price discrimination is generally called third-degree price discrimination in economics (see [8, p. 439]). We call it differentiated pricing and have studied this pricing scheme in [9].

For linear pricing, the interaction between the service provider and the users was modeled as a hierarchical Stackelberg game in [2] and [9], where the service provider announces prices as the leader, and the users respond with usages as the followers. Also, the users play a noncooperative game, with the Nash equilibrium providing a natural solution concept.

Following the framework of [2] and [9], we extended the study to nonlinear pricing in [1], where the charge to a user is allowed to be a nonlinear function of the user's usage. In other words, a nonlinear pricing scheme can be usage-dependent, and in this case, a monopolistic service provider may have price discrimination via quantity discounts. Then, the service provider's pricing policy can be viewed as an *incentive policy*, and the underlying game becomes a *reverse Stackelberg game*. In this game, the service provider first needs to find out the action outcomes that jointly maximize the profit, which is the *team solution*. Then, he solves the *incentive-design problem* to find the optimal incentive policy that induces desired usages from the users such that the team solution is achieved. A more detailed description of the above problem can be found in [1]. Intuitively, compared with linear pricing, which can be seen as a special case of nonlinear pricing, the latter may further improve the service provider's profit.

Nonlinear pricing has been studied extensively in economics, generally under the assumption that the monopoly has incomplete information of user types [10], [11], [12]. Under

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[1] We assume that the cost for the service provider to manage and maintain the network is fixed and thus can be neglected when considering profit maximization.

[2] Among these works, some deal with multiple service providers ([5], [6]), and some allow more than one priority classes ([6], [7]).

this assumption, the monopoly can only provide price discrimination via quantity discounts, while the pricing policy for different users should be the same. This type of pricing scheme is called second-degree price discrimination [8, p. 439], or referred to as nonlinear pricing generally [8, p. 441]. Nonlinear pricing of information goods has been considered in [13]. As in [11], the modeling in [13] is such that each customer's utility function depends on the quantity of goods used by his own, but not on the ones used by others. In other words, there is no presence of network externality, where a good would become more valuable as more customers use it. On the other hand, in [14] and [15], nonlinear pricing for goods that display network externalities, also called positive network effects, has been studied. However, for communication networks to be studied here, a term of congestion cost is included in each user's utility function, such that the increase of one user's usage has a negative impact on the other users' utilities. Therefore, we deal in this paper with nonlinear pricing with the presence of *negative network effects*.

Besides nonlinear pricing under incomplete information, we consider the case where the monopolistic service provider has complete information of user types as well. Then, the service provider can have different pricing policies for different users according to their true types, in addition to providing price discrimination via quantity discounts for each policy. This is called first-degree price discrimination, or perfect price discrimination [8, p. 439], which can achieve Pareto efficiency.

In [1], we have illustrated optimal nonlinear pricing for a single user case. For both the complete information case and the incomplete information case, we have shown that there does not exist an incentive policy which guarantees that the team solution is achieved, i.e., the incentive-design problem is not *incentive controllable*. Rather, we can find an ϵ*-team optimal* incentive policy which enables the service provider to come arbitrarily close to the team solution. Therefore, the problem is ϵ*-incentive controllable*.

In this paper, we continue our study on optimal nonlinear pricing for the multiple user case. In sections 2 and 3, we analyze the complete information case and the incomplete information case, respectively, for which we formulate and solve the incentive-design problem. The paper concludes with a discussion on possible extensions and two appendices.

II. Complete Information

Suppose that a monopolistic network service provider charges users for usage of a communication network to maximize his profit. The set of users is denoted by $N := \{1, \cdots, n\}$. For User i of type w_i, his net utility is

$$F_{w_i}(x_i, \mathbf{x_{-i}}; r_i) := w_i \log(1 + x_i) - \frac{1}{n - x_i - x_{-i}} - r_i, \quad (1)$$

for $0 \leq x_i < n - x_{-i}$, where x_i is User i's usage of bandwidth (called flow here), $\mathbf{x_{-i}}$ is the set of all the other users' flows, $x_{-i} := \sum_{j=1}^n x_j - x_i$, and r_i is the charge to User i by the service provider. This structure is as in [2], where the utility function is logarithmic, and the second term is the congestion cost, which captures the delay in the framework of an $M/M/1$ queue. But here r_i is allowed to be a nonlinear function of x_i. While each user tries to maximize his net utility by choosing his flow (taking the pricing policy and the other users' flows as given), the service provider needs to design optimal pricing policies such that his profit, $\sum_{i \in N} r_i$, is maximized. Note that what differentiates one user from another is captured in the relative values of w_i's, and "complete information" refers to the situation where the service provider knows the true values of all the w_i's.

A. Incentive-Design Problem Formulation

With complete information, the service provider can charge users differently according to their true types. In order to obtain the optimal pricing policy, also called the incentive policy, he needs to first compute the team solution, which is the action outcome that maximizes his profit:

$$\{(x_i^t, r_i^t)\}_{i=1}^n = \arg \max_{\{x_i \geq 0, \sum_{j=1}^n x_j < n, r_i \geq 0\}_{i=1}^n} \sum_{j=1}^n r_j, \quad (2)$$

$$\text{s. t. } F_{w_i}(x_i, \mathbf{x_{-i}}; r_i) \geq F_{w_i}(0, \mathbf{x_{-i}}; 0), \quad i \in N. \quad (3)$$

Note that if a user chooses not to use the network service, then he pays nothing to the service provider, i.e., r_i must be 0 when $x_i = 0$. Thus, (3) guarantees that the users are not worse off than not participating, and can be seen as the individual rationality constraint.

Assume for the moment that the team solution exists and is unique. Then, the next step would be to design a pricing policy for each user, $r_i = \gamma_i(x_i)$, to solve the following incentive-design problem, under which individual users' utility maximizing responses lead to the team solution computed above, that is: for $i \in N$,

$$\arg \max_{0 \leq x_i < n - x_{-i}^t} F_{w_i}(x_i, \mathbf{x^t_{-i}}; \gamma_i(x_i)) = x_i^t, \quad (4)$$

$$\gamma_i(x_i^t) = r_i^t, \quad (5)$$

$$\gamma_i(0) = 0. \quad (6)$$

If there exists a solution to (4) to (6), which is then denoted by $\{\gamma_i^t\}_{i=1}^n$ [3], we say that the incentive-design problem is incentive controllable.

B. Team Solution

1) Optimization Problem Decomposition

Theorem 1: The optimization problem to obtain the team solution, given by (2) and (3), can be equivalently decomposed into the following two problems. First, the optimal flows solve

$$\{x_i^t\}_{i=1}^n = \arg \max_{\{x_i \geq 0, \sum_{j=1}^n x_j < n\}_{i=1}^n} F(x_1, \cdots, x_n), \quad (7)$$

where we define $Q(x_i, x_{-i}; w_i) := w_i \log(1 + x_i) - 1/(n - x_i - x_{-i}) + 1/(n - x_{-i})$ for $i \in N$, and $F(x_1, \cdots, x_n) := \sum_{j=1}^n Q(x_j, x_{-j}; w_j)$. Then, the optimal charges can be obtained from

$$r_i^t = Q(x_i^t, x_{-i}^t; w_i), \quad i \in N. \quad (8)$$

[3]Note that γ_i^t's depend on the values of w_i's. Thus, strictly to say, γ_i^t should be written as $\gamma_i^t(\{w_i\}_{i=1}^n)$. To save notation, we simply write it as γ_i^t, and note that the analysis to follow for the complete information game applies to every fixed $\{w_i\}_{i=1}^n$.

