Automatica, Vol. 17, No. 5, 749–754, 1981
Printed in Great Britain

0005-1098/81/050749-06 $02.00/0
Pergamon Press Ltd
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Brief Paper

Equilibrium Strategies in Dynamic Games with Multi-levels of Hierarchy*

TAMER BAŞAR†

Key Words—Game theory; dynamic games; hierarchical systems; large-scale systems; Stackelberg strategies; hierarchical decision making; optimal systems.

Abstract—This paper considers noncooperative equilibria of three-player dynamic games with three levels of hierarchy in decision making. In this context, first a general definition of a hierarchical equilibrium solution is given, which also accounts for nonunique responses of the players who are not at the top of the hierarchy. Then, a general theorem is proven which provides a set of sufficient conditions for a triple of strategies to be in hierarchical equilibrium. When applied to linear-quadratic games, this theorem provides conditions under which there exists a linear one-step memory strategy for the player (say, $\mathscr{P}1$) at the top of the hierarchy, which forces the other two players to act in such a way so as to jointly minimize the cost function of $\mathscr{P}1$. Furthermore, there exists a linear one-step memory strategy for the second-level player (say, $\mathscr{P}2$), which forces the remaining player to jointly minimize the cost function of $\mathscr{P}2$ under the declared equilibrium strategy of $\mathscr{P}1$. A numerical example included in the paper illustrates the results and the convergence property of the equilibrium strategies, as the number of stages in the game becomes arbitrarily large.

1. *Introduction*

IN TWO-PLAYER dynamic games, existence of a hierarchy in decision making implies that one of the players is in a position to determine his strategy ahead of time, announce it and enforce it on the other player; therefore, the Stackelberg solution is the only possible hierarchical equilibrium solution applicable in such decision problems. In M-player games with more than two levels of hierarchy, however, several different possibilities emerge as to the type of hierarchical equilibrium solution concept to be adopted. We might, for example, have a linear hierarchy, in which case the players announce their strategies in a predetermined order, but one at a time; or we might have the situation in which some of the players are grouped together and they announce their strategies at the same level of hierarchy either cooperatively or in a noncooperative fashion under the Nash equilibrium solution concept. For an account of the available results in the literature on multi-level hierarchical equilibrium solutions, the reader is referred to a recent survey article by Cruz (1978).

In dynamic games with multi-levels of hierarchy, hierarchical equilibrium solution concepts have been introduced heretofore as extensions of the Stackelberg solution concept (which is applicable to two-player games) and under the stipulation that the rational reaction of each follower to every announced strategy(ies) of the leader(s) is unique—in spite of the fact that, under the general closed-loop information structure, the optimal responses will almost always be nonunique. This difficulty is of course circumvented if one confines the analysis to open-loop strategy spaces (Medanic and Radojevic, 1978), or to linear feedback strategy spaces (Medanic, 1977) in linear-quadratic games; but if the players have access to closed-loop information structure, a more general definition that also accounts for nonunique responses is unavoidable. Such a general definition has in fact been given in Başar and Selbuz (1979) within the context of dynamic games with only two-levels of hierarchy in decision making, and in this paper we first extend this equilibrium solution concept to three levels of hierarchy. Then, for the general class of three-player games with three levels of hierarchy, we prove a general theorem (Theorem 1) which provides a set of sufficient conditions for a triple of strategies to be in hierarchical equilibrium.

Inherent in the suffficiency conditions of Theorem 1 is an important feature of the hierarchical equilibrium solution that is akin to the one observed in Başar and Selbuz (1979) for the class of two-player games. Specifically, by announcing an appropriate strategy, the player at the top of the hierarchy can achieve an optimal cost level that is equal to the global minimum of his cost function, and moreover he can force the other two players to collectively minimize the cost function of the player at the second level of hierarchy (under the declared strategy of the leader). We, then, consider a specific application of Theorem 1 to linear-quadratic dynamic games, and obtain recursive expressions for the hierarchical equilibrium strategies which are of the one-step-memory type for the first two players (see Theorem 2). A numerical example included in the paper illustrates the result of Theorem 2 and the convergence properties of the equilibrium strategies as the number of stages in the game becomes arbitrarily large.

2. *The three-level hierarchical equilibrium solution*

To introduce the hierachical equilibrium solution concept for three-player dynamic games with three levels of hierarchy, let us first stipulate that (a) Player 1 ($\mathscr{P}1$) is in a position to announce his strategy ahead of time and enforce it on the other two players, and (b) $\mathscr{P}2$, in view of the announced strategy of $\mathscr{P}1$, has the ability to dictate his strategy on $\mathscr{P}3$ who only thereafter decides on his optimizing strategy. Let us denote the index set $\{1,2,3\}$ by $\mathscr{N}$, the strategy space of $\mathscr{P}i$ by Γ_i $(i\in\mathscr{N})$, and a typical element of Γ_i by γ_i $(i\in\mathscr{N})$. Furthermore, for each triple $\{\gamma_1\in\Gamma_1,\ \gamma_2\in\Gamma_2,\ \gamma_3\in\Gamma_3\}$, let $J_i(\gamma_1,\gamma_2,\gamma_3)$ denote the corresponding unique loss incurred to $\mathscr{P}i$, with the function J_i to be referred to as the cost function of $\mathscr{P}i$ $(i\in\mathscr{N})$, hereafter. In this general framework, the concept of a hierarchical equilibrium strategy for $\mathscr{P}1$ can now be introduced as follows.

