A straightforward computation gives

$$\bar{F}_i^*(y_i^*) = \bar{b}_i, \quad \text{if } y_i^* \in B(r_i; a_i^0)$$
$$= +\infty, \quad \text{otherwise}$$
$$\underline{F}_i^*(z_i^*) = -\underline{b}_i, \quad \text{if } z_i^* \in B(r_i; -a_i^0)$$
$$= +\infty, \quad \text{otherwise.}$$

We substitute these expressions into $F^*(0)$. Then it follows that (S) is robust if and only if

$$0 \leqslant F^*(0) = \min\{H(y,z): (y,z,y_1^*,\cdots,y_n^*,z_1^*,\cdots,z_n^*) \in \Omega_{++}\}$$

where Ω_{++} is described by $\Omega 1$, $\Omega 2$, $\Omega 3'$ and

$$\Omega 4' \quad \|y_i^* - a_i^0\| \leqslant r_i, \qquad i = 1,2,\cdots,n$$
$$\Omega 5' \quad \|z_i^* + a_i^0\| \leqslant r_i, \qquad i = 1,2,\cdots,n.$$

To complete the proof, it suffices to show that $\Omega 3'$–$\Omega 5'$ taken together are equivalent to $\Omega 3$. For fixed $(y,z) \in R^{2n}$, we define the sets

$$M(y,z) = \{(y_1^*, y_2^*, \cdots, y_n^*, z_1^*, z_2^*, \cdots, z_n^*): \Omega 3', \Omega 4', \Omega 5' \text{ hold}\}$$
$$M = \{(y,z): M(y,z) \text{ is nonempty}\}.$$

Since $H(y,z)$ does not depend on the y_i^* and z_i^*, it follows that $F^*(0)$ remains unchanged if $\Omega 3'$–$\Omega 5'$ are replaced by the condition

$$\Omega 3'' \quad (y,z) \in M.$$

It remains to show that $\Omega 3$ and $\Omega 3''$ are equivalent.

$\Omega 3$ implies $\Omega 3''$: Suppose (y,z) satisfies $\Omega 3$. We can assume that

$$\Delta_0(y,z) = \sum_{i=1}^{n} (y_i + z_i) r_i$$

is nonzero. (Otherwise, $y_i^* = a_i^0$, $z_i^* = -a_i^0$ is a point in $M(y,z)$.) To show that $M(y,z)$ is nonempty, we define

$$y_i^* = \frac{a_i^0 \Delta_0(y,z) - r_i w_0(y,z)}{\Delta_0(y,z)}, \qquad i = 1,2,\cdots,n;$$

$$z_i^* = \frac{-a_i^0 \Delta_0(y,z) - r_i w_0(y,z)}{\Delta_0(y,z)}, \qquad i = 1,2,\cdots,n$$

where

$$w_0(y,z) = \sum_{i=1}^{n} (y_i - z_i) a_i^0.$$

Now, it is easily shown that the y_i^* and z_i^* above do indeed satisfy $\Omega 3'$–$\Omega 5'$. Hence, $M(y,z)$ is nonempty and (y,z) satisfies $\Omega 3''$.

$\Omega 3''$ implies $\Omega 3$: Suppose (y,z) satisfies $\Omega 3''$. Then, let $(y_1^*, y_2^*, \cdots, y_n^*, z_1^*, z_2^*, \cdots, z_n^*)$ be a point in $M(y,z)$. Now,

$$\Big\| \sum_{i=1}^{n} (y_i - z_i) a_i^0 \Big\| = \Big\| \sum_{i=1}^{n} y_i (a_i^0 - y_i^*) - z_i (a_i^0 + z_i^*) \Big\| \quad (\text{by } \Omega 3')$$
$$\leqslant \sum_{i=1}^{n} y_i \|a_i^0 - y_i^*\| + z_i \|a_i^0 + z_i^*\|$$
$$\leqslant \sum_{i=1}^{n} (y_i + z_i) r_i \quad (\text{by } \Omega 4' \text{ and } \Omega 5').$$

This completes the proof of the theorem. □

IV. Numerical Example

We take the solution set X to be $L^2[0,1]$ and we consider the perturbed functional equality

$$\int_0^1 a_1(t) x(t)\, dt = b$$
$$a_1 \in B(\delta; a_1^0);\ \delta \geqslant 0;\ a_1^0(t) = 2\sqrt{3}\, t$$
$$b \in [1-\xi, 1+\xi], \qquad \xi \geqslant 0. \qquad (S)$$

