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Closed-Loop Stackelberg Strategies with Applications in the Optimal Control of Multilevel Systems

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Abstract—**This paper develops a new approach to obtain the closed-loop Stackelberg (CLS) solution of an important class of two-person nonzero-sum dynamic games characterized by linear state dynamics and quadratic cost functionals. The new technique makes use of an important property of nonunique representations of a closed-loop strategy, and it relates the CLS solution to a particular representation of the optimal solution of a team problem. It is shown that, under certain conditions, the CLS strategies for the leader are linear and of the one-step memory type, while those of the follower can be realized in linear feedback form. Exact expressions are given for the optimal coefficient matrices involved, which can be determined recursively. These results are then extended to multilevel discrete-time control of linear-quadratic systems which are characterized by one central controller and K second-level controllers. Conditions are obtained under which a one-step memory strategy of the central controller forces the other controllers to a team-optimal solution, while each one of the K second-level controllers is in fact minimizing its own cost function.**

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I. INTRODUCTION

THE Stackelberg solution concept, first introduced in economics in the 1930's within the context of static economic competition [1], has entered the control literature through the works of Chen, Cruz, and Simaan ([2], [3], [4]), who have utilized its dynamic version within the context of hierarchical decision making. The dynamic Stackelberg solution is mostly appropriate in nonzero-sum two-person dynamic (differential) games when one of the players (the leader) has the ability (or enough power) to enforce his strategy on the other player (the follower). Hence, as opposed to the Nash equilibrium solution concept, the roles of the players are not symmetric in this case.

In deterministic differential games, when the players have access to only open-loop information, it is relatively simpler to obtain the necessary conditions that Stackelberg strategies should satisfy, and it is even possible to obtain the optimal strategies explicitly in the case of linear-quadratic (LQ) problems [2], [5]. However, in a

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differential game that characterizes a dynamic situation, it is not very reasonable to assume that the players have access to only open-loop information; and when players have access to dynamic closed-loop information, even though the Stackelberg concept still makes sense, the closed-loop Stackelberg (CLS) solution has not yet been obtained. One of the reasons for this is that the standard techniques of optimal control theory, and in particular dynamic programming, cannot be used to write down even a reasonable set of necessary conditions that the CLS strategies should satisfy. Hence, nonstandard techniques have to be introduced to obtain the CLS solution. However, hitherto this has not been done and derivation of CLS strategies has remained as an unsolved problem. On the other hand, from a conceptual point of view, the CLS solution could have several important applications, especially within the context of optimal control of multilevel systems [6].

Faced with the difficulty of obtaining the CLS solution, even in LQ problems, researchers have taken two different paths: one of these has been to assume specific structures for the closed-loop strategies of the players and to introduce the Stackelberg solution only with regard to the parameters that define those preselected structures. The other one has been to introduce a new hierarchical equilibrium solution concept under dynamic information so that dynamic programming techniques could be applicable, and this has led to the concept of feedback Stackelberg strategies. Medanic's work [17] is a recent example of the former approach, where it has been shown that for LQ nonzero-sum two-person differential games and under a linear feedback structure for the strategies of both players, optimal Stackelberg values of the coefficient matrices depend on the value (or the probability distribution) of the initial state. Feedback Stackelberg strategies on the other hand, have first been introduced by Simaan and Cruz [4], and later considered by several authors, the paper by Gardner and Cruz [7] being one of the most recent works in that area. For more details on these aspects of the Stackelberg solution concept, the reader is referred to two recent survey articles by Cruz [5], [8]. It should be mentioned here that a common feature of all the previous works done on the dynamic Stackelberg problem has been the memoryless (pure feedback) nature of the strategies of the players. This, however, turns out to be a rather severe restriction on the Stackelberg strategy of the leader, and besides, the resulting strategies are not computationally feasible either.

On the other hand, quite recently, there has been a breakthrough in understanding the intimate relation between information patterns and the Nash equilibrium solutions of differential games [9], [10]. In particular, it has been shown that nonzero-sum dynamic (differential)[1] games admit, in general, an uncountable number of the Nash equilibrium solutions under closed-loop information for at least one of the players; and the sole responsibility for occurrence of this phenomenon lies in the fact that every strategy can have several different representations under closed-loop information and at equilibrium [11].

In this paper, we make use of this theory of representations to completely solve for the CLS strategies of a class of LQ two-person dynamic games, and obtain sufficient conditions under which the CLS solution will exist. The derived CLS strategies of the leader are with one-step memory, and those of the follower are in feedback form; and these are obtained under no *a priori* restriction on the structure of the strategies of the players. One important property of the CLS strategies obtained is that they are representations of the optimal team solutions of related team problems. Exact expressions are given in Theorem 1 (Section IV) for the case when only the leader acts at the last stage of the game, and in Theorem 3 (Section VI) for the case when both players act at the last stage. Section II introduces the nonzero-sum dynamic game problem under consideration and provides a definition of the CLS solution concept. In Section III, we give a brief account of reasons why standard optimization techniques cannot be employed to obtain the CLS solutions of dynamic games, and we introduce the new indirect approach used in the paper. In Section V, we discuss one possible application and possible direction of extension of the results given in Section IV. In this context, we obtain the general solution of an optimization problem that involves two-level control of a large scale system, where at the second (lower) level there exist more than one controller. The paper ends with a discussion of the results obtained, with some suggestions for future research, and with an Appendix that provides a proof for Theorem 1. Two numerical examples included in Sections IV and VI illustrate some of the results of the paper, especially for dynamic games with sufficiently large number of stages. Numerical results for these two examples are tabulated in Tables I and II.

II. Problem Formulation and Definitions

Using the conventional notation for discrete-time systems, let θ denote the index set $\{0,1,\cdots,N-1\}$ and define the evolution of the state $x_{(\cdot)}$ over θ by the difference equation

$$x_{n+1}=A(n)x_n+B_1(n)u_n+B_2(n)v_n \tag{1}$$

where $u_{(\cdot)}$ is an r vector controlled by player 1 (the leader), $v_{(\cdot)}$ is an r' vector controlled by player 2 (the follower), $x_{(\cdot)}$ is the state vector of dimension m, $A(\cdot)$, $B_1(\cdot)$, and $B_2(\cdot)$ are matrices of appropriate dimensions. All this information and the value of x_0 are known to both players.

The cost functions for players 1 and 2 are given by $J_1(N,0)$ and $J_2(N,0)$, respectively, where

$$J_1(N,M)=x_N^TQ_1(N)x_N + \sum_{n=M}^{N-1}\left[x_n^TQ_1(n)x_n+u_n^TR_{11}(n)u_n+v_n^TR_{12}(n)v_n\right] \tag{2a}$$

[1] If a differential game is formulated in discrete time, we call it a "dynamic game."

$$J_2(N,M)=x_n^TQ_2(N)x_N + \sum_{n=M}^{N-1}\left[x_n^TQ_2(n)x_n+u_n^TR_{21}(n)u_n+v_n^TR_{22}(n)v_n\right] \quad (2b)$$

with $R_{11}(n)>0$, $R_{12}(n)>0$, $R_{22}(n)>0$, $\forall n\in\theta$, $R_{21}(n)\geqslant 0$, $Q_1(n)\geqslant 0$, $Q_2(N)\geqslant 0$, $\forall n\in\theta$, and $Q_i(N)\geqslant 0$, $i=1,2$.

Each player has access to closed-loop perfect state information. If η_n denotes the set

$$\eta_n=\{x_0,x_1,\cdots,x_n\}, \quad (3)$$

then each η_n will generate an information space Z_n at stage n, as a subset of $\mathcal{R}^{m(n+1)}$. Permissible strategy γ_{1n} for player 1 at stage n will be a measurable mapping from Z_n into $\mathcal{R}^r$. Similarly, player 2 will pick a measurable mapping γ_{2n} from Z_n into $\mathcal{R}^{r'}$. Let us denote the corresponding set of strategies at stage n by $\Gamma_{1,n}$ and $\Gamma_{2,n}$, for players 1 and 2, respectively.

Within the framework of this dynamic game problem, the (CLS) solution concept can be introduced as follows: the leader determines an N-tuple of strategies $\{\gamma_{1n}\in\Gamma_{1,n}, n\in\theta\}$ whose realizations are denoted above by $\{u_n, n\in\theta\}$; and he announces those strategies prior to the start of the game. Then, knowing what the leader's strategies (but *not* the control values) are, the follower selects an N-tuple of strategies $\{\gamma_{2n}\in\Gamma_{2,n}, n\in\theta\}$ that minimize his own cost function $J_2(N,0)$. This choice, of course, affects the value of $J_1(N,0)$, and thus the leader has to take this into account before announcing his strategies; in other words, he must select a set of strategies that minimize $J_1(N,0)$ over the rational reaction set of the follower. To put this into precise mathematical terms, let us first introduce the notation

$$\gamma_i\in\Gamma^i\Rightarrow\{\gamma_{in}\in\Gamma_{i,n}, n\in\theta\}, \qquad i=1,2, \quad (4)$$

$$\tilde{J}_i(\gamma_1,\gamma_2)\triangleq J_i(N,0)|\{u_n=\gamma_{1n}(\eta_n), v_n=\gamma_{2n}(\eta_n), n\in\theta\}, \quad i=1,2, \quad (5)$$

and define the rational reaction set of the follower by

$$R(\gamma_1)=\{\gamma_2^0\in\Gamma^2:\tilde{J}_2(\gamma_1,\gamma_2^0)\leqslant\tilde{J}_2(\gamma_1,\gamma_2),\ \forall\gamma_2\in\Gamma^2\}. \quad (6)$$

If γ_1 is restricted to a proper subset of Γ^1 which is comprised of only affine maps, then $R(\gamma_1)$ will be a singleton for each γ_1 (for the problem posed above). However, for elements of Γ^1 outside the affine class, the reaction set $R(\gamma_1)$ will, in general, not be a singleton. This possibility then forces us to give the following definition as an N-tuple of CLS strategies for the leader.