Definition 1. For the preceding three-player game with three levels of hierarchy (in decision making), $\gamma_1^*\in\Gamma_1$ is a *hierarchical equilibrium strategy* for $\mathscr{P}1$ if

$$J_1^* \triangleq \sup_{\gamma_2\in R_2(\gamma_1^*)}\ \sup_{\gamma_3\in R_3(\gamma_1^*;\gamma_2)} J_1(\gamma_1^*,\gamma_2,\gamma_3)$$

$$= \min_{\gamma_1\in\Gamma_1}\ \sup_{\gamma_2\in R_2(\gamma_1)}\ \sup_{\gamma_3\in R_3(\gamma_1;\gamma_2)} J_1(\gamma_1,\gamma_2,\gamma_3) \quad (1)$$

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*Received 10 April 1980; revised 9 September 1980. The original version of this paper was presented at the 2nd IFAC/IFORS Symposium on Large Scale Systems: Theory and Applications which was held in Toulouse, France during June 1980. The published proceedings of this IFAC meeting may be ordered from: Pergamon Press Ltd., Headington Hill Hall, Oxford OX3 0BW, U.K. This paper was recommended for publication in revised form by Associate Editor D. Tabak.

†Coordinated Science Laboratory, College of Engineering, University of Illinois, Urbana, IL 61801, U.S.A.

where

$$R_2(\gamma_1) \overset{\Delta}{=} \left\{ \xi \in \Gamma_2 : \sup_{\gamma_3 \in R_3(\gamma_1;\xi)} J_2(\gamma_1, \xi, \gamma_3) \leqq \sup_{\gamma_3 \in R_3(\gamma_1;\gamma_2)} J_2(\gamma_1, \gamma_2, \gamma_3) \quad \forall \gamma_2 \in \Gamma_2 \right\} \tag{2a}$$

and

$$R_3(\gamma_1;\gamma_2) \overset{\Delta}{=} \{\xi \in \Gamma_3 : J_3(\gamma_1, \gamma_2, \xi) \leqq J_3(\gamma_1, \gamma_2, \gamma_3) \quad \forall \gamma_3 \in \Gamma_3\}. \tag{2b}$$

Any $\gamma_2^* \in R_2(\gamma_1^*)$ is a corresponding *equilibrium strategy* for $\mathscr{P}2$, and any $\gamma_3^* \in R_3(\gamma_1^*;\gamma_2^*)$ is an *equilibrium strategy* for $\mathscr{P}3$ corresponding to the stategy pair $(\gamma_1^*;\gamma_2^*)$. □

The foregoing definition of a hierarchical equilibrium strategy also takes into account possible nonunique responses of the 'following' players, and in that respect it can be considered as a natural extension of Definition 1 in Başar and Selbuz (1979) to fit the present context. For a given dynamic game problem, it is usually the case that the sets $R_2(\gamma_1)$ and $R_3(\gamma_1;\gamma_2)$ are not singletons for any $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$, since even if the cost functions are strictly convex in the control variables, this feature ceases to hold true when they are expressed in terms of the strategies of the players. This, therefore, makes it necessary that a proper definition of hierarchical equilibrium solution also account for nonunique responses of the players, as in Definition 1 above. For the restricted class of problems for which the sets (2a) and (2b) are singletons, however, the hierarchical equilibrium strategy of $\mathscr{P}1$ naturally satisfies a simpler set of relations as summarized below in Proposition 1.

Proposition 1. For the class of three-player dynamic games covered by Definition 1, if $R_2(\gamma_1)$ and $R_3(\gamma_1;\gamma_2)$ are singletons, then there exist unique mappings $T_2:\Gamma_1 \to \Gamma_2$ and $T_3:\Gamma_1 \times \Gamma_2 \to \Gamma_3$ defined by

$$J_2[\gamma_1, T_2\gamma_1, T_3(\gamma_1, T_2\gamma_1)] \leqq J_2[\gamma_1, \gamma_2, T_3(\gamma_1, \gamma_2)] \quad \forall \gamma_2 \in \Gamma_2 \tag{3a}$$

and

$$J_3[\gamma_1, \gamma_2, T_3(\gamma_1, \gamma_2)] \leqq J_3(\gamma_1, \gamma_2, \gamma_3) \quad \forall \gamma_3 \in \Gamma_3. \tag{3b}$$

Furthermore, the hierarchical equilibrium strategy $\gamma_1^* \in \Gamma_1$ of $\mathscr{P}1$ satisfies

$$J_1[\gamma_1^*, T_2\gamma_1^*, T_3(\gamma_1^*, T_2\gamma_1^*)] \leqq J_1[\gamma_1, T_2\gamma_1, T_3(\gamma_1, T_2\gamma_1)] \quad \forall \gamma_1 \in \Gamma_1. \tag{4}$$

□

It is in general not possible to determine the sets $R_2(\gamma_1)$ and $R_3(\gamma_1;\gamma_2)$ explicitly for all possible elements $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, unless Γ_1 and Γ_2 happen to be the class of open-loop strategies—which is tantamount to having static games. This then clearly rules out the possibility of any direct approach that would yield the hierarchical equilibrium solution in dynamic games, and one has to resort to indirect methods and derivations.

One such indirect approach has been introduced in Başar and Selbuz (1979), wherein hierarchical equilibrium solutions of two-player linear-quadratic dynamic games have been related to optimal solutions of some team problems. We now extend this relation to general three-player dynamic games of the type described earlier (i.e. with three levels of hierarchy in decision making), and in particular, we prove a general theorem (Theorem 1) which provides a set of sufficient conditions for the hierarchical equilibrium solution to satisfy.