First, we observe that the unperturbed system ($\delta = \xi = 0$) has infinitely many solutions (e.g., $x(t) \equiv 1/\sqrt{3}$). We wish to determine the set of perturbations, $\delta \geqslant 0$, $\xi \geqslant 0$, for which (S) is robust. According to the theorem, we must solve

$$\min\{y(1+\xi) - z(1-\xi): (y,z) \in \Omega\}$$

where Ω is described by $y \geqslant 0$; $z \geqslant 0$; $y + z = 1$; and

$$\left[\int_0^1 12 t^2 (y-z)^2\, dt\right]^{1/2} \leqslant (y+z)\delta.$$

Simplifying, we obtain

$$\Omega = \left[\frac{2-\delta}{4}, \frac{2+\delta}{4}\right] \times \left[\frac{2-\delta}{4}, \frac{2+\delta}{4}\right]; \quad \min = \xi - \delta/2; \quad \text{if } 0 \leqslant \delta \leqslant 2;$$
$$= [0,1) \times [0,1); \quad \min = \xi - 1 \quad \text{if } \delta \geqslant 2.$$

We conclude that (S) is robust if and only if

$$(\xi, \delta) \in \{(\xi', \delta'): \xi' \geqslant 1, \delta' \geqslant 2\} \cup \{(\xi', \delta'): 0 \leqslant \delta' \leqslant 2, \xi' \geqslant \delta'/2\}.$$

In general, we may wish to include a *solution restraint set*, X_c, a proper subset of X. Consequently, the definition of robustness can be modified by adding the restriction "$x_0 \in X_c$." In [5], the notion of robustness is interpreted within the context of uncertain dynamical control systems. Loosely speaking, this is accomplished by identifying x, A_i, and B_i above with *input*, *model uncertainty* and *target set*, respectively.

References

[1] D. G. Luenberger, *Introduction to Linear and Nonlinear Programming*. Reading, MA: Addison-Wesley, 1975.
[2] O. L. Mangasarian, *Nonlinear Programming*. New York: McGraw-Hill, 1969.
[3] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1972.
[4] A. D. Ioffe and V. M. Tikhomirov, "Duality of convex functions and extremum problems," *Russian Math. Surveys*, vol. 23, pp. 53–124, 1968.
[5] B. R. Barmish, "A reachability theory for a class of uncertain dynamical systems," in *Proc. 1976 Allerton Conf. on Circuits and Systems.*

Two General Properties of the Saddle-Point Solutions of Dynamic Games

TAMER BAŞAR

***Abstract*—In deterministic team problems every closed-loop representation of an optimal open-loop solution is also optimal. This property, however, no longer holds true when the optimization problem is a zero-sum or a nonzero-sum game. In zero-sum games, two weaker (but still general enough) versions of this statement are valid, which still fail to hold in the case of nonzero-sum games. In this correspondence we state and prove these two general properties of the saddle-point solution in dynamic games.**

I. Introduction

If a two-person dynamic team problem admits an optimal solution in closed-loop policies, then its open-loop representation also constitutes an optimal solution. This property, however, no longer holds true in the Nash solutions of two-person dynamic nonzero-sum games, and no general statement can be made in that case (Başar [1], Başar and Selbuz [3]). On the other hand, in the case of zero-sum dynamic games (ZSDG) certain concrete statements can be made concerning the interplay between the saddle-point solutions and information structures. In this

Manuscript received June 4, 1976.
The author is with the Applied Mathematics Division, Marmara Scientific and Industrial Research Institute, Gebze, Kocaeli, Turkey.

correspondence, we verify two general properties of the saddle-point solution that are shared in common with team solutions, but not with Nash solutions. These results illuminate the close link between any two saddle-point solutions of a ZSDG, obtained under different information patterns.

In the sequel we first give a general formulation of the ZSDG and then make precise the concept of representation of a strategy pair under different information patterns. We state and prove two main features of the saddle-point solution in Section III, and discuss these properties in Section IV.