Definition 1: For the dynamic game posed above, a strategy N-tuple $\gamma_1^*\in\Gamma^1$ constitutes a closed-loop Stackelberg (CLS) strategy N-tuple for the leader if

$$\sup\tilde{J}_1(\gamma_1^*,\gamma_2)\leqslant\sup\tilde{J}_1(\gamma_1,\gamma_2), \qquad \forall\gamma_1\in\Gamma^1 \quad (7)$$

where the supremum is taken over $\gamma_2\in R(\gamma_1^*)$ on the left-hand side (LHS) of (7) and over $\gamma_2\in R(\gamma_1)$ on the right-hand side (RHS). The CLS cost of the leader is then

$$J_1^*=\sup_{R(\gamma_1^*)}\tilde{J}_1(\gamma_1^*,\gamma_2)=\inf_{\Gamma^1}\sup_{R(\gamma_1)}\tilde{J}_1(\gamma_1,\gamma_2). \quad (8)$$

III. A New Indirect Approach for Derivation of CLS Strategies

It can readily be observed from the formulation given in Section II and from Definition 1 that standard optimization techniques cannot be successfully used to obtain the CLS solution; firstly, because $R(\gamma_1)$ is in general not a singleton, and secondly, even if it is a singleton, minimization of $\tilde{J}_1(\gamma_1,\gamma_2)$ over $R(\gamma_1)$ is not a standard optimization problem where dynamic programming could be applicable, since the dynamic nature of the game problem makes it impossible for elements of $R(\gamma_1)$ to be expressed in closed form in terms of γ_1. This, evidently, makes it impossible to write down even a set of necessary conditions that the CLS strategies of the leader should satisfy.

Even though this problem is intractable using standard techniques of dynamic optimization and optimal control theory, it still is a mathematically well-posed problem since the quantity J_1^* defined by (8) is well defined and bounded from below. In fact, we also have the property that "a CLS strategy N-tuple exists if a sequence of policies $\{\gamma_1^{(j)}\}$ under which $\{\sup_{R(\gamma_1^{(j)})}\tilde{J}_1(\gamma_1^{(j)},\gamma_2)\}$ converges to J_1^* admits a limit in Γ^1." Hence, an indirect way of attacking the problem would be to investigate whether it would be possible to determine J_1^* without obtaining the solution and to see further whether there exists an element in Γ^1 for which J_1^* is attained. To this end we can give the following lower bound for J_1^*.

Proposition 1: For the dynamic game problem under consideration,

$$J_1^*\geqslant\min_{\gamma_1\in\Gamma^1}\min_{\gamma_2\in\Gamma^2}\tilde{J}_1(\gamma_1,\gamma_2)=\tilde{J}_1(\gamma_1^l,\gamma_2^l). \quad (9)$$

Proof: This bound readily follows, since $\tilde{J}_1(\gamma_1^l,\gamma_2^l)$ is a global minimum for $\tilde{J}_1$ (which exists by strict convexity). □

This lower bound might at first seem to be rather coarse and not of much use as far as the CLS solution is concerned. However, it will be shown in the next section that this is not so, and the bound could be tight and be achieved by an appropriate strategy N-tuple of the leader under certain conditions on the parameters of the game. Before going into details of its verification, it will be appropriate to discuss here, in general terms, the motivation in arriving such a conclusion.

To this end, we first note that the RHS of (9) defines a joint optimization (team) problem which is of the linear-quadratic nature for the problem formulated in Section II. Hence, it admits a *unique* solution in *feedback* form, which is linear in the current value of the state vector. Let us denote that solution by $\{\gamma_{1n}^l(x_n),\gamma_{2n}^l(x_n), n\in\theta\}$, and its resulting cost by $J_1^l=\tilde{J}_1(\gamma_1^l,\gamma_2^l)$. However, γ_1^l and γ_2^l are

not the only elements of Γ^1 and Γ^2, respectively, that yield the optimal cost J_1^t. Any one of the closed-loop *representations* of those strategies[2] (and there are uncountably many of them) also constitutes a well-defined solution to the problem. Let us denote such strategies of player 1 by

$$\gamma_{1n}^p(\eta_n), \qquad n\in\theta \tag{10}$$

where p stands for a particular representation chosen.

Now the significance of (10) within the context of our problem is the following: no matter what representation player 1 chooses, the minimum of $\tilde{J}_1(\gamma_1^p,\gamma_2)$ over Γ^2 is still attained by the strategy N-tuple $\{\gamma_{2n}^t,\ n\in\theta\}$. But this brings so much flexibility to player 1 that he can change the structure of the cost function of player 2 "drastically" without changing the team values of his own strategies. In particular, there might exist a representation $\gamma_1^{p^*}$ under which the minimum of $\tilde{J}_2(\gamma_1^{p^*},\gamma_2)$ over Γ^2 is attained by the team optimal strategies $\{\gamma_{2n}^t,\ n\in\theta\}$. This thus brings us to the following conclusion.

Proposition 2: If there exists a closed-loop representation $\gamma_1^{p^*}$ of the feedback team-optimal strategies $\{\gamma_{1n}^t,\ n\in\theta\}$ such that

$$\inf_{\Gamma^2}\tilde{J}_2(\gamma_1^{p^*},\gamma_2)=\tilde{J}_2(\gamma_1^{p^*},\gamma_2^t), \tag{11}$$

then $\gamma_1^{p^*}\in\Gamma^1$ constitutes a CLS strategy N-tuple for the leader and $\gamma_2^t\in\Gamma^2$ is a CLS strategy N-tuple for the follower.

Proof: This follows readily since then the bound in (9) is attained. □

In the next section, we show that, under certain conditions on the parameters of the game, there indeed exists such a representation within the class of one-step memory strategies for the leader. The key assumption that we will make at this point is the following.

Assumption 1: At the last stage of the game only the leader acts.

This amounts to having $B_2(N-1)=0$ in the formulation of the problem in Section II, and it will be a running assumption in the next section. We should note, though, that it is possible to obtain the CLS strategies of the dynamic game even in the absence of Assumption 1, and this has been done in Section VI by making use of the optimal solution of a different team problem.

IV. CLS Strategies Under Assumption 1

A. Solution of the Related Team Problem

In order to obtain expressions for the CLS strategies of the dynamic game formulated in Section II and with $B_2(N-1)=0$, we now attempt to utilize Proposition 2. To this end, we first solve the related team problem which is

$$\underset{\{\gamma_{1n}\in\Gamma_{1,n},\,\gamma_{2n}\in\Gamma_{2,n};\ n\in\theta\}}{\text{minimize}}\ \tilde{J}_1(\gamma_1,\gamma_2) \tag{12a}$$

subject to the state dynamics (1) and with

$$B_2(N-1)=0. \tag{12b}$$

Its solution is given below in Lemma 1.

Lemma 1: In feedback strategies, the joint optimization (team) problem (12) admits a unique solution given by

$$\gamma_{1n}^t(x_n)=-L_1(n)x_n, \qquad n\in\theta \tag{13a}$$

$$\gamma_{2n}^t(x_n)=-L_2(n)x_n, \qquad n\in\theta-\{N-1\} \tag{13b}$$

and with minimum cost

$$J_1^t=x_0^TM(0)x_0 \tag{14}$$

where

$$L_1(n)=\left[R_{11}(n)+K_1(n)B_1(n)\right]^{-1}K_1(n)A(n) \tag{15a}$$

$$L_2(n)=\left[R_{12}(n)+K_2(n)B_2(n)\right]^{-1}K_2(n)A(n) \tag{15b}$$

$$\begin{aligned}K_1(n)=B_1^T(n)M(n+1)\big\{&I-B_2(n)\\&\cdot\left[R_{12}(n)+B_2^T(n)M(n+1)B_2(n)\right]^{-1}\\&\cdot B_2^T(n)M(n+1)\big\}\end{aligned} \tag{16a}$$

$$\begin{aligned}K_2(n)=B_2^T(n)M(n+1)\big\{&I-B_1(n)\\&\cdot\left[R_{11}(n)+B_1^T(n)M(n+1)B_1(n)\right]^{-1}\\&\cdot B_1^T(n)M(n+1)\big\}\end{aligned} \tag{16b}$$

and with $M(\cdot)$ defined recursively by

$$\begin{aligned}M(n)=&\,Q_1(n)+F^T(n)M(n+1)F(n)\\&+L_1^T(n)R_{11}(n)L_1(n)\\&+L_2^T(n)R_{12}(n)L_2(n);\\M(N)=&\,Q_1(N)\end{aligned} \tag{17a}$$

where

$$F(n)\triangleq A(n)-B_1(n)L_1(n)-B_2(n)L_2(n). \tag{17b}$$

Proof: The proof follows readily from the standard LQ discrete-time regulator theory by appropriate decomposition. A more direct derivation would be to obtain the person-by-person (pbp) optimal solution of (12) using dynamic programming. Since J_1 is strictly convex, every pbp solution is also team optimal. Hence, Lemma 1 also follows from [10, Theorem 3] as a special case. □

B. Derivation of the CLS Strategies

In the light of Proposition 2, and by making use of Lemma 1, we now attempt to obtain the CLS strategies of the leader in the form of a representation of (13a) and to

[2]For a definition of closed-loop representations of a strategy, the reader is referred to [10], [11], or [12].

find the conditions under which such a solution is valid. In particular, we adopt the one-step memory representation

$$\gamma_{1n}^{P}(x_n, x_{n-1}) = -L_1(n)x_n + P_n[x_n - \bar{x}_n(n-1)], \quad n \in \theta - \{0\} \tag{18a}$$

where P_n is an $(r \times n)$ matrix yet to be determined, and

$$\bar{x}_n(n-1) = F(n-1)x_{n-1} \tag{18b}$$

denotes the team-equilibrium value of x_n in terms of x_{n-1}. Optimum choices for $\{P_n,\ n \in \theta - \{0\}\}$ will then be determined by substitution of (18a) into J_2, minimizing the resulting expression over the controls of the follower, and by comparing the resulting strategies with the team-optimal strategies (13b). The result is given below in Theorem 1 after introducing the following notation.