Preliminary notation. For each $\gamma_1 \in \Gamma_1$, define the subsets $S_1(\gamma_1) \subset \Gamma_2 \times \Gamma_3$ and $S_2(\gamma_1) \subset \Gamma_2 \times \Gamma_3$ by

$$S_i(\gamma_1) = \left\{ (\xi_2, \xi_3) \in \Gamma_2 \times \Gamma_3 : J_i(\gamma_1, \xi_2, \xi_3) = \min_{\gamma_2 \in \Gamma_2} \min_{\gamma_3 \in \Gamma_3} J_i(\gamma_1, \gamma_2, \gamma_3) \right\} \quad i = 1, 2, \tag{5a}$$

and introduce a subset $\tilde{S}_2(\gamma_1)$ of $S_2(\gamma_1)$ by

$$\tilde{S}_2(\gamma_1) = \{(\gamma_2, \gamma_3) \in S_2(\gamma_1) : \gamma_3 \in R_3(\gamma_1;\gamma_2)\}. \tag{5b}$$

Theorem 1. Let there exist a $\gamma_1^* \in \Gamma_1$ such that:

(i) $\tilde{S}_2(\gamma_1^*)$ is nonempty and for every pair $(\xi_2, \xi_3) \in \tilde{S}_2(\gamma_1^*)$

$$\sup_{\gamma_3 \in R_3(\gamma_1^*;\xi_2)} J_2(\gamma_1^*, \xi_2, \gamma_3) = J(\gamma_1^*, \xi_2, \xi_3)$$

(ii) $\tilde{S}_2(\gamma_1^*) \subset S_1(\gamma_1^*)$, and;

(iii) for every $(\xi_2, \xi_3) \in S_1(\gamma_1^*)$, γ_1^* minimizes $J_1(\gamma_1, \xi_2, \xi_3)$ over Γ_1. Then, γ_1^* is a hierarchical equilibrium strategy for $\mathscr{P}1$; and given any pair $(\gamma_2^*, \gamma_3^*) \in \tilde{S}_2(\gamma_1^*)$, γ_2^* is a corresponding equilibrium strategy for $\mathscr{P}2$ and γ_3^* is an equilibrium strategy for $\mathscr{P}3$ corresponding to the pair (γ_1^*, γ_2^*).

Proof. We first note that, since $\tilde{S}_2(\gamma_1^*)$ is not empty, there exists a pair $\{\xi_2 \in \Gamma_2, \xi_3 \in R_3(\gamma_1^*;\xi_2)\}$ which globally minimizes $J_2(\gamma_1^*, \gamma_2, \gamma_3)$ over $\Gamma_2 \times \Gamma_3$, and furthermore by construction every element pair in $\tilde{S}_2(\gamma_1^*)$ has such a property. Then, under condition (i), we have the set equivalence

$$\tilde{S}_2(\gamma_1^*) = \{(\gamma_2, \gamma_3) \in \Gamma_2 \times \Gamma_3 : \gamma_2 \in R_2(\gamma_1^*) \text{ and } \gamma_3 \in R_3(\gamma_1^*;\gamma_2)\} (*) .$$

Now assume, to the contrary, that γ_1^* is not a hierarchical equilibrium strategy. Then, we have from (1), by also making use of the preceding set equivalence relation

$$J_1^* < \sup_{(\gamma_2, \gamma_3) \in \tilde{S}_2(\gamma_1^*)} J_1(\gamma_1^*, \gamma_2, \gamma_3).$$

Furthermore, since $\tilde{S}_2(\gamma_1^*) \subset S_1(\gamma_1^*)$ by (ii), we have the looser bound

$$J_1^* < \sup_{(\gamma_2, \gamma_3) \in S_1(\gamma_1^*)} J_1(\gamma_1^*, \gamma_2, \gamma_3)$$

which is however equal to the global minimum of J_1 over $\Gamma_1 \times \Gamma_2 \times \Gamma_3$ by (iii) and the definition of $S_1(\cdot)$. But since J_1^* is also bounded from below by the same quantity, it follows that the preceding inequalities will have to be replaced by equalities, thereby implying that γ_1^* is indeed a hierarchical equilibrium strategy for $\mathscr{P}1$. Finally, equilibrium property of any element of $\tilde{S}_2(\gamma_1^*)$ follows from equation (*), in view of Definition 1. □

Remark 1. Under the sufficiency conditions of Theorem 1, the strategy γ_1^* of $\mathscr{P}1$ forces the other two players to play in such a way so as also to globally minimize the cost function of $\mathscr{P}1$. The equilibrium strategy γ_2^* of $\mathscr{P}2$, on the other hand, has the property that it forces $\mathscr{P}3$ also to jointly minimize $J_2(\gamma_1^*, \gamma_2, \gamma_3)$. □

We now apply Theorem 1 to three-player linear-quadratic dynamic games defined in discrete time, and obtain explicit equations and conditions for the equilibrium strategies to satisfy.

3. *Three player linear-quadratic dynamic games*

Let θ denote the index set $\{0, 1, \ldots, N-1\}$ and define the evolution of the state $x_{(\cdot)}$ over θ by the difference equation

$$x_{n+1} = A(n)x_n + B_1(n)u_n + B_2(n)v_n + B_3(n)w_n \tag{6}$$

where $u_{(\cdot)}$ is an r_1-vector controlled by $\mathscr{P}1$, $v_{(\cdot)}$ is an r_2-vector controlled by $\mathscr{P}2$, $w_{(\cdot)}$ is an r_3-vector controlled by $\mathscr{P}3$, $x_{(\cdot)}$ is the state vector of dimension m, $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$ and $B_3(\cdot)$ are matrices of appropriate dimensions.