II. Problem Formulation

The class of zero-sum dynamic games (ZSDG) under consideration will be characterized by 1) a general difference equation describing the evolution of the state in discrete time

$$x_{n+1}=f_n(x_n,u_n,v_n), \qquad x_0 \text{ given}, \tag{1a}$$

and 2) a general cost functional

$$J=\phi_N(x_N)+\sum_{n=0}^{N-1}\phi_n(x_n,u_n,v_n). \tag{1b}$$

In this formulation, u_n and v_n represent the control vectors of the minimizer (Player 1) and the maximizer (Player 2), respectively, at the time step n, and are of dimensions r_1 and r_2. x_n is the state vector of dimension m, and f_n, $\phi_N(\cdot)$, and $\phi_n(\cdot,\cdot,\cdot)$ are Borel-measurable functions defined on appropriate dimensional spaces.

At every stage of the game, each player will have access to perfect information about the present and/or past values of the state vector, which we denote by $\eta_i(n)$ for Player i at stage n. In this correspondence our interest lies primarily on three different kinds of information patterns for each player, namely,

1) open-loop (OL): $\eta(n)=\{x_0\}\triangleq\eta^0(n)$;
2) closed-loop no-memory (CLNM): $\eta(n)=\{x_n\}\triangleq\eta^f(n)$;
3) closed-loop (CL): $\eta(n)=\{x^n\}\triangleq\eta^c(n)$;

where

$$x^n\triangleq(x_0',\cdots,x_n')'. \tag{2}$$

The second of these is also known as the *pure-feedback* information pattern.

To delineate the class of admissible control strategies for each player, we first note that each information pattern $\eta_i(n)$ generates an information space $Z_i(n)$ as a subset of an appropriate dimensional Euclidean space. For Player i and for each information pattern 1)–3), we denote this subset respectively by $Z_i^0(n)$, $Z_i^f(n)$, and $Z_i^c(n)$. It should be clear that $Z_i^0(n)$ is actually independent of n and that the dimension of $Z_i^f(n)$ is also independent of n, whereas the dimension of $Z_i^c(n)$ increases as n does. To be more precise, $Z_i^0(n)\subset R^m$, $Z_i^f(n)\subset R^m$, $Z_i^0(n)\subset R^{mn}$, $\forall n=0,\cdots,N-1$. Now, we define $\Gamma_i^0(n)$, $\Gamma_i^f(n)$, and $\Gamma_i^c(n)$ to be the spaces of all Borel-measurable functions mapping, respectively, $Z_i^0(n)$, $Z_i^f(n)$, and $Z_i^c(n)$ into R^{r_i}. Each of these newly defined spaces will be called the admissible strategy space of Player i at stage n under the corresponding information pattern. Denoting the control strategy of Player i at stage n by γ_{in}, we let Γ_i^0 be the N-tuple product space such that $\gamma_i\triangleq(\gamma_{i0}',\gamma_{i1}',\cdots,\gamma_{i(n-1)}')'$ is in Γ_i^0 if and only if $\gamma_{in}\in\Gamma_i^0(n)$, $\forall n=0,\cdots,N-1$. We define the product strategy spaces Γ_i^f and Γ_i^c analogously.

Definition 1: A pair (γ_1^0,γ_2^0) is said to be an *OL-OL saddle-point pair* for the ZSDG under consideration if the following saddle-point inequality is satisfied for all $\gamma_1\in\Gamma_1^0$, $\gamma_2\in\Gamma_2^0$:

$$J(\gamma_1^0,\gamma_2)\leqslant J(\gamma_1^0,\gamma_2^0)\leqslant J(\gamma_1,\gamma_2^0). \tag{3}$$

Similar definitions apply to CL-CL, CLNM-CLNM saddle-point solutions and to all such permutations of OL, CL, and CLNM.

For every pair $(\gamma_1\in\Gamma_1^c,\gamma_2\in\Gamma_2^c)$, we now let $\bar{x}_n(\gamma_1,\gamma_2)$ denote the value of x_n in terms of the initial state x_0 and let $\bar{x}^n(\gamma_1,\gamma_2)$ denote the value of x^n in terms of the initial state x_0, which are determined uniquely and recursively as follows (where we suppress the functional dependence on γ_1 and γ_2 for convenience):

$$\begin{aligned}
&\bar{x}_1=f_0(x_0,\gamma_{10}(x_0),\gamma_{20}(x_0))\\
&\bar{x}^1=(\bar{x}_0',\bar{x}_1')'\\
&\bar{x}_2=f_1(\bar{x}_1,\gamma_{11}(\bar{x}^1),\gamma_{21}(\bar{x}^1))\\
&\bar{x}^2=(\bar{x}^{1\prime},\bar{x}_2')'\\
&\vdots\\
&\bar{x}_{n+1}=f_n(\bar{x}_n,\gamma_{1n}(\bar{x}^n),\gamma_{2n}(\bar{x}^n))\\
&\bar{x}^{n+1}=(\bar{x}^{n\prime},\bar{x}_{n+1}')'.
\end{aligned} \tag{4}$$