Preliminary notation for Theorem 1:

$\Sigma(\cdot)$: an $(r' \times r')$ matrix defined recursively by

$$\begin{aligned}\Sigma(n) &= Q_2(n) + F^T(n)\Sigma(n+1)F(n) \\ &\quad + L_1^T(n)R_{21}(n)L_1(n) \\ &\quad + L_2^T(n)R_{22}(n)L_2(n); \\ \Sigma(N) &= Q_2(N).\end{aligned} \tag{19}$$

Λ_n: an $(r \times m)$ matrix defined recursively by (as a function of $\{P_{n+1}, P_{n+2}, \cdots, P_{N-1}\}$)

$$\begin{aligned}\Lambda_n &= B_1^T(n)P_{n+1}^T\Lambda_{n+1}F(n) - R_{21}(n)L_1(n) \\ &\quad + B_1^T(n)\Sigma(n+1)F(n); \\ \Lambda_N &= 0.\end{aligned} \tag{20}$$

Condition 1: There exists at least one matrix sequence $\{P_{N-1}, P_{N-2}, \cdots, P_1\}$ that satisfies recursively the linear matrix equation

$$\begin{aligned}B_2^T(n-1)P_n^T\Lambda_n F(n-1) &= -B_2^T(n-1)\Sigma(n)F(n-1) \\ &\quad + R_{22}(n-1)L_2(n-1), \qquad n \in \theta\end{aligned} \tag{21}$$

where Λ_n is related to $\{P_{n+1}, \cdots, P_{N-1}\}$ through (20).

Theorem 1: Let Condition 1 be satisfied and let $\{P_{N-1}^*, \cdots, P_1^*\}$ denote one such sequence. Then, under Assumption 1, there exists a CLS solution for the dynamic game of Section II, which is given by

$$\gamma_{1n}^*(x_n, x_{n-1}) = [P_n^* - L_1(n)]x_n - P_n^*F(n-1)x_{n-1}; \quad n \in \theta - \{0\} \tag{22a}$$

$$\gamma_{10}^*(x_0) = -L_1(0)x_0, \tag{22b}$$

$$\gamma_{2n}^*(x_n) = -L_2(n)x_n, \qquad n \in \theta - \{N-1\}, \tag{23}$$

and the CLS cost of the leader and the follower are given, respectively, by

$$J_1^* = x_0^T M(0)x_0, \tag{24a}$$

$$J_2^* = x_0^T \Sigma(0)x_0. \tag{24b}$$

Proof: A proof of the Theorem is provided in the Appendix. □

Remark 1: One of the significances of the CLS solution presented in Theorem 1 is that the coefficient matrices P_n^* can be computed recursively, and starting at $n = N-1$. At each stage n, determination of P_n^* involves the solution of a linear matrix equation (21) which is a function of the coefficient matrices $\{P_i^*, i > n\}$ through Λ_n which is also determined recursively. Yet another significance of the result is that, under Condition 1 and by announcing the strategies (22), the leader can force the other player to an optimum team solution while the latter is in fact minimizing his own cost function. The importance of this result within the context of multilevel control of large scale systems will be elaborated on in Section V.

Remark 2: One other important property of the CLS solution presented in Theorem 1 is that it is also a closed-loop Nash equilibrium solution for the dynamic game under consideration. To see this, we first note the inequality

$$\tilde{J}_2(\gamma_1^*, \gamma_2^*) \leqslant \tilde{J}_2(\gamma_1^*, \gamma_2), \qquad \forall \gamma_2 \in \Gamma^2 \tag{25a}$$

since $\{\gamma_1^*, \gamma_2^*\}$ is a CLS solution. Furthermore, we note the inequality

$$\tilde{J}_1(\gamma_1^*, \gamma_2^*) \leqslant \tilde{J}_1(\gamma_1, \gamma_2^*), \qquad \forall \gamma_1 \in \Gamma^1 \tag{25b}$$

since the CLS solution is also team-optimal under $\tilde{J}_1$. Inequalities (25a) and (25b) considered together thus imply that the given solution is in Nash equilibrium. It is well-known that the dynamic game of Section II in fact admits an uncountable number of closed-loop Nash equilibrium solutions [10], but in this case, since the leader can announce his strategy ahead of time, he can enforce that particular Nash solution that is most favorable to him, which is the one given by (22).

Remark 3: It should be noted that Condition 1 depends on the general parameters of the dynamic game, but in its present form it also depends on the solution of a recursive equation (20). It would, of course, be more convenient to express the said condition directly in terms of the parameters of the problem. But, unfortunately, no such general condition is yet available. Roughly speaking, Condition 1 will be valid if, by employing one-step memory strategies as representations of the team solution γ_1^t, player 1 can manage to "structurally control" the cost function of player 2; in other words, he should be able to enforce the desired quadratic structure on $\tilde{J}_2$ so that its minimum is attained at $\{\gamma_{2n}^t, n \in \theta\}$. If Condition 1 does not hold true, however, then it is still possible to obtain the CLS strategy as a closed-loop representation of the team solution γ_1^t, but now within the class of k-step memory ($k \geqslant 2$) strategies. Such a result has actually been verified within the context of a numerical example considered in [13] (see also [14]), and this verification could easily be extended to the dynamic game problem of Section II and with $B_2(N-1) = 0$, so as to obtain conditions of existence of CLS strategies within this more

general class. But this extension will not be pursued in the present paper.

Remark 4: In the formulation of the dynamic game problem in Section II, we have taken R_{12} to be positive definite; that is, we have assumed that the quadratic cost function of the leader includes terms that assign positive quadratic cost to all the control variables of the follower. As it has been discussed in [14], such an assumption is essential in order to have a well-formulated problem. It has been shown in [13] within the context of a scalar game problem that, in the absence of such an assumption, there does not exist a CLS strategy for the leader. However, it is still possible to determine a closed-loop strategy for the leader that brings his cost arbitrarily close to the related team cost. Such a result was also established earlier in [15] and [16] for specific examples and within the class of linear strategies.

C. A Numerical Example

We now consider a scalar $(N-k)$-stage dynamic game to illustrate the result of Theorem 1, the validity (nonvoidness) of Condition 1, and convergence properties of the CLS solution as $(N-k)$ becomes large.

The linear state equation is defined by

$$x_N = x_{N-1} + u_{N-1} \tag{26a}$$

$$x_{n+1} = x_n + u_n + v_n, \qquad n \leqslant N-2 \tag{26b}$$

and the stationary cost functions are given by

$$J_1 = x_N^2 + \sum_{n=k}^{N-1} x_n^2 + 2u_n^2 + v_n^2 \tag{27a}$$

$$J_2 = x_N^2 + \sum_{n=k}^{N-1} x_n^2 + u_n^2 + 3v_n^2. \tag{27b}$$

For this problem, the relevant terms are determined from

$$L_1(n) = K_1(n)/[2+K_1(n)], \qquad n \leqslant N-1$$

$$L_2(n) = K_2(n)/[1+K_2(n)], \qquad n \leqslant N-2$$

$$K_1(n) = M(n+1)/[1+M(n+1)], \qquad n \leqslant N-1$$

$$K_2(n) = 2M(n+1)/[2+M(n+1)], \qquad n \leqslant N-2$$

$$F(n) = 1 - L_1(n) - L_2(n)$$

$$M(n) = 1 + F^2(n)M(n+1) + 2L_1^2(n) + L_2^2(n); \qquad M(N) = 1.$$

$$\Sigma(n) = 1 + F^2(n)\Sigma(n+1) + L_1^2(n) + 3L_2^2(n); \qquad \Sigma(N) = 1.$$

Furthermore, (20) and (21) can be written as

$$\Lambda_n = P_{n+1}F(n)\Lambda_{n+1} - L_1(n) + \Sigma(n+1)F(n), \qquad \Lambda_N = 0; \tag{28a}$$

$$P_n\Lambda_n F(n-1) = -\Sigma(n)F(n-1) + 3L_2(n-1), \qquad n \leqslant N-1. \tag{28b}$$

It can recursively be shown that $\Lambda_n > 0$, $F(n) > 0$ for all $n \leqslant N-1$, and thus optimal coefficients P_n^* are given by

$$P_n^* = [3L_2(n-1) - \Sigma(n)F(n-1)]/[\Lambda_n F(n-1)]$$

where

$$\Lambda_n = 3L_2(n) - L_1(n), \qquad n \leqslant N-2$$

which follows by substitution of (28b) into (28a).