The cost functions of $\mathscr{P}1$, $\mathscr{P}2$ and $\mathscr{P}3$ in terms of the control vectors are given, respectively, by

$$\bar{J}_1(N,M)=x_N'Q_1(N)x_N+\sum_{n=M}^{N-1}[x_n'Q_1(n)x_n+u_n'u_n+v_n'R_{12}(n)v_n+w_n'R_{13}(n)w_n] \tag{7a}$$

$$\bar{J}_2(N,M)=x_N'Q_2(N)x_N+\sum_{n=M}^{N-1}[x_n'Q_2(n)x_n+u_n'R_{21}(n)u_n+v_n'v_n+w_n'R_{23}(n)w_n] \tag{7b}$$

$$\bar{J}_3(N,M)=x_N'Q_3(N)x_N+\sum_{n=M}^{N-1}[x_n'Q_3(n)x_n+u_n'R_{31}(n)u_n+v_n'R_{32}(n)v_n+w_n'w_n] \tag{7c}$$

with

$$R_{12}(\cdot)>0,\quad R_{13}(\cdot)>0,\quad R_{23}(\cdot)>0,\quad R_{21}(\cdot)\geqq 0,$$
$$R_{31}(\cdot)\geqq 0,\quad R_{32}(\cdot)\geqq 0,\quad Q_i(\cdot)\geqq 0,\quad i\in\mathcal{N}.$$

Each player has access to closed-loop perfect state information. If η_n denotes the set

$$\eta_n=\{x_0,x_1,\ldots,x_n\} \tag{8}$$

then we have $u_n=\gamma_{1,n}(\eta_n)$, $v_n=\gamma_{2,n}(\eta_n)$, $w_n=\gamma_{3,n}(\eta_n)$ where $\gamma_{i,n}$ is an appropriate measurable mapping denoting the strategy of $\mathscr{P}i$ at stage n. We denote the space of such strategies for $\mathscr{P}i$ at stage n by $\Gamma_{i,n}$, denote the entire collection $\{\gamma_{i,0},\gamma_{i,1},\ldots,\gamma_{i,N-1}\}$ as γ_i, and adopt the convention of viewing γ_i as a typical strategy of $\mathscr{P}i$ in the game (which covers the $N-1$ stages) and as an element of Γ_i which is the closed-loop strategy space of $\mathscr{P}i$ constructed appropriately from $\Gamma_{i,n}$, $n\in\theta$. Furthermore, let us finally adopt the notation $J_i(\gamma_1,\gamma_2,\gamma_3)$ to denote the cost function of $\mathscr{P}i$ in terms of the strategies of the players, obtained from $\bar{J}_i(N,0)$ by setting $u_n=\gamma_{1,n}(\eta_n)$, $v_n=\gamma_{2,n}(\eta_n)$, $w_n=\gamma_{3,n}(\eta_n)$, $n\in\theta$.

Now, in order to apply Theorem 1 to the dynamic game problem formulated above, we first assume that $\mathscr{P}2$ and $\mathscr{P}3$ do not act at the last stage of the game, and $\mathscr{P}3$ does not act at the next to the last stage of the game; in other words, $B_2(N-1)=0$, $B_3(N-1)=0$ and $B_3(N-2)=0$. The reason for making such an assumption is [as discussed in Başar and Selbuz (1979) for the specific two-person problem treated there] that otherwise $\mathscr{P}1$ cannot enforce his team solution on the other players, and $\mathscr{P}2$ cannot enforce his desired solution on $\mathscr{P}3$. If this assumption does not hold true, however, the problem is still tractable, but then one has to define a different team problem (instead of $\min_{\gamma_1,\gamma_2,\gamma_3}J_1(\gamma_1,\gamma_2,\gamma_3)$) in a way similar to the analysis of Section VI of Başar'and Selbuz (1979) for the two-player case; but we shall not pursue this extension here.

The related team problem. Since $\bar{J}_1(N,0)$ is a strictly convex functional of the control vectors of the players, the minimization problem

$$\min_{\gamma_1\in\Gamma_1}\ \min_{\gamma_2\in\Gamma_2}\ \min_{\gamma_3\in\Gamma_3} J_1(\gamma_1,\gamma_2,\gamma_3) \tag{9}$$

is well defined. Then, the following lemma readily follows from standard results in optimal control theory.

Lemma 1. The minimization problem (9) admits a unique solution in feedback strategies, which is given by

$$\gamma_{i,n}^f(x_n)=-L_i(n)x_n\quad n\in\theta\quad i\in\mathcal{N} \tag{10}$$

where

$$\begin{bmatrix} L_i(n)\\ \cdots\\ L_2(n)\\ \cdots\\ L_3(n)\end{bmatrix}=[R(n)+B'(n)M(n)B(n)]^{-1}B'(n)M(n)A(n) \tag{11}$$

$$B(n)\triangleq[B_1(n),B_2(n),B_3(n)] \tag{12a}$$

$$R(n)\triangleq\operatorname{diag}[I,R_{12}(n),R_{13}(n)] \tag{12b}$$

and with $M(\cdot)$ defined recursively by

$$M(n)=Q_1(n)+F'(n)M(n+1)F(n)+L_1'(n)L_1(n)+L_2'(n)R_{12}(n)L_2(n)+L_3'(n)R_{13}(n)L_3(n) \tag{13}$$
$$M(N)=Q_1(N)$$

