REFERENCES

[1] P. E. Caines, "A note on the consistency of maximum likelihood estimates for finite families of stochastic processes," *Annals of Statistics*, vol. 3, no. 2, pp. 539–546.
[2] Y. Baram and N. R. Sandell, Jr., "Consistent estimation on finite parameter sets with application to linear system identification," *IEEE Trans. Automat. Contr.*, vol. AC-23, June 1978.
[3] L. Ljung, "On the consistency of prediction error identification methods," in *System Identification, Advances and Case Studies*, R. K. Mehra and D. G. Lainiotis, Eds. New York: Academic, 1976.
[4] P. E. Caines, "Prediction error identification methods for stationary stochastic processes," *IEEE Trans. Automat. Contr.*, vol. AC-21, Aug. 1976.
[5] E. Tse and H. L. Weinert, "Structure determination and parameter identification for multivariable stochastic linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-20, Oct. 1975.
[6] Y. Baram, "Information, consistent estimation and dynamic system identification," Electronic Systems Laboratory, M.I.T., ESL-R-718, Nov. 1976.
[7] Y. Baram and N. R. Sandell, Jr., "An information theoretic approach to dynamical system modeling and identification," *IEEE Trans. Automat. Contr.*, vol. AC-23, Feb. 1978, also in *Proc. of CDC*, New Orleans, LA, Dec. 1978.
[8] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*. New York: Wiley, 1958.
[9] J. L. Doob, *Stochastic Processes*. New York: Wiley, 1953.
[10] W. F. Stout, *Almost Sure Convergence*. New York: Academic, 1974.
[11] A. Segall, "Stochastic processes in estimation theory," *IEEE Trans. Inform. Theory*, vol. IT-22, May 1976.
[12] P. Billingsley, "The Lindberg-Levy theorem for martingales," *Am. Math. Soc. Proc.*, vol. 12, pp. 788–792, 1961.

Equilibrium Solutions in Static Decision Problems with Random Coefficients in the Quadratic Cost

TAMER BAŞAR

Abstract—**This correspondence extends the results of [1] to the case when the weighting terms of the quadratic cost functions also depend on the random state of "Nature." For this class of static two-person quadratic decision problems it is shown that the (Nash) equilibrium strategies are unique under certain conditions which, in general, explicitly depend on the probabilistic structure of the problem.**

I. INTRODUCTION

In this paper we extend the results obtained in [1] to the case when the coefficient matrices in the quadratic cost functions are also random. More specifically, we consider the class of two-person two-criteria decision problems, with static information structure and quadratic objective functionals where the weighting terms are random, and for this class of problems we attempt to obtain a set of sufficient conditions for existence of unique (Nash) equilibria.

Abiding by the terminology and notation of [1], let x denote the unknown state of Nature taken as an n-dimensional second-order random vector, and let u_1 and u_2 denote the decision variables of decision makers (DM) 1 and 2, and taking values in R^{r_1} and R^{r_2}, respectively. Furthermore, let the observation of DMi be given by an m_i-dimensional random vector z_i. Then, the strictly convex quadratic cost function of DMi is defined by

$$J_i(x,u_1,u_2)=d_1^T(x)u_i+\tfrac{1}{2}u_i^TD_{ii}(x)u_i+u_i^TD_{ij}(x)u_j, \qquad j\neq i,\ i,j=1,2 \tag{1a}$$

with

$$D_{ii}(x)>0 \quad \text{a.s. } \mathcal{P}_x \tag{1b}$$

where $\mathcal{P}_x$ denotes the probability measure induced by x on the Borel subsets of R^n. We further assume that each coefficient term is square-integrable under $\mathcal{P}_x$. It should be noted that (1a) defines the most general convex quadratic cost functional in view of Lemma A1 of [1].