Proof: (2) and (3) can be equivalently written as

$$\{(x_i^t, r_i^t)\}_{i=1}^n = \arg \max_{\{x_i \geq 0, \sum_{j=1}^n x_j < n\}_{i=1}^n} \sum_{j=1}^n r_j, \quad \text{s. t. } 0 \leq r_i \leq Q(x_i, x_{-i}; w_i), \quad i \in N.$$

Obviously, to achieve the maximal total revenue, r_i must take the largest possible values on the right-hand side of the constraint. As a result, the above optimization problem can be equivalently decomposed into two problems: first, the optimal flows should solve (7) subject to $Q(x_i, x_{-i}; w_i) \geq 0$, $i \in N$; then, the optimal charges can be obtained from (8).

Next, we show that the constraints $Q(x_i, x_{-i}; w_i) \geq 0$, $i \in N$, are automatically satisfied by the optimal solution to (7). If this is not true, then for some i, $Q(x_i^t, x_{-i}^t; w_i) < 0$, which implies $x_i^t > 0$. Now let $x_i^* = 0$, and $x_j^* = x_j^t$ for $j \neq i$. Then, $r_i^* = 0 > r_i^t$. For $j \neq i$, $r_j^* = Q(x_j^*, x_{-j}^*; w_j) = Q(x_j^t, x_{-j}^t - x_i^t; w_j) \geq Q(x_j^t, x_{-j}^t; w_j) = r_j^t$. Therefore, the objective function F in (7) can be further improved by $\{x_j^*\}_{j=1}^n$, which contradicts the initial hypothesis that $\{x_j^t\}_{j=1}^n$ is the optimal solution. ■

2) Asymptotic Optimal Flows

We now study the case when n is very large. We do this for two reasons. First, an explicit form of the solution is hard to be obtained for general n. Second, this assumption is realistic for communication networks, which often have a large number of users.

First, we solve (7) to obtain the optimal flows. If there exists a positive solution to (7), then the first order condition must be satisfied: for $i \in N$,

$$F_{x_i} = \frac{w_i}{1+x_i} - \frac{n}{(n-\bar{x})^2} + \sum_{j \neq i} \frac{1}{(n-x_{-j})^2} = 0, \quad (9)$$

where $\bar{x} := \sum_{j=1}^n x_j$. To stress the dependence of $\bar{x}$ on n, which is taken as a parameter, we write it as $\bar{x}(n)$. Now, suppose that for some constants a and b, we have

$$\lim_{n \to \infty} \frac{n - \bar{x}(n)}{n^a} = b. \quad (10)$$

Then, the asymptotic solution $\{x_i^t(n)\}_{i=1}^n$ can be obtained by combining (9) and (10). Hereinafter, we let $w_{av}(n) := \sum_{j=1}^n w_j/n$, and assume that $w_{av}(n)$ and $\sum_{j=1}^n {w_j}^2/n$ converge to well-defined limits as $n \to \infty$.

From the discussion on possible values of a and b in Appendix , we know that there are only two possible cases: (i) $0 < a \leq \frac{1}{3}$ and $b > 0$, or (ii) $a = 0$ and $b > 0$. We first discuss (i), assuming that $\lim_{n\to\infty} \frac{x_i(n)}{n-\bar{x}(n)} = 0$ for $i \in N$ (we will show later that the solution in fact satisfies this assumption). Then, $\lim_{n\to\infty} (n - x_{-i}(n))^{-2} = \lim_{n\to\infty} (n-\bar{x}(n))^{-2}(1 + \frac{x_i(n)}{n-\bar{x}(n)})^{-2} = 0$. Thus, by (9), asymptotically, $\frac{w_i}{1+x_i(n)} \sim \frac{w_j}{1+x_j(n)} \sim \frac{n w_{av}(n)}{n+\bar{x}(n)}$ for $i, j \in N$, and hence[4]

$$x_i(n) \sim \frac{w_i}{w_{av}} \left(1 + \frac{\bar{x}(n)}{n}\right) - 1, \quad i \in N. \quad (11)$$

Substituting (10) and (11) back into (9), and using the Taylor series expansion, we get $b^{-2}n^{1-2a} - w_{av}/2 \sim \sum_{j=1}^n (bn^a + \frac{2w_j}{w_{av}} - 1)^{-2} = b^{-2}n^{1-2a} - 2b^{-3}n^{1-3a} + o(n^{1-3a})$. Obviously, this equation holds if and only if $1 - 3a = 0$ and $2b^{-3} = \frac{w_{av}}{2}$, i.e., $a = \frac{1}{3}$ and $b = 4^{\frac{1}{3}} {w_{av}}^{-\frac{1}{3}}$. In conclusion, for case (i), the unique positive solution that satisfies the first order condition is: for $i \in N$,

$$\bar{x}(n) \sim n - 4^{\frac{1}{3}} {w_{av}}^{-\frac{1}{3}} n^{\frac{1}{3}}, \quad (12)$$

$$x_i(n) \sim \frac{2w_i}{w_{av}} - 1 - 4^{\frac{1}{3}} w_i {w_{av}}^{-\frac{4}{3}} n^{-\frac{2}{3}}, \quad (13)$$

if and only if

$$w_i > \frac{w_{av}}{2}, \quad \forall\, i \in N. \quad (14)$$

Note that the above solution satisfies the assumption that $\lim_{n\to\infty} \frac{x_i(n)}{n-\bar{x}(n)} = 0$ for $i \in N$.

Next, we first check the local optimality of the solution given by (12) to (14), and then discuss its global optimality and compare it with the solution in case (ii). Note that for $i, j \in N$, $F_{x_i x_j} = -I(i=j)w_i/(1+x_i)^2 - 2n/(n-\bar{x})^3 + \sum_{k \neq i,\ k \neq j} 2/(n - x_{-k})^3$, where I is the indicator function. By (12) and (13), and using the Taylor series expansion, $F_{x_i x_j}(x_1(n), \cdots, x_n(n)) \sim -I(i=j){w_{av}}^2/(4w_i)$. Hence, $\nabla^2 F < 0$ and F achieves a strictly local maximum at the solution given by (12) to (14). To show that this solution achieves a global maximum, it suffices to show $\nabla^2 F < 0$ everywhere. It can be easily seen that this is true if the total flow is relatively small, say, $\bar{x}(n)$ is below $n - bn^{\frac{1}{3}}$, since $F_{x_i x_j}$, $j \neq i$, vanishes and $F_{x_i x_i}$ is dominated by the first negative term (we can reasonably assume that no individual flow is too large). Unfortunately, we may not be able to verify for the case with a large total flow that the Hessian matrix of F is negative definite globally. Thus, for case (ii), there could exist $a = 0$ and some $b > 0$ such that $\bar{x}(n) \sim n - b$ satisfies the first order condition (9). However, we can rewrite F as $F(x_1, \cdots, x_n) = \sum_{j=1}^n w_j \log(1 + x_j) - \sum_{j=1}^n x_j/[(n-\bar{x})(n-x_{-j})]$. Compared with the solution given by (12) to (14), if the total flow is increased such that $n - \bar{x}(n)$ is bounded, then the second term of F due to congestion decreases to $-\infty$, while the increase of the first log term is bounded and is relatively small. Therefore, we can expect F to decrease. Intuitively, the asymptotic solution given by (12) to (14) reaches a global maximum of F and thus solves the optimization problem (7).