Numerical values for these terms are given in Table I for $n = N-1, \cdots, N-11$, as well as the corresponding CLS strategies from (22) and (23). The tabulated results indicate that there is a rapid convergence in the CLS strategies to the stationary policies

$$\gamma_1^*(x_n, x_{n-1}) = 1.965011x_n - 0.668521x_{n-1}, \tag{29a}$$

$$\gamma_2^*(x_n) = -0.457427x_n \tag{29b}$$

to the nearest six figures.

V. An Application to Multilevel Control of Large-Scale Systems

Consider a central controller and several (say K) secondary (local) controllers with the prime objective being optimal control of a discrete-time system

$$x_{n+1} = A(n)x_n + B_0(n)u(n) + \sum_{i=1}^{K} B_i(n)v_i(n); \quad x_0 \text{ given},$$
$$\dim x_n = m, \quad \dim u = r_0, \quad \dim v_i = r_i, \tag{30a}$$

under the quadratic cost function

$$J_0 = x_N^T Q_0(N) x_N + \sum_{n=0}^{N-1} \Bigg[x_n^T Q_0(n) x_n + u^T(n) R_{00}(n) u(n) + \sum_{i=1}^{K} v_i^T(n) R_{0i}(n) v_i(n) \Bigg], \tag{30b}$$
$$Q_0(\cdot) \geqslant 0, \quad R_{00}(\cdot) > 0, \quad R_{0i}(\cdot) > 0,$$

where $u(\cdot)$ denotes the control vector of the central controller (the leader) and $v_i(\cdot)$, $i = 1, \cdots, K$, denote those of the secondary controllers (the followers). Now, let us further assume that the secondary controllers do not have access to J_0, but select their strategies based on the announced strategies of the central controller and each with the intention of minimizing a "local" cost function

$$J_i = x_N^T Q_i(N) x_N + \sum_{n=0}^{N-1} x_n^T Q_i(n) x_n + u^T(n) R_{i0}(n) u(n) + v_i^T(n) R_{ii}(n) v_i(n) \tag{30c}$$
$$Q_i(\cdot) \geqslant 0, \quad R_{i0}(\cdot) \geqslant 0, \quad R_{ii}(\cdot) > 0,$$

for the ith controller $(i = 1, \cdots, K)$. Actually, each secondary controller might have his own dynamics, but this can still be formulated as in (30a) with all the dynamics

TABLE I
CLS SOLUTION FOR THE NUMERICAL EXAMPLE OF IV-C

n	$L_1(n)$	$L_2(n)$	$F(n)$	P_n^*	$M(n)$	$\Sigma(n)$	$\gamma_{1n}^*(x_n, x_{n-1}) = [P_n^* - L_1(n) \mid x_n - P_n^* F(n-1) x_{n-1}$	$\gamma_{2n}^*(x_n) = -L_2(n) x_n$
N	-	-	-	-	1	1	-	-
N-1	0.2	-	0.8	5.800000	1.72	1.68	$5.600000\, x_{N-1} - 1.620112\, x_{N-2}$	-
N-2	0.240223	0.480447	0.279330	2.131402	1.480447	1.881277	$1.891178\, x_{N-2} - 0.661788\, x_{N-3}$	$-0.480447\, x_{N-2}$
N-3	0.229835	0.459670	0.310494	2.184982	1.459670	1.868083	$1.955146\, x_{N-3} - 0.685053\, x_{N-4}$	$-0.459670\, x_{N-3}$
N-4	0.228824	0.457648	0.313528	2.191492	1.457648	1.865615	$1.962668\, x_{N-4} - 0.687749\, x_{N-5}$	$-0.457648\, x_{N-4}$
N-5	0.228724	0.457449	0.313827	2.193482	1.457449	1.863832	$1.964757\, x_{N-5} - 0.688438\, x_{N-6}$	$-0.457449\, x_{N-5}$
N-6	0.228715	0.457429	0.313856	2.193699	1.457429	1.863633	$1.964984\, x_{N-6} - 0.688512\, x_{N-7}$	$-0.457429\, x_{N-6}$
N-7	0.228714	0.457427	0.313859	2.193722	1.457427	1.863611	$1.965008\, x_{N-7} - 0.688520\, x_{N-8}$	$-0.457427\, x_{N-7}$
N-8	0.228714	0.457427	0.313859	2.193724	1.457427	1.863609	$1.965011\, x_{N-8} - 0.688521\, x_{N-9}$	$-0.457427\, x_{N-8}$
N-9	0.228714	0.457427	0.313859	2.193725	1.457427	1.863608	$1.965011\, x_{N-9} - 0.688521\, x_{N-10}$	$-0.457427\, x_{N-9}$
N-10	0.228714	0.457427	0.313859	2.193725	1.457427	1.863608	$1.965011\, x_{N-10} - 0.688521\, x_{N-11}$	$-0.457427\, x_{N-10}$
N-11	0.228714	0.457427	0.313859		1.457427	1.863608		

collected together and the $Q_i(\cdot)$ matrix in (30c) having "zeros" at the appropriate locations.

Now, the question is whether it is possible for the leader to announce an appropriate strategy (say at most a one-step memory strategy) such that by minimizing their own "local" cost functions, the local controllers also collectively minimize the "central" cost function J_0. This problem can, of course, be formulated within the framework of the CLS solution concept; however, since the cost function of the ith follower will in general be a function of the jth ($j \neq i$) follower, either through $x_{(\cdot)}$ or through the announced strategy of the central controller, we have to introduce an additional equilibrium concept among the K local controllers. The most natural one that fits within this framework is the Nash equilibrium solution concept. We thus seek a set of strategies $\{\gamma_0^* \in \Gamma^0, \gamma_1^* \in \Gamma^1, \cdots, \gamma_K^* \in \Gamma^K\}$ such that

$$J_0(\gamma_0^*, \cdots, \gamma_K^*) \leq J_0(\gamma_0, \gamma_1, \cdots, \gamma_K) \tag{31}$$

$$J_1(\gamma_0^*, \cdots, \gamma_K^*) \leq J_1(\gamma_0^*, \gamma_1, \gamma_2^*, \cdots, \gamma_K^*)$$

$$\vdots$$

$$J_K(\gamma_0^*, \cdots, \gamma_K^*) \leq J_K(\gamma_0^*, \cdots, \gamma_{K-1}^*, \gamma_K) \tag{32}$$

for all $\gamma_0 \in \Gamma^0, \cdots, \gamma_K \in \Gamma^K$, where notation is an appropriate extension of that of (4).

In order to avoid the possibility of nontrivial nonuniqueness of the solution (a technical point that will become clear later), we assume that the strategy sets $\Gamma^1, \cdots, \Gamma^K$ of the secondary controllers are comprised of only feedback control laws (but not necessarily local feedback). Then, conditions under which (31)–(32) will admit an essentially unique solution, and the exact expressions for that solution, are obtained below as a direct extension of Theorem 1. To this end, we first determine the class of control laws that solve (31), as a counterpart of Lemma 1.

Preliminary notation: Definition of matrices which appear in Lemma 2:

$$B(n) = \left[\, B_0(n) \mid B_1(n) \mid \cdots \mid B_K(n) \,\right]_{m \times \bar{r}} \tag{33a}$$

$$\bar{r} = \sum_{i=0}^{K} r_i \tag{33b}$$

$R_0(n)$: an $\bar{r} \times \bar{r}$ matrix block diagonal in terms of $(r_i \times r_i)$ matrices, the $(j+1)$th block given by $R_{0j}(n)$.

(33c)

$$T_i \triangleq [\underbrace{0 \mid \cdots \mid 0}_{r_i \times \sum_{j=0}^{i-1} r_j} \mid I_{r_i} \mid \underbrace{0 \mid \cdots \mid 0}_{r_i \times \sum_{j=i+1}^{K} r_j}]; \qquad (r_i \times \bar{r}) \tag{33d}$$

$L(n)$: an $(\bar{r} \times m)$ matrix given by

$$L(n) = \left[\, R_0(n) + B^T(n) S(n) B(n) \,\right]^{-1} B^T(n) S(n) A(n), \tag{34a}$$

where $S(n)$ is recursively defined by

$$S(n) = Q_0(n) + F^T(n) S(n+1) F(n) + L^T(n) R_0(n) L(n);$$
$$S(N) = Q_0(N) \tag{34b}$$

and

$$F(n) \triangleq A(n) - B(n) L(n)$$
$$\triangleq A(n) - \sum_{i=0}^{K} B_i(n) L_i(n) \tag{34c}$$

where the last expression defines L_i, i.e.,

$$L_i(n) \triangleq T_i L(n). \tag{34d}$$

Lemma 2: In feedback strategies, the joint optimization (team) problem (31) admits a unique solution given by

$$\gamma_{in}^*(x_n) = -L_i(n) x_n, \qquad i = 0, \cdots, K, \; n \in \theta, \tag{35a}$$

and with minimum cost

$$J_0^t = x_0^T S(0) x_0. \tag{35b}$$

Proof: This result is a direct consequence of the standard LQ discrete-time regulator theory. □

We now attempt, as in the case of Theorem 1, to find conditions under which a representation of $\gamma_{0n}^t(\cdot)$, in the one-step memory form

$$\gamma_{0n}^P(x_n, x_{n-1}) = -L_0(n)x_n + P_n[x_n - F(n-1)x_{n-1}], \tag{36}$$

will constitute a CLS solution to the problem described by (31)–(32) while $\{\gamma_1^*, \cdots, \gamma_K^*\}$ are in Nash equilibrium. The result is given below in Theorem 2.