Letting $\mathcal{L}_{r_i}^{z_i}$ denote the Banach space of square-integrable strategies (decision rules) of DMi (see [1] for a precise definition), we define the expected (average) cost for DMi for each $\{\gamma_1\in\mathcal{L}_{r_1}^{z_1},\gamma_2\in\mathcal{L}_{r_2}^{z_2}\}$ by

$$\bar{J}_i(\gamma_1,\gamma_2)=E\left[J_i(x,u_1,u_2)|u_1=\gamma_1(\cdot),u_2=\gamma_2(\cdot)\right], \tag{2}$$

and the problem then is to obtain a set of sufficient conditions for existence of a unique pair of equilibrium strategies

$$\left\{\gamma_1^*\in\mathcal{L}_{r_1}^{z_1},\gamma_2^*\in\mathcal{L}_{r_2}^{z_2}\right\}$$

that satisfy

$$\bar{J}_1(\gamma_1^*,\gamma_2^*)\leq\bar{J}_1(\gamma_1,\gamma_2^*) \tag{3a}$$

$$\bar{J}_2(\gamma_1^*,\gamma_2^*)\leq\bar{J}_2(\gamma_1^*,\gamma_2) \tag{3b}$$

for all $\gamma_i\in\mathcal{L}_{r_i}^{z_i}$, $i=1,2$.

The main results are given in Section II.

II. MAIN RESULTS

As a counterpart of Theorem 1 of [1] we can first prove the following result.

Preliminary notation for Theorem 1:

$$\hat{D}_i(z_i)\triangleq E\left[D_{ii}(x)|z_i\right] \tag{4a}$$

$$\hat{d}_i(z_i)\triangleq E\left[d_i(x)|z_i\right]. \tag{4b}$$

Theorem 1: The pair $\{\gamma_1^*\in\mathcal{L}_{r_1}^{z_1},\gamma_2^*\in\mathcal{L}_{r_2}^{z_2}\}$ satisfies (3) for all $\gamma_i\in\mathcal{L}_{r_i}^{z_i}$ if, and only if, it simultaneously satisfies the following two relations:

$$\hat{D}_i(z_i)\gamma_i^*(z_i)=-\hat{d}_i(z_i)-E\left[D_{ij}(x)\gamma_j^*(z_j)|z_i\right], \qquad i\neq j,\ i,j=1,2. \tag{5}$$

Proof: Since expectation is expectation of conditional expectation with respect to z_i, we first have

$$\bar{J}_i(\gamma_1,\gamma_2)=E\Big[\hat{d}_i^T(z_i)\gamma_i(z_i)+\tfrac{1}{2}\gamma_i^T(z_i)\hat{D}_i(z_i)\gamma_i(z_i)+\gamma_i^T(z_i)E\left[D_{ij}(x)\gamma_j(z_j)|z_i\right]\Big], \qquad i\neq j,\ i,j=1,2.$$

Then, in view of inequalities (3) and the strict convexity assumption (1b), minimization of the above expression with respect to γ_i yields the desired result. □

We can now state the main result of this paper after introducing the following notation.

Preliminary notation for Theorem 2: In the following, $i\neq j$, $i,j=1,2$ unless otherwise stated:

$$N_{ij}(x,z_1,z_2)\triangleq\hat{D}_i^{-1}(z_i)D_{ij}(x)\hat{D}_j^{-1}(z_j). \tag{6a}$$

$\Lambda_i(z_i,\pi_i)$: an $(r_i\times r_i)$-dimensional matrix valued function of z_i for each nonsingular $(r_i\times r_i)$-dimensional matrix $\pi_i(z_i)$ (whose entries are square integrable under $\mathcal{P}_{z_i}$), defined by

$$\Lambda_i(z_i;\pi_i)\triangleq(\pi_i^T)^{-1}E\Big[D_{ji}^T(x)E\Big[N_{ij}^T\pi^T\pi N_{ij}|z_j\Big]D_{ji}(x)|z_i\Big]\pi_i^{-1} \tag{6b}$$

where we have suppressed the arguments of π_i and N_{ij}.

$$k_i(z_i)\triangleq-\hat{D}_i^{-1}(z_i)\hat{d}_i(z_i)+\hat{D}_i^{-1}(z_i)E\Big[D_{ij}(x)\hat{D}_j^{-1}(z_j)\hat{d}_j(z_j)|z_i\Big]. \tag{6c}$$

K_i: a linear operator mapping $\mathcal{L}_{r_i}^{z_i}$ into itself such that for each $\gamma\in\mathcal{L}_{r_i}^{z_i}$

$$K_i\gamma(z_i)\triangleq E\Big[N_{ij}(x,z_1,z_2)E\left[D_{ji}(x)\gamma(z_i)|z_j\right]|z_i\Big] \tag{6d}$$

$$\triangleq E\Big[l_\gamma^i(x,z_1,z_2)|z_i\Big] \tag{6e}$$

where the last expression defines $l_\gamma^i(\cdot)$ for each $\gamma\in\mathcal{L}_{r_i}^{z_i}$.