Definition 2: For any pair of strategies $(\gamma_1\in\Gamma_1^c,\gamma_2\in\Gamma_2^c)$, the pair $(\bar{\gamma}_1\in\Gamma_1^0,\bar{\gamma}_2\in\Gamma_2^0)$ defined uniquely by

$$\bar{\gamma}_{1n}(x_0)\equiv\gamma_{1n}(\bar{x}^n), \qquad \forall x_0\in R^m \tag{5a}$$

$$\bar{\gamma}_{2n}(x_0)\equiv\gamma_{2n}(\bar{x}^n), \qquad \forall x_0\in R^m \tag{5b}$$

is the *unique open-loop representation* of the pair (γ_1,γ_2). Conversely, given a pair of OL strategies $(\bar{\gamma}_1\in\Gamma_1^0,\bar{\gamma}_2\in\Gamma_2^0)$, a pair $(\gamma_1\in\Gamma_1^c,\gamma_2\in\Gamma_2^c)$ that satisfies (5) is a *closed-loop representation* of the pair $(\bar{\gamma}_1,\bar{\gamma}_2)$. Furthermore, if there exists a pair $(\gamma_1\in\Gamma_1^f,\gamma_2\in\Gamma_2^f)$ that satisfies (5), then that pair is a *pure-feedback representation* of the OL strategies $(\bar{\gamma}_1,\bar{\gamma}_2)$.

Remark 1: OL strategies are by definition included in the class of CL strategies and hence according to the above definition an open-loop representation might be equivalent to a closed-loop representation when one deals with open-loop policies. In order to resolve this ambiguity we will from now on refer to a policy $\gamma_i\in\Gamma_i^c$ as a closed-loop strategy only if $\gamma_i\notin\Gamma_i^0$, $i=1,2$, unless otherwise mentioned.

Remark 2: For any given pair of OL strategies $(\gamma_1^0\in\Gamma_1^0,\gamma_2^0\in\Gamma_2^0)$, a closed-loop representation does not necessarily exist, and if it does then it is never unique. This nonuniqueness arises from the property that if $(\gamma_1\in\Gamma_1^c,\gamma_2\in\Gamma_2^c)$ is one such representation, then the pair $(\gamma_1+\beta_1\in\Gamma_1^c,\gamma_2+\beta_2\in\Gamma_2^c)$ is also a CL representation, where β_1 and β_2 are arbitrary elements of Γ_1^c and Γ_2^c, respectively, and with the additional property that their OL representations are identically zero, i.e., $\beta_{1n}(\bar{x}^n)\equiv 0$, $\beta_{2n}(\bar{x}^n)\equiv 0$ for all $n=0,\cdots,N-1$.

III. Main Results

Two important properties of the saddle-point solutions of such zero-sum dynamic games are now given below in Propositions 1 and 2.

Proposition 1: If the zero-sum dynamic game admits a unique pure-feedback saddle-point solution (γ_1^f,γ_2^f), and if (γ_1^0,γ_2^0) is any open-loop saddle-point solution, then

1) $\gamma_1^f(\bar{x}_n)\equiv\gamma_1^0(x_0)$;
2) $\gamma_2^f(\bar{x}_n)\equiv\gamma_2^0(x_0)$;
3) (γ_1^0,γ_2^0) is the unique open-loop saddle-point solution.

Proposition 2: Let (γ_1^f,γ_2^f) be the unique pure-feedback saddle-point solution of the zero-sum dynamic game. Furthermore, let $(\gamma_1^c\in\Gamma_1^c,\gamma_2^c\in\Gamma_2^c)$ be any saddle-point solution in closed-loop strategies. Then,

$$\gamma_1^c(\bar{x}^n)\equiv\gamma_1^f(\bar{x}_n)$$

$$\gamma_2^c(\bar{x}^n)\equiv\gamma_2^f(\bar{x}_n).$$

In the proof of these two propositions we will make use of the following two lemmas.

Lemma 1 (Witsenhausen [6]): Every open-loop saddle-point solution $(\gamma_1^0\in\Gamma_1^0,\gamma_2^0\in\Gamma_2^0)$ and every pure-feedback saddle-point solution $(\gamma_1^f\in\Gamma_1^f,\gamma_2^f\in\Gamma_2^f)$ also constitute a saddle-point solution on $\Gamma_1^c\times\Gamma_2^c$.