and

$$F(n)\triangleq A(n)-B_1(n)L_1(n)-B_2(n)L_2(n)-B_3(n)L_3(n). \tag{14}$$

The minimum (team) cost is

$$J_1^t=x_0'M(0)x_0. \tag{15}$$

□

The solution of the optimal control problem (9), even though it is unique in feedback strategies, is not unique when considered as an element of $\Gamma_1\times\Gamma_2\times\Gamma_3$. To determine the complete characterization of the solution set, let us first denote the value of state $x_{(\cdot)}$ under (10) and as a function of x_0 by $\bar{x}_{(\cdot)}$, i.e. $\bar{x}_{(\cdot)}$ satisfies the difference equation

$$\bar{x}_{n+1}=F(n)\bar{x}_n,\quad \bar{x}_0=x_0. \tag{16}$$

Furthermore, let

$$\bar{\eta}_n\triangleq\{x_0,\bar{x}_1,\ldots,\bar{x}_n\}, \tag{17}$$

and introduce the subset $\bar{\Gamma}_i$ of Γ_i by

$$\bar{\Gamma}_i=\{\gamma_i\in\Gamma_i:\gamma_{i,n}(\bar{\eta}_n)\equiv\gamma_{i,n}^f(\bar{x}_n),\quad n\in\theta\} \tag{18}$$

which is the set of all *representations* of the strategy γ_i^f on the optimal trajectory (16) [see e.g. Başar (1980a) for elaboration on this concept]. Then, we have:

Lemma 2. Every solution of the minimization problem (9) is an element of the product space $\Gamma_1\times\Gamma_2\times\Gamma_3$, and conversely every triplet $\{\gamma_1\in\bar{\Gamma}_1,\gamma_2\in\bar{\Gamma}_2,\gamma_3\in\bar{\Gamma}_3\}$ constitutes a solution to (9). □

And this property of the solution immediately leads to the following conclusion.

Proposition 2. If γ_1 is restricted to $\bar{\Gamma}_1$, then $S_1(\gamma_1)$ introduced by (5a) becomes independent of γ_1, and is given by

$$S_1(\gamma_1)=\bar{\Gamma}_2\times\bar{\Gamma}_3. \quad □$$

Derivation of a one-step-memory hierarchical strategy for $\mathscr{P}1$. Any hierarchical equilibrium strategy for $\mathscr{P}1$ that also satisfies the conditions of Theorem 1 will clearly have to be an element of $\bar{\Gamma}_1$. Hence, to obtain explicit results, we now confine ourselves to a specific class of elements in $\bar{\Gamma}_1$ with a fixed structure. In particular, we fix attention on the one-step-memory strategies

$$\gamma_{1,n}^p(x_n,x_{n-1})=-L_1(n)x_n+P_1(n)[x_n-F(n-1)x_{n-1}];\quad n\in\theta-\{0\}$$

$$\gamma_{1,0}(x_0)=-L_1(0)x_0$$

which are clearly in $\bar{\Gamma}_1$ for every $(r_1\times m)$-matrix sequence $\{P_1(1),\ldots,P_1(N-1)\}$, and we investigate conditions under which there exists one such sequence of matrices for which $S_2(\gamma_1^p)\subset\bar{\Gamma}_2\times\bar{\Gamma}_3$. These conditions are now provided below in Lemma 3.

Preliminary notation for Lemma 3. $\Sigma_2(\cdot)$: an $(m\times m)$-matrix defined recursively by

$$\Sigma_2(n)=Q_2(n)+F'(n)\Sigma_2(n+1)F(n)+L_1'(n)R_{21}(n)L_1(n)$$
$$+L_2'(n)L_2(n)+L_3'(n)R_{23}(n)L_3(n) \qquad (19)$$
$$\Sigma_2(N)=Q_2(N).$$

$\Lambda_1(\cdot)$: an $(r_1\times m)$-matrix defined recursively by (as a function of $\{P_1(n+1),P_1(n+2),\ldots,P_1(N-1)\}$)

$$\Lambda_1(n)=B_1'(n)P_1'(n+1)\Lambda_1(n+1)F(n)-R_{21}(n)L_1(n)$$
$$+B_1'(n)\Sigma_2(n+1)F(n) \qquad (20)$$
$$\Lambda_1(N)=0.$$

Condition 1. There exists at least one matrix-valued function sequence $\{P_1(N-1),P_1(N-2),\ldots,P_1(1)\}$ that satisfies recursively the matrix equations

$$B_2'(n-1)P_1'(n)\Lambda_1(n)F(n-1)=$$
$$-B_2'(n-1)\Sigma_2(n)F(n-1)+L_2(n-1) \qquad (21a)$$
$$B_3'(n-1)P_1'(n)\Lambda_1(n)F(n-1)=$$
$$-B_3'(n-1)\Sigma_2(n)F(n-1)+R_{23}(n-1)L_3(n-1) \qquad (21b)$$

where $\Lambda_1(n)$ is related to $\{P_1(n+1),\ldots,P_1(N-1)\}$ through (20). □

Lemma 3. Let Condition 1 be satisfied and let $\{P_1^*(N-1),\ldots,P_1^*(1)\}$ denote one such sequence. Then, with $\gamma_1^*\in\tilde{\Gamma}_1$ picked as

$$\gamma_{1,n}^*(x_n,x_{n-1})=[P_1^*(n)-L_1(n)]x_n-P_1^*(n)F(n-1)x_{n-1};$$
$$n\in\theta-\{0\} \qquad (22)$$
$$\gamma_{1,0}^*(x_0)=-L_1(0)x_0,$$

we have $S_2(\gamma_1^*)=\tilde{\Gamma}_2\times\tilde{\Gamma}_3$.