Manuscript received January 29, 1978.
The author is with the Applied Mathematics Division, Marmara Scientific and Industrial Research Institute, Gebze, Kocaeli, Turkey.

0018-9286/78/1000-0960$00.75

$y \triangleq (x^T, z_1^T, z_2^T)^T$, an $(n+m_1+m_2)$-dimensional random vector. (6f)

For each $\gamma \in \mathcal{L}_r^\eta$, we finally define the norm

$$\|\gamma\|_r^\eta \triangleq \{E[\gamma^T(\eta)\gamma(\eta)]\}^{1/2}. \tag{6g}$$

Theorem 2: Let there exist either an $(r_1 \times r_1)$-dimensional real nonsingular matrix function $\pi_1(z_1)$ such that for some $\delta < 1$

$$\Lambda_1(z_1;\pi_1) \leq \delta I \quad \text{a.s. } \mathcal{P}_{z_1} \tag{7a}^1$$

or an $(r_2 \times r_2)$-dimensional real nonsingular matrix function $\pi_2(z_2)$ such that for some $\delta < 1$

$$\Lambda_2(z_2;\pi_2) \leq \delta I \quad \text{a.s. } \mathcal{P}_{z_2}, \tag{7b}^1$$

where $\mathcal{P}_{z_i}$ denotes the probability measure induced under the random vector z_i. Then,

1) There exists a *unique* pair of strategies $\{\gamma_1^* \in \mathcal{L}_{r_1}^{z_1}, \gamma_2^* \in \mathcal{L}_{r_2}^{z_2}\}$ that satisfy (3).

2) If (7a) is satisfied for some nonsingular $\pi_1(z_1)$, the unique equilibrium strategy of DM1 is given by

$$\gamma_1^*(z_1) = \lim_n \xi_1^n(z_1) \tag{8a}$$

where lim denotes the limiting operation under the $\mathcal{L}_{r_1}^{\eta_1}$-norm and ξ_1^n denotes a convergent sequence in $\mathcal{L}_{r_1}^{\eta_1}$ defined by

$$\xi_1^n(z_1) = k_1(z_1) + \pi_1^{-1} \sum_{m=1}^{n} (\pi_1 K_1 \pi_1^{-1})^m (\pi_1 k_1(z_1)). \tag{8b}$$

Then, the unique equilibrium strategy of DM2 can be determined from (5) with $i=2$.

3) If instead (7b) is satisfied for some nonsingular $\pi_2(z_2)$, then

$$\gamma_2^*(z_2) = \lim_n \xi_2^n(z_2), \tag{9}$$

where ξ_2^n is defined by (8b) with 1's replaced by 2's and γ_1^* is related to γ_2^* uniquely through (5) with $i=1$.

Proof: To prove part 1) let us start with the result of Theorem 1, multiply both sides of (5) by $\hat{D}_i^{-1}$, and substitute the resulting expression for $i=2$ into the same for $i=1$. This yields the following unique equation that γ_1^* should satisfy:

$$\gamma_1(z_1) = k_1(z_1) + K_1\gamma_1(z_1) \tag{10a}$$

where the required terms are defined by (6c) and (6d). To prove existence of a unique solution $\gamma_1 \in \mathcal{L}_{r_1}^{z_1}$ to (10) is, however, equivalent to proving the existence of a unique solution $\tilde{\gamma}_1 \triangleq \pi_1\gamma_1 \in \mathcal{L}_{r_1}^{z_1}$ to

$$\tilde{\gamma}_1(z_1) = \pi_1 k_1(z_1) + \pi_1 K_1 \pi_1^{-1} \tilde{\gamma}_1(z_1) \triangleq \hat{K}_1 \tilde{\gamma}_1(z_1) \tag{10b}$$

since $\pi_1(z_1)$ is real and nonsingular. Defining a metric $d(\gamma,\beta)$ on $\mathcal{L}_{r_1}^{z_1}$ by the norm $\|\gamma-\beta\|_{r_1}^{z_1}$, we now show that the operator K_1 which maps $\mathcal{L}_{r_1}^{z_1}$ into itself is a contraction mapping under (7a).