Lemma 2 (Owen [5], p. 13): If $\{\gamma_1^c\in\Gamma_1^c,\gamma_2^c\in\Gamma_2^c\}$ and $\{\gamma_3^c\in\Gamma_1^c,\gamma_4^c\in\Gamma_2^c\}$ are two saddle-point solutions, then (γ_1^c,γ_4^c) and (γ_3^c,γ_2^c) are also saddle-point pairs.

Proof of Proposition 1: It follows from Lemma 1 that (γ_1^f,γ_2^f) and (γ_1^0,γ_2^0) also constitute a saddle-point on the enlarged strategy space $\Gamma_1^c\times\Gamma_2^c$. Furthermore, by Lemma 2, (γ_1^0,γ_2^f) and (γ_1^f,γ_2^0) are also saddle-

point pairs on $\Gamma_1^c \times \Gamma_2^c$, i.e.,

$$J(\gamma_1^0, \gamma_2) \leq J(\gamma_1^0, \gamma_2^f) \leq J(\gamma_1, \gamma_2^f), \qquad \forall \gamma_1 \in \Gamma_1^c,\ \gamma_2 \in \Gamma_2^c,$$

and an analogous pair of inequalities for (γ_1^f, γ_2^0). Now, let us consider the right-hand side inequality

$$J(\gamma_1^0, \gamma_2^f) = \min_{\Gamma_1^c} J(\gamma_1, \gamma_2^f).$$

This is a standard optimum control problem in discrete time with a cost function

$$\hat{J}(\gamma_1) = \phi_N(x_N) + \sum_{n=0}^{N-1} \hat{\phi}_n(x_n, u_n).$$

$$\hat{\phi}_n(x_n, u_n) \triangleq \phi_n(x_n, u_n, \gamma_{2n}^f(x_n)), \qquad u_n = \gamma_{1n}(x^n),$$

and state equation

$$x_{n+1} = \hat{f}_n(x_n, u_n) \triangleq f_n(x_n, u_n, \gamma_{2n}^f(x_n)).$$

Globally optimal solution of this problem over Γ_1^c can be obtained (if it exists) via dynamic programming and as an element of Γ_1^f. But by hypothesis (since (γ_1^f, γ_2^f) is a saddle-point pair) there exists a unique element in Γ_1^f (namely, γ_1^f) that solves this problem. Since every closed-loop policy admits a unique open-loop representation, it follows that there exists a unique solution to that optimum control problem over Γ_1^0, which is given by $\gamma_1^0(x_0) = \gamma_1^f(\bar{x}_n)$. This completes the proof of 1). Part 2) can likewise be proven. To prove part 3), we now assume that the open-loop saddle-point is nonunique that is, there exist two open-loop saddle-point pairs (γ_1^0, γ_2^0) and $(\hat{\gamma}_1^0, \hat{\gamma}_2^0)$ with $\hat{\gamma}_i^0 \neq \gamma_i^0$ for at least one i (say, $i = 1$). Then $(\hat{\gamma}_1^0, \gamma_2^f)$ is also a saddle-point on $\Gamma_1^c \times \Gamma_2^c$ by Lemmas 1 and 2. But $\hat{\gamma}_1^0(x_0) \equiv \gamma_1^f(\bar{x}_n) \equiv \gamma_1^0(x_0)$ by 1). Therefore, the open-loop saddle-point is unique.

Proof of Proposition 2: If (γ_1^c, γ_2^c) is a saddle-point pair, then it follows from Lemmas 1 and 2 that (γ_1^c, γ_2^f) and (γ_1^f, γ_2^c) are also saddle-point pairs. Since (γ_1^c, γ_2^f) is a saddle-point solution we have

$$J(\gamma_1^c, \gamma_2^f) = \min_{\Gamma_1^c} J(\gamma_1, \gamma_2^f).$$

The right-hand side of this equality defines an optimum control problem, and as in the proof of Proposition 1, every globally optimal solution of this optimization problem can be obtained via dynamic programming, and by hypothesis γ_1^f is the unique element in Γ_1^f that renders the global minimum. Hence, every solution in Γ_1^c has the same open-loop representation as γ_1^f, that is, $\gamma_1^c(\bar{x}^n) \equiv \gamma_1^f(\bar{x}_n)$. Similarly, $\gamma_2^c(\bar{x}^n) \equiv \gamma_2^f(\bar{x}_n)$.