Proof. The fact that (γ_2^f,γ_3^f) minimizes $J_2(\gamma_1^*,\gamma_2,\gamma_3)$ over $\Gamma_2\times\Gamma_3$ follows from Theorem 1 of Başar and Selbuz (1979) by viewing $\mathscr{P}2$ and $\mathscr{P}3$ as a single player minimizing the cost function J_2 under the announced strategy γ_1^* of $\mathscr{P}1$, and by an appropriate decomposition of equation (21) of that reference into two linear matrix equations [which are (21a) and (21b) above]. Now, since $J_2(\gamma_1^*,\gamma_2,\gamma_3)$ is a strictly convex function of γ_2 and γ_3 when these strategies are restricted to open-loop policies, it follows that the minimization problem $\min_{\gamma_2\in\Gamma_2}\min_{\gamma_3\in\Gamma_3} J_2(\gamma_1^*,\gamma_2,\gamma_3)$ admits a unique open-loop solution; furthermore, the pair (γ_2^f,γ_3^f) is a particular closed-loop representation of these open-loop strategies since it also minimizes $J_2(\gamma_1^*,\gamma_2,\gamma_3)$ and the open-loop solution is unique. Consequently, the set of all minimizing solutions of $J_2(\gamma_1^*,\gamma_2,\gamma_3)$ is comprised of all representations of (γ_2^f,γ_3^f), which is precisely the product set $\tilde{\Gamma}_2\times\tilde{\Gamma}_3$. Hence, $S_2(\gamma_1^*)=\tilde{\Gamma}_2\times\tilde{\Gamma}_3$.

The next step in the derivation now is to determine a $\gamma_2^*\in\tilde{\Gamma}_2$ such that $R_3(\gamma_1^*;\gamma_2^*)\subset\tilde{\Gamma}_3$. To this end, we again confine ourselves to a specific structure, namely to linear one-step-memory strategies in $\tilde{\Gamma}_2$, which can be expressed as

$$\gamma_{2,n}^*(x_n,x_{n-1})=-L_2(n)x_n+P_2(n)[x_n-F(n-1)x_{n-1}];$$
$$n=N-2,\ldots,1$$
$$\gamma_{2,0}(x_0)=-L_2(0)x_0$$

where the $(r_2\times m)$-matrix sequence $\{P_2(1),\ldots,P_2(N-2)\}$ is yet to be determined. Conditions under which there exists one such sequence so that $\mathscr{P}3$ is forced to choose an element of $\tilde{\Gamma}_3$ are provided below in Lemma 4.

Preliminary notation for Lemma 4. $\Sigma_3(\cdot)$: an $(m\times m)$-matrix defined recursively by

$$\Sigma_3(n)=Q_3(n)+F'(n)\Sigma_3(n+1)F(n)+L_1'(n)R_{31}(n)L_1(n)$$
$$+L_2'(n)R_{32}(n)L_2(n)+L_3'(n)L_3(n) \qquad (23)$$
$$\Sigma_3(N)=Q_3(N).$$

$\Lambda_2(\cdot)$ and $\Lambda_3(\cdot)$: $(r_1\times m)$- and $(r_2\times m)$-matrices defined recursively by

$$\Lambda_2(n)=B_1'(n)[P_1^{*\prime}(n+1)\Lambda_2(n+1)+P_2'(n+1)\Lambda_3(n+1)]$$
$$\times F(n)-R_{31}(n)L_1(n)+B_1'(n)\Sigma_3(n+1)F(n);$$
$$\Lambda_2(N)=0 \qquad (24a)$$
$$\Lambda_3(n)=B_2'(n)[P_1^{*\prime}(n+1)\Lambda_2(n+1)+P_2'(n+1)\Lambda_3(n+1)]F(n)$$
$$-R_{32}(n)L_2(n)+B_2'(n)\Sigma_3(n+1)F(n);\quad \Lambda_3(N)=0$$
$$(24b)$$

where $\{P_1^*(1),\ldots,P_1^*(N-1)\}$ is the matrix sequence determined in Lemma 3.

Condition 2. There exists at least one matrix-valued function $\{P_2(N-2),\ldots,P_2(1)\}$ that satisfies recursively the linear matrix equation

$$B_3'(n-1)[P_1^{*\prime}(n)\Lambda_2(n)+P_2'(n)\Lambda_3(n)]F(n-1)=$$
$$-B_3'(n-1)\Sigma_3(n)F(n-1)+L_3(n-1). \qquad (25)$$

□

Lemma 4. Let Condition 2 be satisfied and let $\{P_2^*(N-2),\ldots,P_2^*(1)\}$ denote one such sequence. Then, with $\gamma_2^*\in\tilde{\Gamma}_2$ picked as

$$\gamma_{2,n}^*(x_n,x_{n-1})=[P_2^*(n)-L_2(n)]x_n-P_2^*(n)F(n-1)x_{n-1};$$
$$n=N-2,\ldots,1$$
$$\gamma_{2,0}^*(x_0)=-L_2(0)x_0, \qquad (26)$$

we have $R_3(\gamma_1^*;\gamma_2^*)=\tilde{\Gamma}_3$, and hence condition (i) of Theorem 1 holds.