For every $\gamma, \beta \in \mathcal{L}_{r_1}^{z_1}$,

$$\begin{aligned} d(\hat{K}_1\gamma, \hat{K}_1\beta) &= \|\hat{K}_1\gamma - \hat{K}_1\beta\|_{r_1}^{z_1} = \|\pi_1 K_1 \pi_1^{-1}(\gamma-\beta)\|_{r_1}^{z_1} \\ &= \|\pi_1 E\left[l_{\pi_1^{-1}\gamma}^1 - l_{\pi_1^{-1}\beta}^1 | z_1\right]\|_{r_1}^{z_1} \leq \|\pi_1\left(l_{\pi_1^{-1}\gamma}^1 - l_{\pi_1^{-1}\beta}^1\right)\|_{r_1}^{y} \\ &= d\left(\pi_1 l_{\pi_1^{-1}\gamma}^1, \pi_1 l_{\pi_1^{-1}\beta}^1\right) \end{aligned} \tag{11a}^2$$

where the inequality follows from the nonexpansive property of conditional expectation (see [1, Lemma 1]), and the notation is that of (6d)–(6g). Now, using a well-known property of conditional expectation under the expectation operator, we have

$$\begin{aligned} &\left[d\left(\pi_1 l_{\pi_1^{-1}\gamma}^1, \pi_1 l_{\pi_1^{-1}\beta}^1\right)\right]^2 \\ &= E\Big[E\Big[(\gamma(z_1)-\beta(z_1))^T(\pi_1^{-1})^T D_{21}^T(x)|z_2\Big] \\ &\quad \cdot N_{12}^T \pi_1^T \pi_1 N_{12} E\left[D_{21}(x)\pi_1^{-1}(\gamma(z_1)-\beta(z_1))|z_2\right]\Big] \\ &= E\Big[E\Big[(\gamma(z_1)-\beta(z_1))^T(\pi_1^{-1})^T D_{21}^T|z_2\Big] E\left[N_{12}^T\pi_1^T\pi_1 N_{12}|z_2\right] \\ &\quad \cdot E\left[D_{21}\pi_1^{-1}(\gamma(z_1)-\beta(z_1))|z_2\right]\Big] \\ &\equiv E\Big[E\Big[(\gamma(z_1)-\beta(z_1))^T(\pi_1^{-1})^T D_{21}^T|z_2\Big] M^T(z_2)M(z_2) \\ &\quad \cdot E\left[D_{21}\pi_1^{-1}(\gamma(z_1)-\beta(z_1))|z_2\right]\Big] \end{aligned} \tag{11b}$$

where in the last identity we have made use of the fact that it is possible to find a matrix $M(z_2)$ with square-integrable entries and with the property

$$M^T(z_2)M(z_2) \equiv E\left[N_{12}^T(x,z_1,z_2)\pi_1^T\pi_1 N_{12}(x,z_1,z_2)|z_2\right] \tag{11c}$$

since the latter expression is nonnegative definite a.s. $\mathcal{P}_{z_2}$, and the conditional expectation possesses the said properties. Actually, more than one such $M(\cdot)$ will in general exist, but any of them will be equally acceptable at this stage. Hence, taking over from (11b), we have

$$\begin{aligned} d\left(\pi_1 l_{\pi_1^{-1}\gamma}^1, \pi_1 l_{\pi_1^{-1}\beta}^1\right) &= \|M(z_2)E\left[D_{21}(x)\pi_1^{-1}(\gamma(z_1)-\beta(z_1))|z_2\right]\|_{r_1}^{z_2} \\ &= \|E\left[M(z_2)D_{21}(x)\pi_1^{-1}(\gamma(z_1)-\beta(z_1))|z_2\right]\|_{r_1}^{z_2} \\ &\leq \|M(z_2)D_{21}(x)\pi_1^{-1}(\gamma(z_1)-\beta(z_1))\|_{r_1}^{y} \\ &= \Big\{E\Big[(\gamma-\beta)^T(\pi_1^{-1})^T E \\ &\quad \cdot \left[D_{21}^T(x)M^T(z_2)M(z_2)D_{21}(x)|z_1\right]\pi_1^{-1}(\gamma-\beta)\Big]\Big\}^{1/2} \\ &\leq \delta\left\{E\left[(\gamma-\beta)^T(\gamma-\beta)\right]\right\}^{1/2} = \delta d(\gamma,\beta) \end{aligned}$$