Preliminary notation for Theorem 2:

$\Sigma_i(\cdot)$: an $(r_i \times r_i)$ matrix defined recursively by

$$\begin{aligned}\Sigma_i(n) &= Q_i(n) + F^T(n)\Sigma_i(n+1)F(n) \\ &\quad + L_0^T(n)R_{i0}(n)L_0(n) \\ &\quad + L_i^T(n)R_{ii}(n)L_i(n); \\ \Sigma_i(N) &= Q_i(N) \qquad i = 1, \cdots, K.\end{aligned} \tag{37}$$

$\Lambda_i(n)$: an $(r \times m)$ matrix defined recursively by (as a function of $\{P_{n+1}, P_{n+2}, \cdots, P_{N-1}\}$)

$$\begin{aligned}\Lambda_i(n) &= B_0^T(n)P_{n+1}^T\Lambda_i(n+1)F(n) - R_{i0}L_0(n) \\ &\quad + B_0^T(n)\Sigma_i(n+1)F(n); \\ \Lambda_i(N) &= 0.\end{aligned} \tag{38}$$

Condition 2: There exists at least one matrix sequence $\{P_{N-1}, P_{N-2}, \cdots, P_1\}$ that satisfies recursively the K-tuple of linear matrix equations

$$\begin{aligned}B_i^T(n-1)P_n^T\Lambda_i(n)F(n-1) &= -B_i^T(n-1)\Sigma_i(n)F(n-1) \\ &\quad + R_{ii}(n-1)L_i(n-1); \qquad n \in \theta - \{0\}\end{aligned} \tag{39}$$

jointly for all $i = 1, \cdots, K$, where $\Lambda_i(n)$ is related to $\{P_{n+1}, \cdots, P_{N-1}\}$ through (38).

Theorem 2: Let Condition 2 be satisfied and let $\{P_{N-1}^*, \cdots, P_1^*\}$ denote one such sequence. Then, there exists a solution to the multilevel control problem introduced in this section and defined by inequalities (31)–(32). An optimal strategy for the central controller is given by

$$\gamma_{0n}^*(x_n, x_{n-1}) = [P_n^* - L_0(n)]x_n - P_n^*F(n-1)x_{n-1}; \qquad n \in \theta - \{0\}, \tag{40a}$$

$$\gamma_{00}^*(x_0) = -L_0(0)x_0, \tag{40b}$$

and the optimal strategies of the secondary level controllers are given uniquely by

$$\gamma_{in}^*(x_n) = -L_i(n)x_n, \qquad n \in \theta - \{N-1\}. \tag{41}$$

The optimal cost of the central controller is

$$J_0^* = x_0^T S(0) x_0, \tag{42a}$$

while that of the ith controller $(i = 1, \cdots, K)$ is

$$J_i^* = x_0^T \Sigma_i(0) x_0. \tag{42b}$$

Proof: Now let us fix the control strategies of $K-1$ secondary level controllers (except the jth one) at (41) and consider inequality (31) together with that of (32) corresponding to the jth secondary level controller. Then these two inequalities define a CLS solution in the light of Remark 2, and thus Condition 2, (40) and (41) for $i = j$ follow directly from Theorem 1 [the equilibrium strategies of the other controllers enter the problem only through $F(\cdot)$]. But since j was arbitrary, the result follows, provided that we write a condition like that of Condition 1 for each secondary controller, which is Condition 2. Finally, the uniqueness of (41) follows since under (40), inequalities (32) define an LQ K-player nonzero-sum game, whose Nash solution is well known to be unique and linear in feedback strategies [10], [11]. Note that if memory strategies were also allowed for the secondary level controllers, this result would no longer be valid. □

Remark 5: In general, for Condition 2 to be valid, one will have to require that $r_0 \geq \sum_{i=1}^K r_i$, so that the central controller can have enough flexibility and "power" to force all the other controllers to the optimal team solution while minimizing their own cost functions. Yet another point that should be stressed is that in the formulation of the multilevel control problem we assumed the secondary (local) controllers to have access to the entire current state vector. However, a more reasonable assumption would be to think of each local controller with his own dynamics and having access to only his own state vector. For a specific model, if it turns out that the solution presented in Theorem 2 possesses this property (i.e., if the jth controller's local state $(j = 1, \cdots, K)$ is sufficient information), then this requirement is automatically satisfied. If that is not the case, such a local feedback solution will have to be enforced. This also brings us to the model proposed by Cruz in [6], whose solution will be considered in a forthcoming paper, using the techniques developed here.

VI. CLS Strategies in the Absence of Assumption 1

A. Derivation of a Tight Lower Bound and Solution of the Related Team Problem

Let us now consider the CLS solution of the multistage LQ two-person dynamic game formulated in Section II in the absence of Assumption 1. That is, the dynamic game is defined, in discrete time, by the state dynamics (2), and with $B_2(N-1)$ not necessarily equal to zero. Now, when the follower also acts at the last stage $(N-1)$, the optimal team cost of Proposition 1 no longer constitutes a tight (attainable) lower bound for J_1. It is, however, possible to introduce a different team problem whose optimal cost provides such a bound. To this end, we first note the following property of the rational reaction set $R(\gamma_1)$.

Proposition 3: For the dynamic game problem formulated in this section, the reaction set $R(\gamma_1)$ defined by (6), has the property that, for each $\gamma_{1,N-1}\in\Gamma_{1,N-1}$, the last r'-dimensional component of $\gamma_2^0\in R(\gamma_1)$ is uniquely given as

$$\begin{aligned}\gamma^0_{2,N-1}(x_{N-1},\gamma_{1,N-1})&=-\left[R_{22}(N-1)+B_2^T(N-1)Q_2(N)B_2(N-1)\right]^{-1}\\&\cdot B_2^T(N-1)Q_2(N)\\&\cdot\left[A(N-1)x(N-1)+B_1(N-1)\gamma_{1,N-1}(\eta_{N-1})\right].\end{aligned}\tag{43}$$

Proof: This result follows directly from the definition of $R(\gamma_1)$, as given by (6), and by minimizing $\tilde{J}_2(\gamma_1,\gamma_2)$ over $\gamma_{2,N-1}\in\Gamma_{2,N-1}$, since the selection of $\gamma_1\in\Gamma^1$ does not affect the structure of this minimization problem. □

Hence, regardless of what strategy $\gamma_1\in\Gamma^1$ the leader announces, the follower's reaction at the last stage will always be a unique linear function of x_{N-1} and $\gamma_{1,N-1}(\cdot)$. Now, substituting (43) into $J_1(N,M)$ and $J_2(N,M)$ for v_{N-1}, we obtain, respectively,

$$\begin{aligned}\hat{J}_1(N,M)&=\hat{x}_N^T\hat{Q}_1(N)\hat{x}_N+x_{N-1}^TQ_1(N-1)x_{N-1}\\&+u_{N-1}^TR_{11}(N-1)u_{N-1}\\&+\sum_{n=M}^{N-2}\left[x_n^TQ_1(n)x_n+u_n^TR_{11}(n)u_n+v_n^TR_{12}(n)v_n\right]\end{aligned}\tag{44a}$$

$$\begin{aligned}\hat{J}_2(N,M)&=\hat{x}_N^T\hat{Q}_2(N)\hat{x}_N+x_{N-1}^TQ_2(N-1)x_{N-1}\\&+u_{N-1}^TR_{21}(N-1)u_{N-1}\\&+\sum_{n=M}^{N-2}\left[x_n^TQ_2(n)x_n+u_n^TR_{21}(n)u_n+v_n^TR_{22}(n)v_n\right]\end{aligned}\tag{44b}$$

where

$$\hat{x}_N=A(N-1)x_{N-1}+B_1(N-1)u_{N-1},\tag{45a}$$

$$\begin{aligned}\hat{Q}_1(N)&=\left[I-B_2(N-1)T(N-1)\right]^T\\&\cdot Q_1(N)\left[I-B_2(N-1)T(N-1)\right]\\&+T^T(N-1)R_{12}(N-1)T(N-1),\end{aligned}\tag{45b}$$

$$\hat{Q}_2(N)=Q_2(N)\left[I-B_2(N-1)T(N-1)\right],\tag{45c}$$

$$\begin{aligned}T(N-1)\triangleq&\left[B_2^T(N-1)Q_2(N)B_2(N-1)+R_{22}(N-1)\right]^{-1}\\&\cdot B_2^T(N-1)Q_2(N).\end{aligned}\tag{45d}$$

Further introducing the notation

$$\hat{\gamma}_2\in\hat{\Gamma}^2\Rightarrow\{\gamma_{2n}\in\Gamma_{2,n},\ n=0,\cdots,N-2\},\tag{46a}$$

$$\begin{aligned}\tilde{J}_i(\gamma_1,\hat{\gamma}_2)\triangleq\hat{J}_i(N,0)\big|\{&u_n=\gamma_{1n}(\eta_n),\ n\in\theta;\\&v_m\in\gamma_{2m}(\eta_m),\ m=0,\cdots,N-2\},\end{aligned}\tag{46b}$$

and defining

$$\hat{R}(\gamma_1)=\left\{\hat{\gamma}_2^0\in\hat{\Gamma}^2:\tilde{J}_2(\gamma_1,\hat{\gamma}_2^0)\leqslant\tilde{J}_2(\gamma_1,\hat{\gamma}_2),\ \forall\hat{\gamma}_2\in\hat{\Gamma}^2\right\},\tag{46c}$$

it follows directly from Proposition 3 that the CLS cost for the leader, as given by (8), can equivalently be written as

$$\hat{J}_1^*=\sup_{\hat{R}(\gamma_1^*)}\tilde{J}_1(\gamma_1^*,\hat{\gamma}_2)=\inf_{\Gamma^1}\sup_{\hat{R}(\gamma_1)}\tilde{J}_1(\gamma_1,\hat{\gamma}_2),\tag{47a}$$

a lower bound for which is definitely provided by the optimal team cost

$$\hat{J}_1^*\geqslant\inf_{\Gamma^1}\inf_{\Gamma^2}\tilde{J}_1(\gamma_1,\hat{\gamma}_2)=\tilde{\tilde{J}}_1(\hat{\gamma}_1^t,\hat{\gamma}_2^t),\tag{47b}$$

as in the case of (9). This, in fact, provides an attainable tight lower bound on the CLS cost of the leader, as to be shown in the next subsection under certain conditions. To this end, we first attempt to determine the optimal team solution $\{\hat{\gamma}_1^t,\hat{\gamma}_2^t\}$ of $\tilde{\tilde{J}}_1$. But this optimization problem is precisely the one considered in Section IV-A with only $Q_1(N)$ replaced by $\hat{Q}_1(N)$, and thus its solution follows directly from Lemma 1, which we quote here for the sake of completeness.