Proof. The fact that γ_3^f minimizes $J_3(\gamma_1^*,\gamma_2^*,\gamma_3)$ over Γ_3 follows from Theorem 1 of Başar and Selbuz (1979), this time by viewing $\mathscr{P}1$ and $\mathscr{P}2$ as a single player with the announced strategies (22) and (25), and by an appropriate decomposition of equation (20) of that reference into two coupled recursive equations [which are (24a) and (24b)]. Because of the strict convexity of $J_3(\gamma_1^*,\gamma_2^*,\gamma_3)$, the subset of Γ_3 that minimizes this quantity is the class of all representations of γ_3^f on the trajectory of (16), which is precisely $\tilde{\Gamma}_3$—a result which follows from reasoning quite analogous to the one used in the proof of Lemma 3. Then, clearly, $R_3(\gamma_1^*;\gamma_2^*)=\tilde{\Gamma}_3$, which further implies that $\bar{S}_2(\gamma_1^*)$ is nonempty since $\{\gamma_2^*\}\times\tilde{\Gamma}_3\subset S_2(\gamma_1^*)$. Moreover, the latter part of condition (i) of Theorem 1 holds since $J_2(\gamma_1^*,\gamma_2^*,\xi)$ has a constant value for all $\xi\in\tilde{\Gamma}_3$. □

We now finally have

Theorem 2. Under Conditions 1 and 2, γ_1^* as defined by (22) provides a hierarchical equilibrium strategy for $\mathscr{P}1$ in the three-person linear-quadratic dynamic game under consideration, γ_2^*, as defined by (26), is a corresponding equilibrium strategy for $\mathscr{P}2$, and any $\gamma_3\in\tilde{\Gamma}_3$ is an equilibrium strategy for $\mathscr{P}3$ corresponding to the pair (γ_1^*,γ_2^*).

Proof. Condition (i) of Theorem 1 is fulfilled because of Lemma 4. Condition (ii) is also fulfilled, by Lemma 3 and Proposition 2. Finally Condition (iii) is fulfilled by Proposition 2, since $\gamma_1^*\in\tilde{\Gamma}_1$. Consequently, Theorem 2 follows directly from Theorem 1. □

Remark 2. The solution presented in Theorem 2 is not the only hierarchical equilibrium solution that the three-person linear-quadratic dynamic game admits, since we have restricted our investigation at the outset to a specific class of strategies for $\mathscr{P}1$, namely to linear one-step-memory strategies in $\tilde{\Gamma}_1$. To explore the possibilities for other (structurally different) equilibrium solutions, we could, for example, adopt the following more general representation of (10) (with $i=1$) for $\mathscr{P}1$

$$\gamma_{1,n}(x_n, x_{n-1}, x_{n-2}) = -L_1(n)x_n + P_1(n)[x_n - F(n-1)x_{n-1}] + P_3(n)[x_{n-1} - F(n-2)x_{n-2}]$$

which incorporates a two-step memory. We could also allow the coefficient matrices $P_1(\cdot)$ and $P_3(\cdot)$ to depend on the initial state of the game in which case equations of the form (21) will be replaced by vector equations obtained by multiplying (21a) and (21b) from the right by $\bar{x}_{n-1}$,which follows directly from equation (A-9) of Başar and Selbuz (1979). Such structural forms will then clearly lead to less stringent conditions than Conditions 1 and 2. For $\mathscr{P}2$, also, we can adopt a more general class of elements in $\tilde{\Gamma}_2$ than the one-step memory structure (25), which would lead to a further relaxation of Condition 2. The corresponding values of the cost functions (J_1, J_2 and J_3), however, will be the same under all these different representations, since all different hierarchical equilibrium solutions that satisfy the sufficiency conditions of Theorem 1 are basically representations of the team strategies (10). These optimal cost values will in fact be $J_1^* = x_0'M(0)x_0$, $J_2^* = x_0'\Sigma_2(0)x_0$, $J_3^* = x_0'\Sigma_3(0)x^0$. □

We now provide a specific example to illustrate Theorem 2 and the convergence properties of the hierarchical equilibrium solution as the number of stages in the game becomes arbitrarily large.

4. *A scalar example*

Let the state dynamics of an $(N-k)$-stage dynamic game be described by

$$\left.\begin{aligned} x_N &= x_{N-1} + u_{N-1} \\ x_{N-1} &= x_{N-2} + u_{N-2} + 2v_{N-2} \\ &\;\;\vdots \\ x_{n+1} &= x_n + u_n + 2v_n + w_n, \quad n \leq N-2 \end{aligned}\right\} \tag{27}$$

and the cost functions be given as

$$J_1 = x_N^2 + \sum_{n=k}^{N-1} x_n^2 + u_n^2 + 4v_n^2 + w_n^2 \tag{28a}$$

$$J_2 = x_N^2 + \sum_{n=k}^{N-1} 3x_n^2 + v_n^2 + (1/4)w_n^2 \tag{28b}$$

$$J_3 = x_N^2 + \sum_{n=k}^{N-1} 2x_n^2 + w_n^2. \tag{28c}$$

First, the solution of the minimization problem (9) can readily be computed (from Lemma 1), with the relevant quantities tabulated in Table 1. It should be noted that as $N-k$ becomes arbitrarily large, the feedback optimal team strategies converge to the stationary values

$$\gamma_{1,n}^f(x_n) = -0.2637626x_n$$

$$\gamma_{2,n}^f(x_n) = -0.1318813x_n$$

$$\gamma_{3,n}^f(x_n) = -0.2637626x_n.$$

We then compute $\Sigma_2(\cdot)$ and $\Sigma_3(\cdot)$, and record the corresponding values in columns 1 and 2, respectively, of Table 2. With these values at hand, we consider the difference equation (20) and the relations (21a) and (21b) together, and solve recursively for the unique $\Lambda_1(\cdot)$ and $\mathscr{P}_1(\cdot)$ whose corresponding values are recorded in columns 3 and 4, respectively, of Table 2. Finally, we solve for $\Lambda_2(\cdot)$, $\Lambda_3(\cdot)$ and $P_2(\cdot)$, iteratively from (24) and (25), and again obtain unique values, which are listed respectively in columns 5, 6 and 7 of Table 2.