3) Asymptotic Team Solution

Now we know that the optimal flows for the asymptotic team solution, $\bar{x}^t(n)$ and $x_i^t(n)$ for $i \in N$, are given by (12) and (13) subject to (14). Then, we can calculate the optimal charges by substituting (12) and (13) in (8) and obtain that for $i \in N$,

$$r_i^t(n) \sim w_i \log(2w_i/w_{av}) - (4^{\frac{1}{3}} w_i {w_{av}}^{-\frac{1}{3}} + 4^{-\frac{2}{3}} {w_{av}}^{\frac{2}{3}}) n^{-\frac{2}{3}}. \quad (15)$$

We are also interested in evaluating the total profit, $\bar{r} := \sum_{j=1}^n r_j$. Using the Taylor series expansion, $\bar{r}^t(n) = F(x_1^t(n), \cdots, x_n^t(n)) \sim \sum_{j=1}^n w_j \log(2w_j/w_{av}) - 3(4^{-\frac{2}{3}} {w_{av}}^{\frac{2}{3}}) n^{\frac{1}{3}} + {w_{av}}^{-1} \sum_{j=1}^n {w_j}^2/n - 3w_{av}/4$. Since $\sum_{j=1}^n 2w_j/w_{av} = 2n$, finally we have

$$\bar{r}^t(n) \sim \sum_{j=1}^n w_j \log\left(\frac{2w_j}{w_{av}}\right) \geq w_{av}(\log 2)n, \quad (16)$$

where the equality holds if and only if $w_i = w_{av}$ for $i \in N$.

[4]Henceforth, we suppress the dependence of $w_{av}(n)$ on n, and simply write it as w_{av}.

C. Solution of the Incentive-Design Problem

Given the team solution, the incentive-design problem is to find $\{\gamma_i\}_{i=1}^n$ such that (4) to (6) are satisfied. However, by (6), (5) and (8), $F_{w_i}(0, \mathbf{x}^t_{-\mathbf{i}}; \gamma_i(0)) = F_{w_i}(x_i^t, \mathbf{x}^t_{-\mathbf{i}}; \gamma_i(x_i^t)) = -1/(n - x^t_{-i})$. Therefore, it is always possible that the users may choose not to participate, and thus the problem is not incentive controllable. Rather, we can show that it is ϵ-incentive controllable.

Remember that we define $Q(x_i, x_{-i}; w_i) := w_i \log(1 + x_i) - 1/(n - x_i - x_{-i}) + 1/(n - x_{-i})$. Then, (4) can be expressed as

$$\gamma_i(x_i) \geq Q(x_i, x^t_{-i}; w_i), \quad 0 \leq x_i < n - x^t_{-i}. \tag{17}$$

Clearly, any $\{\gamma_i^t\}_{i=1}^n$ such that γ_i^t goes through $(0, 0)$ and (x_i^t, r_i^t) and falls above or on $Q(x_i, x^t_{-i}; w_i)$, for $i \in N$, solves the incentive-design problem given by (4) to (6). Next, we make a small "dip" around the team solution, that is, substitute (x_i^t, r_i^t) of γ_i^t with $(x_i^t, r_i^t - \epsilon_i)$, where ϵ_i is an arbitrarily small positive amount [5]. Then, $(x_i^t, r_i^t - \epsilon_i)$ becomes the unique utility maximizing point for User i and thus he must stick to this point. As a result, the service provider achieves an ϵ-team optimal solution, $\{\gamma_i^{t\epsilon}\}_{i=1}^n$.

Now we discuss some possible examples of $\{\gamma_i^t\}_{i=1}^n$. Note that $Q_{x_i} = w_i/(1 + x_i) - 1/(n - x_i - x_{-i})^2$, and $Q_{x_i x_i} = -w_i/(1 + x_i)^2 - 2/(n - x_i - x_{-i})^3 < 0$. Since Q is strictly concave, there does not exist a linear function γ_i such that γ_i goes through $(0, 0)$ and (x_i^t, r_i^t), and satisfies (17). This implies that the classical Stackelberg version of the pricing problem cannot admit a solution that is team optimal or ϵ-team optimal. On the other hand, if we can find some x_i^* such that $Q_{x_i}(x_i^*, x^t_{-i}; w_i) = 0$, then $Q(x_i, x^t_{-i}; w_i)$ reaches the unique global maximum $Q(x_i^*, x^t_{-i}; w_i)$ at x_i^*. So, we can let γ_i^t take the value of $Q(x_i^*, x^t_{-i}; w_i)$ everywhere except that $\gamma_i^t(0) = 0$ and $\gamma_i^t(x_i^t) = r_i^t$, and this γ_i^t satisfies (17).

Based on the above discussion, we can derive an ϵ-team optimal incentive policy for the asymptotic team solution. From (12) and (13), $Q_{x_i}(0, x^t_{-i}(n); w_i) \sim w_i > 0$ and $Q_{x_i}(x_i^t(n), x^t_{-i}(n); w_i) \sim w_{av}/2 > 0$. Furthermore, $\lim_{x_i \to n - x^t_{-i}(n)} Q_{x_i}(x_i, x^t_{-i}(n); w_i) = -\infty$. Therefore, there exists a unique $x_i^*(n) \in (x_i^t(n), n - x^t_{-i}(n))$ such that $Q_{x_i}(x_i^*(n), x^t_{-i}(n); w_i) = 0$. Solving this for $x_i^*(n)$, we get $x_i^*(n) \sim 4^{\frac{1}{3}} w_{av}{}^{-\frac{1}{3}} n^{\frac{1}{3}} - 2^{\frac{1}{3}} w_i{}^{-\frac{1}{2}} w_{av}{}^{-\frac{1}{6}} n^{\frac{1}{6}}$, and

$$\begin{aligned} &Q(x_i^*(n), x^t_{-i}(n); w_i) \\ &\quad \sim w_i \log(4^{\frac{1}{3}} w_{av}{}^{-\frac{1}{3}} n^{\frac{1}{3}} - 2^{\frac{1}{3}} w_i{}^{-\frac{1}{2}} w_{av}{}^{-\frac{1}{6}} n^{\frac{1}{6}}). \end{aligned}$$

In conclusion, in the large user population regime, one possible incentive policy for the service provider, $\{\gamma_i^{t\epsilon}\}_{i=1}^n$, leading to an ϵ-team optimal solution, is to let $\gamma_i^{t\epsilon}(0) = 0$, $\gamma_i^{t\epsilon}(x_i^t(n)) = r_i^t(n) - \epsilon_i$, and $\gamma_i^{t\epsilon}$ take very large values at all other points, which must exceed $Q(x_i^*(n), x^t_{-i}(n); w_i)$ as obtained above, for $i \in N$.