Remark 3: It should be obvious that we cannot claim uniqueness of the pair (γ_1^c, γ_2^c) since there are uncountably many ordered pairs in $\Gamma_1^c \times \Gamma_2^c$ with identical open-loop representations, and some of these could be saddle-point pairs.

IV. Certain Comments and Discussion

The two properties of the saddle-point solution, that we have outlined in Propositions 1 and 2, are also shared in common with optimal (team) solutions of two-person dynamic team problems. Moreover, it is well-known that for dynamic team problems, the statements of Propositions 1 and 2 can actually be made stronger.

Proposition 1': If a two-person dynamic team problem admits a unique pure-feedback optimal team solution (γ_1^f, γ_2^f), then it admits a unique open-loop team solution (γ_1^0, γ_2^0) which can be obtained through the relations

$$\gamma_1^0(x_0) = \gamma_1^f(\bar{x}_n)$$

$$\gamma_2^0(x_0) = \gamma_2^f(\bar{x}_n).$$

Proposition 2': If (γ_1^c, γ_2^c) is any optimal team solution in closed-loop strategies, then its open-loop representation is also an optimal team solution.

Even though these two properties are valid for team solutions, they are not shared in common with saddle-point solutions, since existence of a saddle-point solution in feedback or more general closed-loop strategies does not necessarily imply existence of an open-loop saddle-point solution (Başar [2], Bensoussan [4]). On the other hand, existence of a saddle-point in open-loop policies could necessarily imply existence of a saddle-point in pure-feedback strategies since there are as yet no results available in the literature to disprove this conjecture.

For the Nash solutions of nonzero-sum games, however, none of these properties remain valid (see for example [1], [3]). Hence, we can say that Propositions 1 and 2 provide two properties of saddle-point solutions, that are shared in common with team solutions but not with Nash solutions.

References

[1] T. Başar, "A counterexample in linear-quadratic games: Existence of nonlinear Nash solutions," *J. Optimiz. Theory Appl.*, vol. 14, no. 4, 1974.

[2] T. Başar, "Some thoughts on saddle-point conditions and information structures in zero-sum differential games," *J. Optimiz Theory Appl.*, vol. 18, no. 1, 1976.

[3] T. Başar and H. Selbuz, "Properties of Nash solutions of a two-stage nonzero-sum game," *IEEE Trans. Automat. Contr.*, vol. AC-21, Feb. 1976.

[4] A. Bensoussan, "Saddle-points of convex concave functionals-with applications to linear quadratic differential games," in *Differential Games and Related Topics*, H. W. Kuhn and G. P. Szego, Eds. Amsterdam, Holland: North-Holland, 1971.

[5] G. Owen, *Game Theory* Philadelphia, PA: Saunders, 1968.

[6] H. S. Witsenhausen, "On the relations between the values of a game and its information structure," *Inform. Contr.*, vol. 19, no. 3, 1971.

On Computation of the Canonical Pencil of a Linear System

DAVID JORDAN AND LOUIS F. GODBOUT, JR.

***Abstract*—An algorithm for computing the canonical form of a singular pencil is presented. This algorithm, which is based on well-known linear algebra techniques, offers efficiency and error analysis advantages over previous algorithms presented by Thorp and Gantmacher. These results are useful in exposing the structure of linear multivariable systems.**

Introduction

Kronecker's pencil theory [4], [5] has found recent application in such linear system theoretic areas as model matching [10], pole placement [3], and modeling [2]. These results have been achieved primarily as a result of the convenient structure of the canonical form of a singular pencil first introduced by Gantmacher [4] and applied to linear systems by Thorp [10] and Morse [7]. Both Gantmacher [4] and Thorp [11] have presented methods for computing canonical form, the latter including a computational algorithm. This correspondence presents an alternate algorithm based on well known, efficient linear algebra techniques.

Given the n-state, m-input, r-output linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{1}$$

the corresponding system pencil is

$$\Gamma(\lambda) = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}. \tag{2}$$

Further, there exist nonsingular matrices P and Q of the form

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} P_{11}^{-1} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} \tag{3}$$

Manuscript received June 11, 1976. This work was supported in part by the NSF under Grant GK-37841.

The authors are with the Department of Electrical Engineering and Computer Science, University of Connecticut, Storrs, CT 06268.