where the first inequality again follows from [1, Lemma 1], whereas the second inequality follows from (7a) by making use of (11c) and (6b). Hence, we have shown that for some $\delta < 1$

$$d(\hat{K}_1\gamma, \hat{K}_1\beta) \leq \delta d(\gamma,\beta),$$

and this implies that $\hat{K}_1$ is a contraction mapping, which in turn establishes the existence of a unique fixed point of $\hat{K}_1$ by direct application of Banach's classical contraction mapping principle. This thus completes the proof of 1) under (7a). A proof under (7b) can likewise be constructed by first writing down an expression for γ_2 similar to (10a) and by going through similar steps.

Since $\hat{K}_1$ is a contraction operator under (7a), 2) follows (as in [2]) directly from (10b) and the property of Picard iterates in a fixed-point space [3]. 3) can likewise be proven since $\hat{K}_2$ (defined through (10b) with 1's interchanged by 2's) is also a contraction mapping under (7b). □

Remark 1: When the coefficient terms of the quadratic cost function are nonrandom, then it is known that there exist sufficient conditions of existence and uniqueness which are independent of the probabilistic structure of the problem [1]. If the weighting coefficients are random, however, it should be observed from (7) that this property, in general, ceases to hold true.

III. Conclusion

In this correspondence, we have extended the results of [1] to the case when the weighting terms of the quadratic cost function are also random. In this case, the set of sufficiency conditions derived depends explicitly on the probabilistic structure of the problem, and the unique equilibrium strategies can be realized as the limit of an infinite sequence in an appropriate Banach space.

References

[1] T. Başar, "Equilibrium solutions in two-person quadratic decision problems with static information structures," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 320–328, June 1975.

[1] Here I denotes the identity matrix of appropriate dimensions.

[2] Metric $d(\cdot,\cdot)$ is defined here likewise but on $\mathcal{L}_{r_1}^{y}$.

[2] ——, "Decentralized multicriteria optimization of linear stochastic systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 233–243, Apr. 1978.
[3] L. A. Liusternik and V. J. Sobolev, *Elements of Functional Analysis*. New York: Ungar, 1961.

Error Analysis of Three-Dimensional Linear Diffusion Equations When Approximated to Differential-Difference Equations

G. B. MAHAPATRA

Abstract—**This correspondence attempts to investigate the error involved when three-dimensional linear diffusion equations are approximated to differential difference equations.**

I. Introduction and Problem Statement

Consider the linear three-dimensional diffusion equation

$$\frac{\partial\theta}{\partial t}=\sum_{i=1}^{3}\frac{\partial^2\theta}{\partial x_i^2};\qquad x_i\in(0,1);\ \theta=\theta(x_1,x_2,x_3,t) \tag{1}$$

with initial condition

$$\theta(x_1,x_2,x_3,0)=\theta_0(x_1,x_2,x_3) \tag{2}$$

and boundary conditions

$$\theta=0,\qquad \text{for } x_1,x_2,x_3=1.0 \text{ and } x_2,x_3=0 \tag{3}$$

$$\theta=u(t),\qquad \text{for } x_1=0. \tag{4}$$

Denoting the nodal points (ih,jh,kh) along spatial coordinates as (i,j,k) with the mesh size $h=1/N$, N being the number of grids along any spatial coordinate, (1) is replaced by finite difference technique [2], [4]

$$\begin{aligned}\frac{dT(i,j,k,t)}{dt}&=(1/h)^2[T(i+1,j,k,t)+T(i-1,j,k,t)\\&\quad+T(i,j+1,k,t)+T(i,j-1,k,t)+T(i,j,k+1,t)\\&\quad+T(i,j,k-1,t)-6T(i,j,k,t)];\\&=L(T);\qquad i,j,k=1,2,\cdots,(N-1).\\&\qquad T(i,j,k,0)=\theta_0(i,j,k)\end{aligned} \tag{5}$$

where true solution $\theta(i,j,k,t)$ is approximated to $T(i,j,k,t)$ and $L(T)$ represents the right-hand side (R.H.S.) of (5), L being an operator. Such formulations have been employed in control studies of diffusion systems [1]–[3]. Error analysis is available when discretization is made along both spatial and time coordinates in numerical studies [4]. But it is not yet clear how much error is involved between $\theta(i,j,k,t)$ and $T(i,j,k,t)$ when (1)–(4) are replaced by (5). An attempt is made in this note to predict the error in terms of the grid points N and time t using state-space theory.