Hence, this dynamic game problem admits a linear hierarchical equilibrium solution within the class of one-step-memory strategies for $\mathscr{P}1$ and $\mathscr{P}2$; and as the number of stages in the game becomes large, there is a rather fast convergence to the stationary policies

$$\gamma_{1,n}^*(x_n, x_{n-1}) = -43.59159x_n + 43.32783x_{n-1}$$

$$\gamma_{2,n}^*(x_n, x_{n-1}) = 19.68395x_n - 19.95771x_{n-1}$$

$$\gamma_{3,n}^*(x_n) = 0.263763x_n$$

to the nearest five decimal places. The corresponding optimal cost values are

$$J_1^* = 1.26376x_0^2$$

$$J_2^* = 3.17300x_0^2$$

$$J_3^* = 2.16383x_0^2.$$

5. *Concluding remarks*

Several extensions of the results of this paper are possible. Firstly, as discussed in Remark 2, Conditions 1 and 2 can be made less stringent by considering a more general class of representations of the team strategies (10); in this context one may also include nonlinear representations of (10) by following the lines of Tolwinski (1981) in the two player case.

TABLE 1. SOLUTION OF THE TEAM PROBLEM (9) FOR THE SCALAR EXAMPLE

n	$L_1(n)$	$L_2(n)$	$L_3(n)$	F(n)	M(n)
N	–	–	–	–	1
N-1	0.5	–	–	0.5	1.5
N-2	0.375	0.1875	–	0.25	1.375
N-3	0.2682926	0.1341463	0.2682926	0.1951219	1.2682926
N-4	0.2639593	0.1319797	0.2639593	0.2081218	1.2639593
N-5	0.2637711	0.1318856	0.2637711	0.2086864	1.2637711
N-6	0.2637629	0.1318815	0.2637629	0.2087110	1.2637629
N-7	0.2637626	0.1318813	0.2637626	0.2087121	1.2637625
N-8	0.2637626	0.1318813	0.2637626	0.2087121	1.2637625
N-9	0.2637626	0.1318813	0.2637626	0.2087121	1.2637625

TABLE 2. NUMERICAL VALUES OF THE RELEVANT QUANTITIES THAT DETERMINE THE EQUILIBRIUM STRATEGIES OF THE PLAYERS FOR THE SCALAR EXAMPLE

n	$\Sigma_2(n)$	$\Sigma_3(n)$	$\Lambda_1(n)$	$P_1^*(n)$	$\Lambda_2(n)$	$\Lambda_3(n)$	$P_2^*(n)$
N	1	1	0	–	0	0	–
N-1	3.25	2.25	0.5	-5.75	0.5	0	–
N-2	3.2382812	2.140625	0.09375	-25.375006	-0.15625	-0.31250	15.137503
N-3	3.1592801	2.1534799	0.0670731	-42.374769	0.2682926	0.5365852	19.576519
N-4	3.1716804	2.1629518	0.0659898	-43.274732	0.2639593	0.5279186	19.934466
N-5	3.1729143	2.1637717	0.0659427	-43.325063	0.2637711	0.5275422	19.956505
N-6	3.1729984	2.1638253	0.0659407	-43.327685	0.2637629	0.5275258	19.957646
N-7	3.1730035	2.1638285	0.0659406	-43.327828	0.2637626	0.5275252	19.957687
N-8	3.1730037	2.1638286	0.0659406	-43.327831	0.2637626	0.5275252	19.957711
N-9	3.1730037	2.1638286	0.0659406	-43.327831	0.2637626	0.5275252	19.957711

Secondly, both Definition 1 and the result of Theorem 1 can readily be extended to general M-player dynamic games which incorporate a linear hierarchy. A counterpart of Theorem 1 in this general framework will lead to equilibrium strategies with the following property: The equilibrium strategy of the player at the top of hierarchy (i.e. $\mathcal{P}1$) forces the player at the second level of hierarchy (i.e. $\mathcal{P}2$) to force $\mathcal{P}3$ to force $\mathcal{P}4$...to force $\mathcal{P}M$ to minimize collectively the cost functional of $\mathcal{P}1$.

As another possible extension, one can allow more than one player at each level of hierarchy, with the players at each such level playing according to the Nash equilibrium solution concept among themselves [i.e. a counterpart of the analysis of Section V of Başar and Selbuz (1979) is possible in the present framework]. Furthermore, since the general framework of Theorem 1 pertains to games in normal (strategic) form, it also covers stochastic games, and therefore the sufficiency condition of Theorem 1 can also be applied to stochastic dynamic games; one such application has in fact been presented in Başar (1981). Finally, Theorem 1 may find direct applications in three-player differential games (i.e. dynamic games defined in continuous time) so as to obtain, for instance, the counterpart of the two-player results of Başar and Olsder (1980) and Papavassilopoulos and Cruz (1980) in case of three players with linear hierarchy.

Before concluding, we mention, as a word of caution for the reader, that the sufficiency condition of Theorem 1 may not always be satisfied in three-player dynamic games; in such cases the global minimum value of J_1 cannot be realized as the hierarchical (Stackelberg) equilibrium cost value of $\mathcal{P}1$, and one has to derive new tighter bounds on J_1. For a discussion on this issue in two-player dynamic games we refer the reader to a recent article by Başar (1980b).

Acknowledgement—Some part of the research reported in this paper was performed while the author was on sabbatical leave at Twente University of Technology, Enschede, Netherlands, whose support he gratefully acknowledges. The paper was submitted while the author was with Marmara Research Institute, Gebze, Kocaeli, Turkey.

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