D. Asymptotic Team Solution vs Stackelberg Game Solution

With optimal nonlinear pricing, the service provider almost achieves the asymptotic team solution, given by (12) to (16). Now we want to compare the total profit, given by (16), with the optimal total profit attainable by linear pricing, to see by how much it is increased by nonlinear pricing. Note that for this complete information case, the service provider can charge users according to their true types, and the optimal asymptotic Stackelberg game solution for differentiated (linear) pricing has been obtained in [9], where the total profit is

$$\bar{r}^d(n) \sim w_{av} n - \frac{1}{2}(v_{av}^{\frac{1}{2}})^2 n, \tag{18}$$

where $v_{av}^{\frac{1}{2}} := \sum_{j=1}^n w_j^{\frac{1}{2}}/n$. It can be shown that $\bar{r}^t(n) > \bar{r}^d(n)$, and when $w_i = w_{av}$, $\forall i \in N$, $\bar{r}^t(n)/(nw_{av}) - \bar{r}^d(n)/(nw_{av})$ reaches the minimum $0.6931 - 0.5 = 0.1931$ (the proof is given in Appendix). We can see that in the case of all the users having the same type, in the high population regime, nonlinear pricing leads to a profit improvement of $\bar{r}^t(n)/\bar{r}^d(n) - 1 \sim 38.62\%$ as compared with the best linear pricing scheme. In fact, from the following numerical results, we can see that for other cases where the users have different types, the more diverse the types are, the more advantageous is nonlinear pricing to the service provider, and a higher profit improvement can be expected.

Example 2.1: We first normalize the user types as $p_i := w_i/w_{av}$, $i \in N$. Note that $\sum_{j=1}^n p_j/n = 1$, and by (14), $p_i > \frac{1}{2}$, $\forall i \in N$. For different values of p_i's, we compute the normalized asymptotic optimal profits for nonlinear pricing and linear pricing, $\bar{r}^t(n)/(nw_{av})$ and $\bar{r}^d(n)/(nw_{av})$, respectively, and then evaluate the profit improvement, $\bar{r}^t(n)/\bar{r}^d(n) - 1$. The results are shown in Table I, which validate the above conclusion.

TABLE I
RESULTS FOR EXAMPLE 2.1

users : p_i's	$\frac{\bar{r}^t(n)}{nw_{av}}$	$\frac{\bar{r}^d(n)}{nw_{av}}$	$\frac{\bar{r}^t(n)}{\bar{r}^d(n)} - 1$
n:1	~ 0.6931	~ 0.5	$\sim 38.62\%$
$\frac{n}{3}$:$\frac{5}{4}$, $\frac{n}{3}$:1, $\frac{n}{3}$:$\frac{3}{4}$	~ 0.7142	~ 0.5053	$\sim 41.34\%$
$\frac{n}{2}$:$\frac{5}{4}$, $\frac{n}{2}$:$\frac{3}{4}$	~ 0.7247	~ 0.5079	$\sim 42.69\%$
$\frac{n}{3}$:$\frac{4}{3}$, $\frac{n}{3}$:1, $\frac{n}{3}$:$\frac{2}{3}$	~ 0.7309	~ 0.5096	$\sim 43.43\%$
$\frac{n}{2}$:$\frac{4}{3}$, $\frac{n}{2}$:$\frac{2}{3}$	~ 0.7498	~ 0.5143	$\sim 45.79\%$
$\frac{n}{3}$:$\uparrow \frac{3}{2}$, $\frac{n}{3}$:1, $\frac{n}{3}$:$\downarrow \frac{1}{2}$	$\uparrow 0.7804$	$\uparrow 0.5225$	$\uparrow 49.36\%$
$\frac{n}{2}$:$\uparrow \frac{3}{2}$, $\frac{n}{2}$:$\downarrow \frac{1}{2}$	$\uparrow 0.8240$	$\uparrow 0.5335$	$\uparrow 54.45\%$
1:$\uparrow \frac{n+1}{2}$, $n-1$:$\downarrow \frac{1}{2}$	$\uparrow \infty$	$\uparrow 0.75$	$\uparrow \infty$

Also, notice that the total flow and individual flows for the asymptotic Stackelberg game solution as given in [9] are $\bar{x}^d(n) \sim n - 2(v_{av}^{\frac{1}{2}})^{-\frac{2}{3}} n^{\frac{1}{3}}$, and $x_i^d(n) \sim 2w_i^{\frac{1}{2}}/v_{av}^{\frac{1}{2}} - 1 - 2w_i^{\frac{1}{2}}(v_{av}^{\frac{1}{2}})^{-\frac{5}{3}} n^{-\frac{2}{3}}$ for $i \in N$. Comparing this with (12) and (13), and since it can be easily verified that $(v_{av}^{\frac{1}{2}})^2 \leq w_{av}$, we have $\bar{x}^t(n) > \bar{x}^d(n)$, $x_i^t(n) > x_i^d(n)$ if $w_i \geq (w_{av}/v_{av}^{\frac{1}{2}})^2$, and $x_i^t(n) < x_i^d(n)$ if $w_i < (w_{av}/v_{av}^{\frac{1}{2}})^2$. Therefore, in the asymptotic case, the optimal nonlinear pricing scheme, compared with the optimal linear pricing scheme, make the total flow and the flows of those users with relatively large w_i's increase, but the flows of users with relatively small w_i's decrease.

[5] This method can be seen in Example 7.4 of [16] and is also applied in [1].

III. Incomplete Information

Next, we study the incomplete information game for the multiple user case, with User i's utility function as given in (1). Suppose that all the users have identically and independently distributed types, each belonging to a set of m possible types, i.e., $w_i = w^l$ with probability $q_l \in (0,1)$, for $l \in M := \{1, \cdots, m\}$, where $\sum_{l=1}^m q_l = 1$. We thus deal here with a discrete distribution. Let us assume, without any loss of generality, that $w^1 > \cdots > w^m > 0$. With incomplete information, the service provider only knows this distribution, but not the users' true types. Furthermore, we assume that each user's true type is private information to himself such that the other users only know the distribution too.

A. Incentive-Design Problem Formulation

With incomplete information, the service provider's objective is to maximize the expected total profit. Also, he cannot have price discrimination for different users according to their true types. Thus, he should have the same pricing policy, γ, for any user. As a result, the team solution is the same for all the users, which consists of m optimal flow-charge pairs, one pair for each possible user type [6]:

$$\{(x^{lt}, r^{lt})\}_{l=1}^m = \arg \max_{\{0 \le x^l < 1, r^l \ge 0\}_{l=1}^m} n \sum_{l=1}^m q_l r^l, \quad (19)$$

$$\text{s. t. } \mathcal{F}(w^l, x^l, r^l; \{x^{l_i}\}) \ge \mathcal{F}(w^l, 0, 0; \{x^{l_i}\}), \quad (20)$$

$$\mathcal{F}(w^l, x^l, r^l; \{x^{l_i}\}) \ge \mathcal{F}(w^l, x^k, r^k; \{x^{l_i}\}), \quad (21)$$

for $l, k \in M$ and $l \neq k$, where we denote for convenience

$$\mathcal{F}(w, x, r; \{y^{l_i}\}) := \sum_{\{l_i\}_{i=1}^{n-1}} \{(\prod_{i=1}^{n-1} q_{l_i}) F_w(x, \{y^{l_i}\}_{i=1}^{n-1}; r)\}.$$

Here, we require $x^l < 1$ for $l \in M$, because in any case the total flow cannot exceed the total capacity n for the congestion cost in (1) to be well defined. (20) guarantees that the users are not worse off than not participating, and (21) is the incentive compatibility constraint such that a user with a certain type cannot benefit by pretending to be of other types.