II. Error Analysis

Let the error between true solution and truncated solution be given by

$$E(i,j,k,t)=\theta(i,j,k,t)-T(i,j,k,t). \tag{6}$$

From Taylor series expansion of $\theta(x_1,x_2,x_3,t)$,

$$\frac{d\theta(i,j,k,t)}{dt}=L(\theta)-2V(t) \tag{7}$$

where

$$\begin{aligned}V(t)&=\frac{h^2}{4!}\sum_{m=1}^{3}\frac{\partial^4\theta}{\partial x_m^4}+\frac{h^4}{6!}\sum_{m=1}^{3}\frac{\partial^6\theta}{\partial x_m^6}\\&\quad+\frac{h^6}{8!}\sum_{m=1}^{3}\frac{\partial^8\theta}{\partial x_m^8}+\cdots\Bigg|_{i,j,k}\end{aligned} \tag{8}$$

$$=\frac{h^2}{4!}W_1(t)+\frac{h^4}{6!}W_2(t)+\frac{h^6}{8!}W_3(t)+\cdots \tag{9}$$

and

$$W_n(t)=\left[\sum_{m=1}^{3}\frac{\partial^{2n+2}\theta}{\partial x_m^{2n+2}}\right]_{i,j,k};\qquad n=1,2,3,\cdots. \tag{10}$$

From (5), (6), and (7), error equation is given by

$$\frac{dE(i,j,k,t)}{dt}=L(E)-2V(t) \tag{11}$$

and with initial conditions from (5) and (6)

$$E(i,j,k,0)=0 \tag{12}$$

and boundary conditions from (3), (4) and (6)

$$E=0,\qquad \text{for } i,j,k=0 \text{ and } N. \tag{13}$$

Equations (11)–(13) can be written in state variable form:

$$\dot{Y}=AY(t)-BV(t) \tag{14}$$

with initial condition

$$Y(0)=0. \tag{15}$$

$Y(t)$ is a column vector of order $(N-1)^3$; each element representing $E(i,j,k,t)$; A is a square matrix of size $(N-1)^3$; B is a diagonal square matrix of order $(N-1)^3$, each element $=2$; $V(t)$ is a column vector, each element being evaluated at (i,j,k).

Solving (14) and (15)

$$Y(t)=-2\int_0^t P\exp(D(t-\tau))P^{-1}V(\tau)\,d\tau \tag{16}$$

where P and D are modal matrix and diagonalized eigenvalue matrix of A, respectively. It can be seen that A matrix is the same corresponding to (5) and, therefore, it is symmetric matrix having negative real eigenvalues λ_i, $i=1,2,\cdots,(N-1)^3$.

Let the partial derivatives (p.d.) of θ in x_i evaluated at the nodal point (i,j,k) in (8) be all bounded. From (8) and (9),

$$\|V\|\leq\frac{h^2}{4!}\|W_1\|+\frac{h^4}{6!}\|W_2\|+\frac{h^6}{8!}\|W_3\|+\cdots \tag{17}$$

$$=M\quad\text{(a positive finite quantity).} \tag{18}$$

From (16), (17), and (18),

$$\|Y(t)\|\leq 2M\|P^{-1}\|\,\|P\|\int_0^t\|\exp(D(t-\tau))\|\,d\tau \tag{19}$$

$$\leq 2M\|P^{-1}\|\,\|P\|(N-1)^{3/2}t;\qquad 0<t<\infty \tag{20}$$

and

$$\|\exp(D(t-\tau))\|=\left[\sum_{i=1}^{(N-1)^3}\exp(2\lambda_i(t-\tau))\right]^{1/2} \tag{21}$$

$$\leq(N-1)^{3/2}. \tag{22}$$

Manuscript received March 28, 1977; revised April 22, 1978.
The author is with the Department of Electrical Engineering, Roorkee University, Roorkee, India.

0018-9286/78/1000-0962$00.75