Lemma 3: In feedback strategies, the joint optimization (team) problem defined by the RHS of (47b) admits a unique solution given by

$$\hat{\gamma}_{1n}^t(x_n)=-\hat{L}_1(n)x_n,\qquad n\in\theta\tag{48a}$$

$$\hat{\gamma}_{2n}^t(x_n)=-\hat{L}_2(n)x_n,\qquad n\in\theta-\{N-1\}\tag{48b}$$

and with minimum cost

$$\hat{J}_1^t=x_0^T\hat{M}(0)x_0\tag{48c}$$

where

$$\hat{L}_1(n)=\left[R_{11}(n)+\hat{K}_1(n)B_1(n)\right]^{-1}\hat{K}_1(n)A(n),\qquad n\in\theta\tag{49a}$$

$$\hat{L}_2(n)=\left[R_{12}(n)+\hat{K}_2(n)B_2(n)\right]^{-1}\hat{K}_2(n)A(n),\qquad n\leqslant N-2\tag{49b}$$

$$\hat{K}_1(N-1)=B_1^T(N-1)\hat{Q}_1(N)\tag{50a}$$

$$\begin{aligned}\hat{K}_1(n)=B_1^T(n)\hat{M}(n+1)\Big\{&I-B_2(n)\\&\cdot\left[R_{12}(n)+B_2^T(n)\hat{M}(n+1)B_2(n)\right]^{-1}\\&\cdot B_2^T(n)\hat{M}(n+1)\Big\},\qquad n\leqslant N-2.\end{aligned}\tag{50b}$$

$$\begin{aligned}\hat{K}_2(n)=B_2^T(n)\hat{M}(n+1)\Big\{&I-B_1(n)\\&\cdot\left[R_{11}(n)+B_1^T(n)\hat{M}(n+1)B_1(n)\right]^{-1}\\&\cdot B_1^T(n)\hat{M}(n+1)\Big\},\qquad n\leqslant N-2\end{aligned}\tag{50c}$$

and with $\hat{M}(\cdot)$ defined recursively by

$$\hat{M}(N-1)=Q_1(N-1)+\hat{F}^T(N-1)\hat{Q}_1(N)\hat{F}(N-1)+\hat{L}_1^T(N-1)R_{11}(N-1)\hat{L}_1(N-1), \quad (51a)$$

$$\hat{M}(n)=Q_1(n)+\hat{F}^T(n)\hat{M}(n+1)\hat{F}(n)+\hat{L}_1^T(n)R_{11}(n)\hat{L}_1(n)+\hat{L}_2^T(n)R_{12}(n)\hat{L}_2(n); \quad n\leqslant N-2 \quad (51b)$$

where

$$\hat{F}(N-1)\triangleq A(N-1)-B_1(N-1)\hat{L}_1(N-1),$$
$$\hat{F}(n)\triangleq A(n)-B_1(n)\hat{L}_1(n)-B_2(n)\hat{L}_2(n); \quad n\leqslant N-2. \quad (51c)$$

B. Exact Expressions for the CLS Strategies

It should be noted that the Stackelberg problem defined in terms of the cost functions $\hat{J}_1$ and $\hat{J}_2$ is structurally similar to the one introduced in Section IV, since we have converted the original Stackelberg game into an equivalent one which involves the action of only the leader at the last stage[3]. Thus, a particular representation

$$\hat{\gamma}_{1n}^p(x_n,x_{n-1})=-\hat{L}_1(n)x_n+\hat{P}_n[x_n-\bar{x}_n(n-1)] \quad (52)$$

of the team strategies $\{\hat{\gamma}_{1n}^l(\cdot)\}$ will constitute a CLS solution for the leader, under a condition quite analogous to Condition 1. Because of the equivalence between the present Stackelberg problem and the one considered in Section IV, with only $Q_i(N)$ replaced by $\hat{Q}_i(N)$ and $\hat{x}(N)$ standing for $x(N)$ in the present formulation, all the results of Section IV-B hold true in the present case with the appropriate modifications. Hence, expressions for the CLS strategies of the game problem of this section follow directly from Theorem 1, as given below in Theorem 3.

Preliminary notation for Theorem 3:

$\hat{\Sigma}(\cdot)$: an $(r'\times r')$ matrix defined recursively by

$$\hat{\Sigma}(N-1)=Q_2(N-1)+\hat{F}^T(N-1)\hat{Q}_2(N)\hat{F}(N-1)+\hat{L}_1^T(N-1)R_{21}(N-1)\hat{L}_1(N-1), \quad (53a)$$

$$\hat{\Sigma}(n)=Q_2(n)+\hat{F}^T(n)\hat{\Sigma}(n+1)\hat{F}(n)+\hat{L}_1^T(n)R_{21}(n)\hat{L}_1(n)+\hat{L}_2^T(n)R_{22}(n)\hat{L}_2(n); \quad n\leqslant N-2. \quad (53b)$$

$\hat{\Lambda}_n$: an $(r\times m)$ matrix defined recursively by (as a function of $\{\hat{P}_{n+1},\hat{P}_{n+2},\cdots,\hat{P}_{N-1}\}$)

$$\hat{\Lambda}_n=B_1^T(n)\hat{P}_{n+1}^T\hat{\Lambda}_{n+1}\hat{F}(n)-R_{21}(n)\hat{L}_1(n)+B_1^T(n)\hat{\Sigma}(n+1)\hat{F}(n);$$
$$\hat{\Lambda}_N=0. \quad (54)$$

[3]Note that $\hat{J}_1$ and $\hat{J}_2$ no longer depend on v_{N-1}.

Condition 3: There exists at least one matrix sequence $\{\hat{P}_{N-1},\hat{P}_{N-2},\cdots,\hat{P}_1\}$ that satisfies recursively the linear matrix equation

$$B_2^T(n-1)\hat{P}_n^T\hat{\Lambda}_n\hat{F}(n-1)=-B_2^T(n-1)\hat{\Sigma}(N)\hat{F}(n-1)+R_{22}(n-1)\hat{L}_2(n-1); \quad n\in\theta \quad (55)$$

where $\hat{\Lambda}_n$ is related to $\{\hat{P}_{n+1},\cdots,\hat{P}_{N-1}\}$ through (54).

Theorem 3: Let Condition 3 be satisfied and let $\{\hat{P}_{N-1}^*,\cdots,\hat{P}_1^*\}$ denote one such sequence. Then, there exists a CLS solution for the dynamic game originally formulated in Section II, which is given by

$$\hat{\gamma}_{1n}^*(x_n,x_{n-1})=[\hat{P}_n^*-\hat{L}_1(n)]x_n-\hat{P}_n^*\hat{F}(n-1)x_{n-1}; \quad n\in\theta-\{0\} \quad (56a)$$

$$\hat{\gamma}_{10}^*(x_0)=-\hat{L}_1(0)x_0, \quad (56b)$$

$$\hat{\gamma}_{2,N-1}^*(x_{N-1})=-T(N-1)\cdot[A(N-1)-B_1(n-1)\hat{L}_1(n-1)]x_{N-1}\triangleq-\hat{L}_2(n-1)x_{N-1}, \quad (57a)$$

$$\gamma_{2n}^*(x_n)=-\hat{L}_2(n)x_n; \quad n\in\theta-\{N-1\}, \quad (57b)$$

and the CLS costs of the leader and the follower are given, respectively, by

$$\hat{J}_1^*=x_0^T\hat{M}(0)x_0, \quad (58a)$$
$$\hat{J}_2^*=x_0^T\hat{\Sigma}(0)x_0. \quad (58b)$$