Assume that the team solution exists. Then, the incentive-design problem is to find γ such that for $l \in M$,

$$\arg \max_{x \ge 0} \mathcal{F}(w^l, x, \gamma(x); \{x^{l_i t}\}) = x^{lt}, \quad (22)$$

$$\gamma(x^{lt}) = r^{lt}, \quad (23)$$

$$\gamma(0) = 0. \quad (24)$$

If there exists a solution to (22) to (24), which we then denote by γ^t, we say that the incentive-design problem is incentive controllable.

B. Team Solution

1) Optimization Problem Decomposition

As for the complete information game, here the optimization problem for the team solution, given by (19) to (21), can also be decomposed such that the optimal flows are obtained first, followed by the optimal charges accordingly.

[6]To save notation, we write $\sum_{\{l_i\}_{i=1}^{n-1}=(1,\cdots,1)}^{(m,\cdots,m)}$ simply as $\sum_{\{l_i\}_{i=1}^{n-1}}$.

Lemma 1: $x^{1t} \ge \cdots \ge x^{mt}$.

Proof: Fix $l, k \in M$ such that $l < k$. By assumption, $w^l > w^k$. From (21), $\mathcal{F}(w^l, x^l, r^l; \{x^{l_i}\}) \ge \mathcal{F}(w^l, x^k, r^k; \{x^{l_i}\})$, and $\mathcal{F}(w^k, x^k, r^k; \{x^{l_i}\}) \ge \mathcal{F}(w^k, x^l, r^l; \{x^{l_i}\})$. By summing up these two inequalities and canceling and combining terms as possible, we finally obtain that $(w^l - w^k)[\log(1+x^l) - \log(1+x^k)] \ge 0$, which implies $x^l \ge x^k$, since $w^l > w^k$. ∎

Lemma 2: Suppose (21) is satisfied. If (20) holds for $l = m$, then it automatically holds for $l < m$.

Proof: Fix $l \in M$ and $k \in M$ such that $l < k$. Then, $w^l > w^k$ implies $w^l \log(1+x^k) \ge w^k \log(1+x^k)$, or equivalently, $F_{w^l}(x^k, \{x^{l_i}\}_{i=1}^{n-1}; r^k) \ge F_{w^k}(x^k, \{x^{l_i}\}_{i=1}^{n-1}; r^k)$. Combined with (21), this implies $\mathcal{F}(w^l, x^l, r^l; \{x^{l_i}\}) \ge \mathcal{F}(w^k, x^k, r^k; \{x^{l_i}\})$. Thus, the left-hand side of (20) is the smallest with $l = m$, while the right-hand side remains the same for $l \in M$. So we only need (20) to hold for $l = m$. ∎

Proposition 1: The optimization problem for the team solution, (19) to (21), is equivalent to the following optimization problem (P1): $\max n \sum_{l=1}^m q_l r^l$ subject to

$$1 > x^1 \ge \cdots \ge x^m \ge 0; \quad (25)$$

$$r^l \ge 0, \quad l \in M; \quad (26)$$

$$r^m \le w^m \log(1+x^m) - D(x^m; \{x^{l_i}\}) + D(0; \{x^{l_i}\}); \quad (27)$$

$$r^l - r^{l+1} + D(x^l; \{x^{l_i}\}) - D(x^{l+1}; \{x^{l_i}\}) \in \left[w^{l+1} \log \frac{1+x^l}{1+x^{l+1}}, w^l \log \frac{1+x^l}{1+x^{l+1}}\right], \quad l < m, \quad (28)$$

where we denote

$$D(x; \{x^{l_i}\}) := \sum_{\{l_i\}_{i=1}^{n-1}} (\prod_{i=1}^{n-1} q_{l_i}) \frac{1}{n - x - \sum_{i=1}^{n-1} x^{l_i}}.$$

Proof: (25) and (26) come directly from (19) and Lemma 1. By Lemma 2, we can equivalently write (20) as (27). Next, we show that (21) can be equivalently written as (28). From (21), we know that for $l, k \in M$ and $l \neq k$,

$$r^l - r^k \le w^l \log \frac{1+x^l}{1+x^k} - D(x^l; \{x^{l_i}\}) + D(x^k; \{x^{l_i}\}). \quad (29)$$

Label (29) for specific l and k as $(29_{l,k})$. Next, fix some l, k, $h \in M$ such that $l < k < h$. Adding $(29_{l,k})$ to $(29_{k,h})$, we get $r^l - r^h \le w^l \log[(1+x^l)/(1+x^k)] + w^k \log[(1+x^k)/(1+x^h)] - D(x^l; \{x^{l_i}\}) + D(x^h; \{x^{l_i}\}) \le w^l \log[(1+x^l)/(1+x^h)] - D(x^l; \{x^{l_i}\}) + D(x^h; \{x^{l_i}\})$. The last inequality comes from the facts that $w^l > w^k$ and $x^k \ge x^h$. Thus, $(29_{l,k})$ and $(29_{k,h})$ imply $(29_{l,h})$. Similarly, $(29_{h,k})$ and $(29_{k,l})$ imply $(29_{h,l})$. Therefore, (21) can be finally reduced to (28). Note that $w^l > w^{l+1}$ and $x^l \ge x^{l+1}$ guarantee that the upper bound is no less than the lower bound for $r^l - r^{l+1}$ in (28). ∎

Proposition 2: The optimization problem P1 in Proposition 1 is equivalent to the following optimization problem (P2):

$\max n \sum_{l=1}^{m} q_l r^l$ subject to:

$$1 > x^1 \geq \cdots \geq x^m \geq 0; \tag{30}$$

$$r^m = w^m \log(1+x^m) - D(x^m;\{x^{l_i}\}) + D(0;\{x^{l_i}\}); \tag{31}$$

$$r^l = w^l \log(1+x^l) - \sum_{k=l}^{m-1}(w^k - w^{k+1})\log(1+x^{k+1}) - D(x^l;\{x^{l_i}\}) + D(0;\{x^{l_i}\}), \quad l < m. \tag{32}$$

Proof: First, define another optimization problem (P2.1), which is the same as P2, except that the equality in (31) is replaced by "$\leq$" to become (31.1) and the equality in (32) is replaced by "$\leq$" to become (32.1).

Obviously, if $\{(x^l, r^l)\}_{l=1}^m$ is feasible for P2, i.e., satisfies the optimization constraints (30) to (32), it must also satisfy (31.1) and (32.1) and thus is feasible for P2.1. So, the maximum of P2 cannot exceed the maximum of P2.1. On the other hand, it can be easily seen that the optimal solution to P2.1 must make r^l's as large as possible, i.e., equalities hold for (31.1) and (32.1). Hence, it is also feasible for P2, which implies that the maximum of P2 is no less than the maximum of P2.1. Therefore, the maxima of P2 and P2.1 are the same and the two problems are equivalent.

Next, we show that P1 is equivalent to P2 and P2.1. Note that (31.1) is just (27) and (32.1) can be easily deduced from (27) and (28) inductively. So, any feasible solution for P1 is also feasible for P2.1. As a result, the maximum of P1 cannot exceed the maximum of P2.1 and P2.

For the reverse direction, obviously (31) and (32) imply (27) and (28), with equalities for (27) and for the upper bounds in (28). Now we only need to show that the optimal solution to P2 should also satisfy (26). This can be proved by contradiction.

Suppose that $\{(x^{lt}, r^{lt})\}_{l=1}^m$ is optimal for P2. If $r^{mt} < 0$, then $x^{mt} > 0$, since otherwise $x^{mt} = 0$ implies $r^{mt} = 0$. Let $x^{m*} = 0$ and $x^{l*} = x^{lt}$ for $l \in \{1, \cdots, m-1\}$. Then, $\sum_{i=1}^{n-1} x^{l_i *} \leq \sum_{i=1}^{n-1} x^{l_i t}$. From (31) and (32), we have $r^{m*} = 0 > r^{mt}$, and $r^{l*} = w^l \log(1+x^{lt}) - \sum_{k=l}^{m-2}(w^k - w^{k+1})\log(1+x^{(k+1)t}) - D(x^{lt};\{x^{l_i *}\}) + D(0;\{x^{l_i *}\}) > r^{lt}$, for $l \in \{1, \cdots, m-1\}$. This contradicts the assumption that $\{(x^{lt}, r^{lt})\}_{l=1}^m$ is optimal. Thus, we must have $r^{mt} \geq 0$.

Now if there exists some $l \in \{1, \cdots, m-1\}$ such that $r^{lt} < 0$ and $r^{kt} \geq 0$ for $k \in \{l+1, \cdots, m\}$, then $r^{lt} < 0 \leq r^{(l+1)t}$ implies $x^{lt} > x^{(l+1)t}$, since otherwise $x^{lt} = x^{(l+1)t}$ implies $r^{lt} = r^{(l+1)t}$. Let $x^{l*} = x^{(l+1)t}$ and $x^{h*} = x^{ht}$ for $h \in M$ and $h \neq l$. So, $x^{l*} = x^{(l+1)*}$ and $\sum_{i=1}^{n-1} x^{l_i *} \leq \sum_{i=1}^{n-1} x^{l_i t}$. Again, from (31) and (32), $r^{m*} \geq r^{mt}$, $r^{h*} \geq r^{ht}$, for $h \in \{l+1, \cdots, m-1\}$, $r^{l*} = r^{(l+1)*} \geq r^{(l+1)t} \geq 0 > r^{lt}$, and $r^{h*} = w^h \log(1+x^{ht}) - \sum_{k \neq l-1, k=h}^{m-1}(w^k - w^{k+1})\log(1+x^{(k+1)t}) - (w^{l-1} - w^l)\log(1+x^{(l+1)t}) - D(x^{ht};\{x^{l_i *}\}) + D(0;\{x^{l_i *}\}) > r^{ht}$ for $h \in \{1, \cdots, l-1\}$. This contradicts the assumption that $\{(x^{lt}, r^{lt})\}_{l=1}^m$ is optimal. Therefore, we must have $r^{lt} \geq 0$ for $l \in M$ and (26) is satisfied.

In conclusion, the optimal solution to P2 is also feasible for P1 and thus the maximum of P1 is no less than the maximum of P2. So, P1 and P2 have the same maximum and they are equivalent. ■

Finally, we have the following decomposition result, which follows directly from Proposition 2:

Theorem 2: The optimization problem to obtain the team solution, given by (19) to (21), is equivalent to

$$\{x^{lt}\}_{l=1}^m = \arg \max_{1 > x^1 \geq \cdots \geq x^m \geq 0} nG(x^1, \cdots, x^m), \tag{33}$$

where $G(x^1, \cdots, x^m) := H(x^1, \cdots, x^m) - \sum_{l=1}^m q_l D(x^l;\{x^{l_i}\}) + D(0;\{x^{l_i}\})$. Note that we denote $H(x^1, \cdots, x^m) := q_1 w^1 \log(1+x^1) + \sum_{l=2}^m v^l \log(1+x^l)$, where $v^l := (\sum_{k=1}^l q_k) w^l - (\sum_{k=1}^{l-1} q_k) w^{l-1}$ for $l \in \{2, \cdots, m\}$. Then, the optimal charges, r^{mt} and r^{lt} for $l < m$, can be calculated from the optimal flows, $\{x^{lt}\}_{l=1}^m$, by (31) and (32), respectively.

2) Near-Optimal Asymptotic Team Solution

Now we first solve (33) for the optimal flows. However, even for the asymptotic case, it is hard to obtain the solution explicitly. So, we obtain a near-optimal solution as follows.

Note that $H \geq G$. Let $\{x^{lh}\}_{l=1}^m$ maximize $nH(x^1, \cdots, x^m)$ subject to $1 \geq x^1 \geq \cdots \geq x^m \geq 0$. This can be easily solved inductively for $l \in M$ from 1 to m. First, it is obvious that $x^{1h} = 1$. Then, for $l = 2$, if $v^2 > 0$, then $x^{2h} = 1$; otherwise, we must have $x^{2h} = x^{3h}$ and need to combine the second term with the third one to get a new $v^3 = (\sum_{k=1}^3 q_k) w^3 - q_1 w^1$. Next, for $l = 3$, if $v^3 > 0$, then $x^{3h} = 1$; otherwise, $x^{3h} = x^{4h}$ and we need to revise v^4 by adding v^3. Similarly, we can proceed, until $l = m$: if $v^m > 0$, then $x^{mh} = 1$; otherwise, $x^{mh} = 0$. Finally, $l_h = \arg\max_{k \in M} \sum_{l=1}^k q_l w^k$, $x^{lh} = 1$ for $l \in M_H := \{1, \cdots, l_h\}$, and $x^{lh} = 0$ for $l \in M_L := \{l_h + 1, \cdots, m\}$. Write the corresponding maximum of the problem as nH_{max}, which must be greater than the maximum of (33), nG_{max}.