C. A Numerical Example

To illustrate the result of Theorem 3, the validity (nonvoidness) of Condition 3, and the convergence properties of the CLS solution in the case of an infinite-horizon problem, we now consider the same example treated in Section IV-C but with the follower also acting at the last stage. That is, we consider the scalar $(N-k)$-stage dynamic game described by the linear state equation

$$x_{n+1}=x_n+u_n+v_n; \quad n\leqslant N-1, \quad (59)$$

and the stationary cost functions

$$J_1=x_N^2+\sum_{n=k}^{N-1}x_n^2+2u_n^2+v_n^2 \quad (60a)$$

$$J_2=x_N^2+\sum_{n=k}^{N-1}x_n^2+u_n^2+3v_n^2. \quad (60b)$$

The relevant terms, in this case, are determined from

$$\hat{L}_1(n)=\hat{K}_1(n)/[2+\hat{K}_1(n)], \quad n\leqslant N-1$$
$$\hat{L}_2(n)=\hat{K}_2(n)/[1+\hat{K}_2(n)], \quad n\leqslant N-2$$
$$\hat{L}_2(N-1)=(1/4)(1-\hat{L}_1(N-1)).$$

$$\hat{K}_1(n)=\hat{M}(n+1)/[1+\hat{M}(n+1)], \qquad n\leqslant N-1$$
$$\hat{K}_2(n)=2\hat{M}(n+1)/[2+\hat{M}(n+1)], \qquad n\leqslant N-2$$
$$\hat{F}(N-1)=1-\hat{L}_1(N-1)$$
$$\hat{F}(n)=1-\hat{L}_1(n)-\hat{L}_2(n), \qquad n\leqslant N-2$$
$$\hat{M}(N-1)=1+(5/8)\hat{F}^2(N-1)+2\hat{L}_1^2(N-1)$$
$$\hat{M}(n)=1+\hat{F}^2(n)\hat{M}(n+1)+2\hat{L}_1^2(n), \qquad n\leqslant N-2$$
$$\hat{\Sigma}(N-1)=1+(3/4)\hat{F}^2(N-1)+\hat{L}_1^2(N-1),$$
$$\hat{\Sigma}(n)=1+\hat{F}(n)\hat{\Sigma}(n+1)+\hat{L}_1^2(n)+3\hat{L}_2^2(n), \qquad n\leqslant N-2.$$

Furthermore, (54) and (55) can be written as

$$\hat{\Lambda}_n=\hat{P}_{n+1}\hat{F}(n)\hat{\Lambda}_{n+1}-\hat{L}_1(n)+\hat{\Sigma}(n+1)\hat{F}(n), \qquad \Lambda_N=0; \quad (61a)$$

$$\hat{P}_n\hat{\Lambda}_n\hat{F}(n-1)=-\hat{\Sigma}(n)\hat{F}(n-1)+3\hat{L}_2(n-1), \qquad n\leqslant N-1. \quad (61b)$$

It can recursively be shown that $\hat{\Lambda}_n>0$, $\hat{F}(n)>0$ for all $n\leqslant N-1$, and thus optimal coefficients $\hat{P}_n^*$ are given by

$$\hat{P}_n^*=[3\hat{L}_2(n-1)-\hat{\Sigma}(n)\hat{F}(n-1)]/[\hat{\Lambda}_n\hat{F}(n-1)]$$

where

$$\hat{\Lambda}_{N-1}=-\hat{L}_1(N-1)$$
$$\hat{\Lambda}_n=3\hat{L}_2(n)-\hat{L}_1(n), \qquad n\leqslant N-2.$$

Numerical values for these terms are given in Table II for $n=N-1,\cdots,N-11$, as well as the corresponding CLS strategies from Theorem 3. The tabulated results indicate that there is rapid convergence to precisely the same stationary policies as in Example IV-C. This strongly suggests the conclusion that, as far as the stationary CLS strategies of an infinite-horizon game are concerned, it does not really make any difference whether the follower acts at the "last" stage of the game or not. In other words, in either case, the leader can force the follower to a strategy that also minimizes his own cost function (in the long run).

VII. Concluding Remarks

In this paper, we have developed a new indirect approach towards the closed-loop Stackelberg (CLS) solution of dynamic nonzero-sum games, which allows us to obtain the solution of an important subclass of such problems previously unsolved. The new approach makes use of the concept of representation of control strategies, whose significance has been verified earlier within the context of Nash equilibrium solutions.

In Section II, we have formulated the dynamic game problem and introduced the CLS solution concept. In Section III, we have discussed our new approach, and have applied it, in Section IV, to the dynamic game problem, under the assumption that only the leader acts at the last stage. Under Condition 1, a one-step memory representation of the optimal team solution γ_1^t turns out to be a CLS strategy for the leader. It should be noted that, if Condition 1 does not hold true, it is possible to allow a k-step memory strategy ($k\geqslant 2$) and obtain the solution under less stringent conditions (as it is done in [13] and [14] for a specific example considered). But this does not bring in anything conceptually new, and thus we have avoided to include this more general solution in the paper, which also necessitates inclusion of extensive notation. A scalar multistage dynamic game example included in Section IV-C illustrates the nonvoidness of Condition 1, and the convergence properties of the solution as the number of stages in the game increases.

In Section V, we have discussed possible extensions to multilevel control of LQ large-scale systems. In particular, we solve the problem of a central controller and K secondary level controllers, each with a different cost function and the central controller being in a position to enforce his strategies to the other K controllers. Under the assumption that the other K controllers adopt the Nash equilibrium solution among themselves, we have obtained conditions under which a one-step memory strategy of the central controller forces the other controllers to a team-optimal solution while minimizing their own cost functions. This model includes the case when each secondary level controller has his own dynamics, but does not in general include the case when they might only use local feedback. Though, the latter problem is also tractable within the framework of a model proposed in [6] and will be considered in a forthcoming paper.

Yet another problem of considerable significance is the one in which the central controller interacts with the system less frequently than do the local controllers, that is to say, there are two different time scales. To some extent, this problem is also solvable using the techniques of this paper, but it seems that the central controller will have to use at least a k-step memory strategy (where k is the frequency of his interaction with the system) in order to force the local controllers to the desired optimal-team solution.

In Section VI, we have considered the case when the follower also acts at the last stage, and have obtained the closed-loop Stackelberg solution of this more general problem, again allowing only a one-step memory strategy to the leader. It is shown, within this context, that the CLS strategy of the leader is again a representation of the solution of a team problem, but not the one defined in terms of only the cost function of the leader. An example considered in Section VI-C illustrates different aspects of the solution, and in particular, the property that as long as the CLS solution of an infinite-horizon problem is concerned we can, without any loss of generality, assume the follower not acting at the "very last stage" and consider

TABLE II
CLS SOLUTION FOR THE NUMERICAL EXAMPLE OF VI-C

n	$\hat{L}_1(n)$	$\hat{L}_2(n)$	$\hat{F}(n)$	$\hat{P}_n^*$	$\hat{M}(n)$	$\hat{\Sigma}(n)$	$\hat{\gamma}_{1n}^*(x_n, x_{n-1}) = [\hat{P}_n^* - \hat{L}_1(n)]x_n - \hat{P}_n^*\hat{F}(n-1)x_{n-1}$	$\hat{\gamma}_{2n}^*(x_n) = -\hat{L}_2(n)x_n$
N	-	-	-	-	0.625	0.75	-	-
N-1	0.161290	0.209678	0.838710	6.245829	1.491675	1.553590	$6.084538\, x_{N-1} - 1.929206\, x_{N-2}$	$-0.209678\, x_{N-1}$
N-2	0.230376	0.460747	0.308879	2.208664	1.460747	1.838158	$1.978287\, x_{N-2} - 0.692128\, x_{N-3}$	$-0.460747\, x_{N-2}$
N-3	0.228877	0.457754	0.313370	2.194851	1.457754	1.861508	$1.965975\, x_{N-3} - 0.688769\, x_{N-4}$	$-0.457754\, x_{N-3}$
N-4	0.228730	0.457459	0.313811	2.193802	1.457459	1.863441	$1.965072\, x_{N-4} - 0.688535\, x_{N-5}$	$-0.457459\, x_{N-4}$
N-5	0.228715	0.457430	0.313855	2.193729	1.457430	1.863596	$1.965014\, x_{N-5} - 0.688521\, x_{N-6}$	$-0.457430\, x_{N-5}$
N-6	0.228714	0.457427	0.313859	2.193725	1.457427	1.863607	$1.965011\, x_{N-6} - 0.688521\, x_{N-7}$	$-0.457427\, x_{N-6}$
N-7	0.228714	0.457427	0.313859	2.193725	1.457427	1.863608	$1.965011\, x_{N-7} - 0.688521\, x_{N-8}$	$-0.457427\, x_{N-7}$
N-8	0.228714	0.457427	0.313859	2.193725	1.457427	1.863608	$1.965011\, x_{N-8} - 0.688521\, x_{N-9}$	$-0.457427\, x_{N-8}$
N-9	0.228714	0.457427	0.313859	2.193725	1.457427	1.863608	$1.965011\, x_{N-9} - 0.688521\, x_{N-10}$	$-0.457427\, x_{N-9}$
N-10	0.228714	0.457427	0.313859	2.193725	1.457427	1.863608	$1.965011\, x_{N-10} - 0.688521\, x_{N-11}$	$-0.457427\, x_{N-10}$
N-11	0.228714	0.457427	0.313859		1.457427	1.863608		

the CLS solution of the infinite-stage game as the limit of the CLS solutions of such finite-stage games.

In conclusion, we believe that the approach developed in this paper is applicable to several optimal control and coordination problems in large scale systems that involve decentralization, multicontrollers, multitime scale, and multiobjectives.

APPENDIX

In this Appendix, we provide a proof for Theorem 1 which was stated in Section IV-B. To this end, we first substitute the strategies of the leader given by (22) into $J_2(N,i)$ and denote the resulting expression by $\tilde{J}_i^N(v_{N-1},\cdots,v_i)$, which is a strictly convex and quadratic function of $\{v_{N-1},\cdots,v_i\}$ for each $i\leqslant N-1$. By Proposition 2, the proof then amounts to showing that the minimum of $\tilde{J}_0^N(v_{N-1},\cdots,v_0)$ is attained by (23) (and also by the trivial solution $v_{N-1}^*=0$), since (22a) is a closed-loop representation of the team solution (14a), and (23) is the same as (14b).