Let $\tilde{x}^{lt} = 1 - \delta$ for $l \in M_H$, and $\tilde{x}^{lt} = 0$ for $l \in M_L$, where $\delta = an^{-b}$ for some $a > 0$ and $0 < b < 1$ (we will show in a while that $b = \frac{2}{3}$ may be the best choice). Then, we can easily see that $n\tilde{H} = nH(\tilde{x}^{1t}, \cdots, \tilde{x}^{mt}) = nH_{max} - n(\sum_{k=1}^{l_h} q_k) w^{l_h} \log[2/(2-\delta)]$. Also, since $\tilde{x}^{lt} \leq 1 - \delta$ for all $l \in M$, $n\tilde{G} = nG(\tilde{x}^{1t}, \cdots, \tilde{x}^{mt}) = n\tilde{H} - n\sum_{l=1}^m q_l D(\tilde{x}^{lt};\{\tilde{x}^{l_i t}\}) + nD(0;\{\tilde{x}^{l_i t}\}) \geq nH_{max} - n(\sum_{k=1}^{l_h} q_k) w^{l_h} \log[2/(2-\delta)] - n(1-\delta)/[n\delta(n\delta + 1 - \delta)]$. Using the Taylor series expansion and substituting an^{-b} for δ, we get $n\log[2/(2-\delta)] \sim an^{1-b}/2$ and $n(1-\delta)/[n\delta(n\delta + 1 - \delta)] \sim a^{-2} n^{2b-1}$, both not comparable with nH_{max} for large n with $0 < b < 1$ (the best choice may be $b = \frac{2}{3}$ such that $1 - b = 2b - 1 = \frac{1}{3}$). Thus, asymptotically, $nH_{max}(n) \geq nG_{max}(n) \geq n\tilde{G}(n) \geq nH_{max}(n)$, which implies that $n\tilde{G}(n) \sim nH_{max}(n) \sim n\tilde{H}(n) \sim nG_{max}(n)$. Thus, we have shown that $\{\tilde{x}^{lt}\}_{l=1}^m$ obtains a near-optimal value of (33), which converges to the optimum as the number of users gets large.

Next, we substitute $\{\tilde{x}^{lt}\}_{l=1}^m$ into (31) and (32) to obtain the near-optimal charges. Actually, we can verify that as $n \to \infty$, the last two terms of (31) or (32) vanish. Finally, we obtain a near-optimal team solution as follows:

$$l \in M_H: \quad \tilde{x}^{lt} = 1 - \delta, \quad \tilde{r}^{lt} = w^{l_h} \log(2 - \delta); \tag{34}$$

$$l \in M_L: \quad \tilde{x}^{lt} = 0, \quad \tilde{r}^{lt} = 0. \tag{35}$$

C. Near-Optimal Asymptotic Solution of the Incentive-Design Problem

Given the near-optimal asymptotic team solution as obtained in (34) and (35), the incentive-design problem is to find γ such that (22) to (24) are satisfied. By (23) and (24), $\gamma(0) = 0$ and $\gamma(1-\delta) = w^{l_h}\log(2-\delta)$. However, for $l \in M_H$, $\mathcal{F}(w^l, 0, \gamma(0); \{\tilde{x}^{l_i t}\}) \sim 0$, and $\mathcal{F}(w^l, 1-\delta, \gamma(1-\delta); \{\tilde{x}^{l_i t}\}) \sim (w^l - w^{l_h})\log(2-\delta)$. Especially, for a user with the type l_h, both equal 0. So, he may not stick to the flow $1-\delta$, and the problem is not incentive controllable.

Next, we show that the problem is ϵ-incentive controllable by obtaining an ϵ-team optimal incentive policy, $\tilde{\gamma}^{t\epsilon}$. First, let $\tilde{\gamma}^{t\epsilon}(0) = 0$ and $\tilde{\gamma}^{t\epsilon}(1-\delta) = w^{l_h}\log(2-\delta) - \epsilon$, where ϵ is some arbitrarily small positive number. Then, for $l \in M_H$, $\mathcal{F}(w^l, 1-\delta, \tilde{\gamma}^{t\epsilon}(1-\delta); \{\tilde{x}^{l_i t}\}) \sim (w^l - w^{l_h})\log(2-\delta) + \epsilon > 0$. Furthermore, for $\tilde{x}^{lt}$ to be the unique solution to (22), we need: for $l \in M_H$, $\mathcal{F}(w^l, 1-\delta, \tilde{\gamma}^{t\epsilon}(1-\delta); \{\tilde{x}^{l_i t}\}) > \mathcal{F}(w^l, x, \tilde{\gamma}^{t\epsilon}(x); \{\tilde{x}^{l_i t}\})$ for $0 < x < 1$ and $x \neq 1-\delta$; for $l \in M_L$, $\mathcal{F}(w^l, 0, 0; \{\tilde{x}^{l_i t}\}) > \mathcal{F}(w^l, x, \tilde{\gamma}^{t\epsilon}(x); \{\tilde{x}^{l_i t}\})$ for $0 < x < 1$. Since $w^1 > \cdots > w^m > 0$, the above requirements can be further simplified as: for $0 < x < 1-\delta$, $\tilde{\gamma}^{t\epsilon}(x) > w^{l_h}\log(1+x) - \epsilon$ and $\tilde{\gamma}^{t\epsilon}(x) > w^{l_h+1}\log(1+x)$; for $1-\delta < x < 1$, $\tilde{\gamma}^{t\epsilon}(x) > w^1\log(1+x) - (w^1 - w^{l_h})\log(2-\delta) - \epsilon$ and $\tilde{\gamma}^{t\epsilon}(x) > w^{l_h+1}\log(1+x)$; and $\epsilon < (w^{l_h} - w^{l_h+1})\log(2-\delta)$. In conclusion, $\tilde{\gamma}^{t\epsilon}$ is an ϵ-team optimal incentive policy that almost achieves the near-optimal asymptotic team solution, if $\tilde{\gamma}^{t\epsilon}(0) = 0$, $\tilde{\gamma}^{t\epsilon}(1-\delta) = w^{l_h}\log(2-\delta) - \epsilon$, and the above requirements are satisfied (for instance, we can let $\tilde{\gamma}^{t\epsilon}(x) = w^1\log(1+x)$ for $0 < x < 1$ and $x \neq 1-\delta$).

D. Complete Information vs Incomplete Information

Next, we compare the near-optimal asymptotic team solution under incomplete information, given by (34) and (35), with the asymptotic team solution under complete information, given by (12) to (16). Specifically, we are interested in studying how the service provider's profits are compared in the two cases. For incomplete information, the near-optimal asymptotic expected total profit is

$$n\sum_{l=1}^{m} q_l \tilde{r}^{lt} = n\tilde{H}(n) \sim n\sum_{l=1}^{l_h} q_l w^{l_h}\log 2, \qquad (36)$$

where $l_h = \arg\max_{k\in M}\sum_{l=1}^{k} q_l w^k$. Asymptotically, we can assume that the true realization of the user types is consistent with the distribution, i.e., nq_l users have the type w^l, for $l \in M$. In that case, (36) just gives the total profit under incomplete information. Also under this assumption, by (16) and (14), the total profit for the complete information case is $\bar{r}^t(n) \sim n\sum_{l=1}^{l_c} q_l w^l \log(2w^l/w_{av}^{l_c})$, where $w_{av}^{l_c} := \sum_{l=1}^{l_c} q_l w^l / \sum_{l=1}^{l_c} q_l$, and l_c is the maximum number in M such that $w^{l_c} > w_{av}^{l_c}/2$. Intuitively, we know that the optimal total profit for the service provider with complete information must be larger than that with incomplete information, though the analytical proof does not seem to be within reach. In the following, we provide some numerical results to evaluate the profit loss due to incomplete information, which are shown in Table II. Generally, the more diverse the user types are, the higher is the difference between the profits of the service provider under complete and incomplete information.