We now show that (23) minimizes $\tilde{J}_0^N(v_{N-1},\cdots,v_0)$. Since $\tilde{J}_0^N$ is quadratic and strictly convex, it is both necessary and sufficient that $\{\gamma_{2n}^*\}$ be a stage-by-stage optimal solution, i.e., the set of inequalities

$$\tilde{J}_0^N(v_i^*,\{v^*\}_i)\leqslant\tilde{J}_0^N(v_i,\{v^*\}_i) \tag{A-1a}$$

should be satisfied for all $v_i=\gamma_{2i}(\cdot)\in\Gamma_{2,i}$, and all $i=0,\cdots,N-2$, where $v_n^*=\gamma_{2n}^*(x_n)$, and $\{v^*\}_i$ denotes the entire sequence $\{v_n^*,\ n\in\theta\}$ with only v_i^* missing. Because of the additive nature of the cost function, (A-1a) can equivalently be written as

$$\tilde{J}_i^N(v_{N-2}^*,\cdots,v_i^*)\leqslant\tilde{J}_i^N(v_{N-2}^*,\cdots,v_{i+1}^*,v_i), \tag{A-1b}$$

for all $v_i=\gamma_{2i}(\cdot\in\Gamma_{2,i}$, and all $i=0,\cdots,N-2$. (It should be noted that we have suppressed the v_{N-1} dependence of $\tilde{J}_i^N$ since v_{N-1} only enters the cost function but not the state dynamics and thus its optimal value is trivially zero.) We also note that on the RHS of (A-1b), $\tilde{J}_i^N$ is not only a function of v_i but also of x_i, which we take at its equilibrium value x_i^* because of the RHS of (A-1a) [i.e., the sequence $\{v_0,\cdots,v_{i-1}\}$ that determines x_i is already at equilibrium while treating (A-1b)].

The proof thus will be completed if we can show that

$$\nabla_{v_i}\tilde{J}_i^N(v_{N-2}^*,\cdots,v_{i+1}^*,v_i)\big|_{v_i=v_i^*}=0,\qquad \forall i=0,\cdots,N-2. \tag{A-2b}$$

Now, for an arbitrary i, $\tilde{J}_i^N(v_{N-2}^*,\cdots,v_{i+1}^*,v_i)\triangleq J_i(v_i)$ is given by

$$\begin{aligned}J_i(v_i)&=\tilde{x}_N^TQ(N)\tilde{x}_N\\&\quad+\sum_{n=i+1}^{N}\left[\tilde{x}_n^TQ(n)\tilde{x}_n+\tilde{u}_n^TR_{21}(n)\tilde{u}_n+\tilde{v}_n^TR_{22}(n)\tilde{v}_n\right]\\&\quad+x_i^{*T}Q(n)x_i^*+\tilde{u}_i^TR_{21}(i)\tilde{u}_i+v_i^TR_{22}(i)v_i\end{aligned} \tag{A-3}$$

where

$$\begin{aligned}\tilde{x}_N&=F(N-1)\tilde{x}_{N-1}+B_1(N-1)P_{N-1}^*\\&\quad\cdot\left[\tilde{x}_{N-1}-F(N-2)\tilde{x}_{N-2}\right]\\&\ \ \vdots\\\tilde{x}_{i+2}&=F(i+1)\tilde{x}_{i+1}+B_1(i+1)P_{i+1}^*\left[\tilde{x}_{i+1}-F(i)x_i^*\right]\\\tilde{x}_{i+1}&=A(i)x_i-B_1(i)L_1(i)x_i^*+B_2(i)v_i\end{aligned} \tag{A-4}$$

and

$$\tilde{u}_n=\begin{cases}-L_1(n)\tilde{x}_n+P_n^*\left[\tilde{x}_n-F(n-1)\tilde{x}_{n-1}\right], & n\geqslant i+1\\-L_1(i)x_i^*, & n=i\end{cases} \tag{A-5}$$

with $\tilde{x}_i = x_i^*$. Now, if the last row relation in (A-4) is used recursively in the remaining relations of (A-4), we can express $\tilde{x}_n$, $n \geq i+2$, in terms of $\tilde{x}_{i+1}$ and x_i^*, and if this is used in (A-5), $\tilde{u}_n$, $n \geq i+1$ can be expressed in terms of the same. The resulting expressions are

$$x_n = \mathcal{F}(n,i+1)\tilde{x}_{i+1} + \mathcal{K}(n,i+1)B_1(i+1) \cdot P_{i+1}^*[\tilde{x}_{i+1} - F(i)x_i^*], \qquad n \geq i+2, \quad \text{(A-6a)}$$

$$u_n = -L_1(n)\mathcal{F}(n,i+1)\tilde{x}_{i+1} + [Y(n,i+1) - L_1(n)\mathcal{K}(n,i+1)B_1(i+1)] \cdot P_{i+1}^*[\tilde{x}_{i+1} - F(i)x_i^*], \qquad n \geq i+1 \quad \text{(A-6b)}$$

where

$$\mathcal{F}(n,i) \triangleq F(n-1)\cdots F(i); \qquad \mathcal{F}(n,n) = I \quad \text{(A-7a)}$$

$$\mathcal{K}(n,i) = \mathcal{K}(n,i+1)B_1(i+1)P_{i+1}^* + \mathcal{F}(n,i+1) \quad \text{(A-7b)}$$

$$\mathcal{K}(n,n-1) = I, \qquad \mathcal{K}(n,l) = 0, \quad n \leq l.$$

$$Y(n,i) = P_n^* B_1(n-1)\cdots P_{i+1}^* B_1(i); \qquad Y(n,n) = I. \quad \text{(A-7c)}$$

Now (A-2b) can readily be computed since the only dependence on v_i is through $\tilde{x}_{i+1}$, whose relation is the last row of (A-4). Differentiation and evaluation at $v_i = v_i^*$ gives the following expression for (A-2b) after some straightforward though extensive manipulations:

$$\frac{1}{2}\cdot \nabla_{v_i} \tilde{J}_i^N \Big|_{v_i = v_i^*} = \Big[B_2^T(i) P_{i+1}^{*T} \Big\{ B_1^T(i+1)\mathcal{K}^T(N,i+1)Q_2(N)\mathcal{F}(N,i) + \sum_{k=i+1}^{N-1} \big[B_1^T(i+1)\mathcal{K}^T(k,i+1)Z(k)\mathcal{F}(k,i) - Y^T(k,i+1)R_{21}(k)L_1(k)\mathcal{F}(k,i) \big] \Big\} + B_2^T(i)\Sigma(i+1)F(i) - R_{22}(i)L_2(i) \Big] x_i^* \quad \text{(A-8)}$$

where

$$Z(k) \triangleq L_1^T(k)R_{21}(k)L_1(k) + L_2^T(k)R_{22}(k)L_2(k) + Q_2(k).$$

Some further manipulations prove that (A-8) can equivalently be written as

$$\big[B_2^T(i) P^{*T}_{i+1} \Lambda^*_{i+1} F_i + B_2^T(i)\Sigma(i+1)F(i) - R_{22}(i)L_2(i) \big] x_i^* \quad \text{(A-9)}$$

where Λ_{i+1}^* satisfies the recursive equation (20) with P_n replaced by P_n^*. But by Condition 1 [(21)], expression (A-9) identically vanishes, which proves that v_i^* minimizes $\tilde{J}_0^N$ at stage i and is in equilibrium with $\{v^*\}_i$. Since i was an arbitrary integer, the proof of the main part of the Theorem follows. Expressions (24), on the other hand, follow from standard results of LQ regulator theory, by mere substitution.

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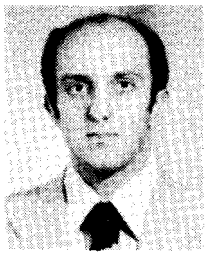

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Distributed Parameter State Estimation and Optimal Feedback Control — An Experimental Study in Two Space Dimensions

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Abstract—**Both optimal and suboptimal distributed parameter state estimation algorithms were applied in real time to a heated cylindrical ingot. Following this, deterministic and stochastic feedback controllers were implemented. Detailed studies of the effects of variables such as sensor choice, measurement error, model accuracy, controller and estimator parameters, and computational approximations on the control system performance are reported. The results show that the control scheme is robust to measurement noise and performs well even with only a single thermocouple sensor provided to the state estimator. Real-time computational requirements were modest and indicate that practical industrial implementation of such state estimation and control schemes is feasible.**

I. Introduction

THERE are a large number of industrial processes which are distributed in space (e.g., processes involving heating, drying, chemical reaction, etc.), and thus should offer scope for profitable application of distributed parameter systems theory. However, in spite of a large and growing theoretical literature in this area (cf. [1]), there have been no reported real-time industrial applications of distributed parameter state estimation and control theory, and only a small number of laboratory studies have been carried out [2]. One of the present limitations to widespread application of real-time distributed parameter systems theory is the lack of experience in real-time implementation and the dearth of experimental studies showing the benefits and limitations of such methods.

In this paper we present the results of a very detailed experimental study in which real-time state estimation, deterministic optimal and suboptimal feedback control, as well as stochastic feedback control were applied to a two-dimensional laboratory process. A comprehensive study of the design variables, and their influence on the control systems performance was made. Among the variables studied were 1) number and location of sensors, 2) level of measurement errors, 3) model accuracy, 4) controller and estimator weighting parameters, and 5) computational approximations. As a test process, a cylindrical steel ingot placed in a three-zone furnace was chosen because it has the essential features of ingot heating processes of importance in steelmaking [2]–[5], and it has two space dimensions. Previous laboratory studies with one-dimensional distributed systems (e.g., [6], [7]) showed good results; however, the present study provides first results on the control system performance and real-time computational requirements for a representative two-dimensional process. Earlier results of a state estimation study with a full set of axial sensors were reported in [4]

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