TABLE II

PROFIT LOSS DUE TO INCOMPLETE INFORMATION

m	q_l's	w^l's	$\tilde{H}(n) \sim$	$\frac{\bar{r}^t(n)}{n} \sim$	profit loss
1	1	w_{av}	$w_{av}\log 2$	$w_{av}\log 2$	0
3	$\frac{1}{3}$	$\frac{5}{4},\ 1,\ \frac{3}{4}$	0.5199	0.7142	27.21%
2	$\frac{1}{2}$	$\frac{5}{4},\ \frac{3}{4}$	0.5199	0.7247	28.26%
3	$\frac{1}{3}$	$\frac{4}{3},\ 1,\ \frac{2}{3}$	0.4621	0.7309	36.78%
2	$\frac{1}{2}$	$\frac{4}{3},\ \frac{2}{3}$	0.4621	0.7498	38.37%
3	$\frac{1}{3}$	$\uparrow\frac{3}{2},\ 1,\ \downarrow\frac{1}{2}$	0.4621	$\uparrow$ 0.7804	$\uparrow$ 40.79%
2	$\frac{1}{2}$	$\uparrow\frac{3}{2},\ \downarrow\frac{1}{2}$	$\uparrow$ 0.5199	$\uparrow$ 0.8240	$\uparrow$ 36.91%

IV. EXTENSIONS

Results of this paper can be extended in several directions. First, for the incomplete information game, it would be of interest to compare the results obtained here with the classical Stackelberg game solution for linear pricing, and evaluate the gain from adoption of nonlinear pricing. Second, it would be useful to extend the analysis to general utility functions for the users, replacing the special utility functions (1) adopted in this paper.

APPENDIX

Obviously, since $\bar{x}(n) < n$, we must have either $a = 1$ and $0 < b \leq 1$, or $a < 1$ and $b > 0$. We can further restrict this set of possible values of a and b.

Proposition 3: $a \geq 0$.

Proof: (9) is equivalent to $w_i/(1+x_i) = (n-\bar{x})^{-2} + A_i(x_1,\cdots,x_n)$, where $A_i(x_1,\cdots,x_n) := \sum_{j\neq i} x_j(n-\bar{x}+n-x_{-j})(n-\bar{x})^{-2}(n-x_{-j})^{-2} > 0$, for $i \in N$. Thus, $w_i > w_i/(1+x_i(n)) > (n-\bar{x}(n))^{-2}$. If there exist some $a < 0$ and $b > 0$ such that (10) holds, then $\lim_{n\to\infty}(n-\bar{x}(n))^{-2} = \infty$, which is impossible. Hence, we must have $a \geq 0$. ■

Proposition 4: $a < 1$ (i.e., $\lim_{n\to\infty}\frac{\bar{x}(n)}{n} = 1$).

Proof: Suppose that for $a = 1$ and some $0 < b \leq 1$, (10) holds. Then, $\lim_{n\to\infty}(n-\bar{x}(n))^{-2} = \lim_{n\to\infty} 2n(n-\bar{x}(n))^{-3} = 0$. Since $0 < A_i(x_1(n),\cdots,x_2(n)) < 2\bar{x}(n)(n-\bar{x}(n))^{-3} < 2n(n-\bar{x}(n))^{-3}$ for $i \in N$, $\lim_{n\to\infty} w_i/(1+x_i(n)) = \lim_{n\to\infty} A_i(x_1(n),\cdots,x_n(n)) = 0$, and so we have $\lim_{n\to\infty} x_i(n) = \infty > 1$. Consequently, $\lim_{n\to\infty}\bar{x}(n) > n$, which is impossible since $\bar{x}(n) < n$. Therefore, a cannot be 1. ■

Proposition 5: If there exist some a, $0 < a < 1$, and $b > 0$ such that (10) holds, then $a \leq \frac{1}{3}$.

Proof: If the assumption holds, $\lim_{n\to\infty}(n-\bar{x}(n))^{-2} = 0$. If $a > \frac{1}{3}$, then $\lim_{n\to\infty} 2n(n-\bar{x}(n))^{-3} = 0$. Similarly as in the proof for the previous proposition, $\lim_{n\to\infty}\bar{x}(n) > n$, which is impossible. Thus, $a \leq \frac{1}{3}$. ■

In summary, there are only two possible cases: (i) $0 < a \leq \frac{1}{3}$ and $b > 0$; (ii) $a = 0$ and $b > 0$.

For convenience, define $p_i := w_i/w_{av}$ for $i \in N$. Then, $\sum_{j=1}^{n} p_j = n$, and by (14), $p_i > \frac{1}{2}$, $\forall i \in N$. Now we want to evaluate $\bar{r}^t(n)/(nw_{av}) - \bar{r}^d(n)/(nw_{av}) \sim \sum_{j=1}^{n} p_j\log(2p_j)/n - 1 + \frac{1}{2}(p_{av}^{\frac{1}{2}})^2 =: P(p_1,\cdots,p_{n-1})$, where $p_{av}^{\frac{1}{2}} := \sum_{j=1}^{n} p_j^{\frac{1}{2}}/n$, subject to $p_i > \frac{1}{2}$, $\forall i \in N/\{n\}$, and $p_n = n - \sum_{j=1}^{n-1} p_j > \frac{1}{2}$. Then, for $i \in N/\{n\}$, $P_{p_i} = \log p_i/n - \log p_n/n + p_{av}^{\frac{1}{2}}(p_i^{-\frac{1}{2}} - p_n^{-\frac{1}{2}})/(2n)$, and for

$i, k \in N/\{n\}$, $P_{p_i p_k} = I(i = k)[1 - p_{av}^{\frac{1}{2}}/(4p_i^{\frac{1}{2}})]/(np_i) + [1 - p_{av}^{\frac{1}{2}}/(4p_n^{\frac{1}{2}})]/(np_n) + (p_i^{-\frac{1}{2}} - p_n^{-\frac{1}{2}})(p_k^{-\frac{1}{2}} - p_n^{-\frac{1}{2}})/(4n^2)$, where I is the indicator function. Since $p_i > \frac{1}{2}$, $\forall i \in N$, the last term of $P_{p_i p_k}$ is much smaller (as $n \to \infty$) than the other terms, and thus can be ignored. Also, it can be verified that $(p_{av}^{\frac{1}{2}})^2 \leq \sum_{j=1}^{n} p_j/n = 1$, and consequently, $p_{av}^{\frac{1}{2}}/(4p_i^{\frac{1}{2}}) < 1/(4\sqrt{1/2}) < 1$ for $i \in N$. As a result, the Hessian matrix of P is positive definite, which means that P is strictly convex in the region restricted by the constraints. Note that when $p_i = 1$, $\forall i \in N$, $P_{p_i} = 0$ for $i \in N$, and thus P achieves the minimum $0.6931 - 0.5 = 0.1931 > 0$ at this point.

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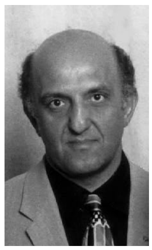

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