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# Distributed Algorithms for the Computation of Noncooperative Equilibria*†

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*For a general class of nonquadratic convex noncooperative games, there exist distributed algorithms employing iteration in the policy space and accurate/inaccurate search techniques, which yield the unique stable Nash equilibrium*

**Key Words**—Game theory, noncooperative equilibria, numerical methods, distributed algorithms, inaccurate search techniques

**Abstract**—In this paper, a general class of nonquadratic convex Nash games is studied, from the points of view of existence, stability and iterative computation of noncooperative equilibria Conditions for contraction of general nonlinear operators are obtained, which are then used in the stability study of such games These lead to existence and uniqueness conditions for stable Nash equilibrium solutions, under both global and local analysis Also, convergence of an algorithm which employs inaccurate search techniques is verified It is shown in the context of a fish war example that the algorithm given is in some aspects superior to various algorithms found in the literature, and is furthermore more meaningful for real world implementation

## 1 INTRODUCTION

DISTRIBUTED ITERATIVE SCHEMES are important and highly relevant in the computation of equilibria in noncooperative games, particularly because they could be implemented in a real world game situation when the game is incomplete (Harsanyi, 1967, 1968a, b) In such incomplete games, and with distributed computation, the agents do not have to know each other's cost functionals and private information, as well as the parameters and subjective probability distributions adopted by the others, they only have to communicate to each other their tentative decisions during each phase of computation Such naturally also carry computational advantages over centralized schemes

Some earlier results on this class of problems are the following Perhaps the first serious algorithmic work done on Nash games was by Rosen (1965), who studied a gradient-type algorithm for convex games More recently, a contraction mapping theorem has been given by Bertsekas (1983) for general asynchronous distributed algorithms for teams and equation solving problems A recent paper by Papavassilopoulos (1986) deals with various updating schemes for stochastic quadratic adaptive Nash games Results concerning noncooperative games with inconsistent probabilistic models can be found in Başar (1985) and Başar and Li (1985)

In this paper, the general class of nonquadratic convex Nash games is considered, first for two players, and subsequently for $N$ players In Theorem 1 a general condition for contraction of general nonlinear operators is obtained, which can be applied to the stability problems arising in such convex games Existence and uniqueness conditions for stable Nash equilibria are given, under both global and local analysis Then the result obtained here on uniqueness and stability is compared with Rosen's (1965) result which applies to similar types of static games Some algorithms and learning schemes using both accurate and inaccurate searches are also developed It is shown in the context of a fish war example that the algorithm given in this paper is in some aspects superior to both Newton's algorithm and the gradient algorithm for this class of problems, and is furthermore more meaningful for real world implementation

The paper is organized as follows In Section 2, the problem is formulated in a Hilbert space setting

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In Section 3, conditions for existence and uniqueness of globally and locally stable Nash equilibria are provided In Section 4, an inaccurate search algorithm is presented and its convergence is analyzed Section 5 is devoted to an example chosen from the field of economics which enables the authors to illustrate numerically various aspects of the accurate and inaccurate search algorithms and to compare them with other existing algorithms In Section 6, the results from two player games are extended to $n$-player games In Section 7, possible extensions are considered and concluding remarks provided

## 2 PROBLEM FORMULATION

In this section, the problem is formulated in a Hilbert space setting, and the Nash equilibrium and stable Nash equilibrium concepts are introduced

Let $N = \{1, 2\}$ be the player set, $U$ and $V$ be the strategy spaces of decision makers 1 and 2, respectively, and $J^i$ $U \times V \to R$ be the cost functional for decision maker $i$ (DM$i$), $i \in N$ Then the normal form of the game is described by $\{U, V\}$ and $\{J^1, J^2\}$, where it is assumed that $U$ and $V$ are appropriate Hilbert spaces with inner products $\langle\ ,\ \rangle_u$ and $\langle\ ,\ \rangle_v$, respectively

*Definition* 2 1 A pair of strategies $(u^N \in U, v^N \in V)$ constitutes a *Nash equilibrium solution* if

$$J^1(u^N, v^N) \leqslant J^1(u, v^N)$$

$$J^2(u^N, v^N) \leqslant J^2(u^N, v)$$

for all $u \in U$ and $v \in V$

Define the sequence $\{u_k, v_k\}$ by the following algorithm

$$u_{k+1} = \arg\min_{u \in U} J^1(u, v_k),$$

$$v_{k+1} = \arg\min_{v \in V} J^2(u_{k+1}, v), \quad v_0 \in \mathrm{V}, \qquad (2\ 1)$$

or equivalently,

$$v_{k+1} = \arg\min_{v \in V} J^2(u_k, v),$$

$$u_{k+1} = \arg\min_{u \in U} J^1(u, v_{k+1}) \quad u_0 \in U \qquad (2\ 1')$$

*Definition* 2 2 A Nash equilibrium solution is *stable* if it can be obtained as the limit of (2 1) or (2 1′) for any initial point It is *locally stable* if convergence is valid for all initial conditions in some $\varepsilon$-neighborhood of the equilibrium solution □

Now some structure is brought on $J^i$, in particular, assume that $J^i(u, v)$ is second order continuously Fréchet differentiable (Luenberger, 1969), $J^1(u, v)$ is strongly convex in $u$, and $J^2(u, v)$ is strongly convex in $v$ Furthermore, assume that for each $v \in V$ there exists a $u_v \in U$ such that

$$J^1(u_v, v) \leqslant J^1(u, v), \quad \text{for all } u \in U, \qquad (2\ 2a)$$

and likewise, for each $u \in U$ there exists a $v_u \in V$ such that

$$J^2(u, v_u) \leqslant J^2(u, v), \quad \text{for all } v \in V \qquad (2\ 2b)$$

Note that a set of sufficient conditions for (2 2a) and (2 2b) when $U$ and $V$ are finite dimensional is that for every sequence $\{u_k\} \in U$ and $\{v_k\} \in V$, with $\lim_k \|u_k\| = \lim_k \|v_k\| = +\infty$, and every $u \in U$, $v \in V$

$$\lim_{k \to \infty} J^1(u_k, v) = \infty \quad \text{and} \quad \lim_{k \to \infty} J^2(u, v_k) = \infty$$

When $U$ and $V$ are infinte-dimensional Hilbert spaces, a similar condition applies, with the unboundedness requirement on $J^1$ and $J^2$ replaced by some appropriate coercivity growth bound (Ekeland and Temam, 1974), which is not given here because its specific form will not be needed in the analysis to follow

The problem addressed below is the derivation of conditions under which algorithm (2 1) [or equivalently (2 1′)] converges to a Nash equilibrium when the cost functions $J^i$ are of the type given above Towards this end first note that since $J^1(u, v)$ and $J^2(u, v)$ are strictly convex and Fréchet differentiable in $u$ and $v$, respectively, under the further conditions (2 2a) and (2 2b) the following are the necessary and sufficient conditions for a Nash equilibrium to satisfy

$$\nabla_u J^1(u, v) = 0$$

$$\nabla_v J^2(u, v) = 0$$

By strong convexity and continuous second-order Fréchet differentiability, the operators $\nabla^2_{u^2} J^1$ and $\nabla^2_{v^2} J^2$ exist, are continuous and are strongly positive (i e $\langle h, \nabla^2_{u^2} J^1 h \rangle_u > 0$, for all $h \in U$, $h \neq 0$, and $\langle h, \nabla^2_{v^2} J^2 h \rangle_v > 0$ for all $h \in V$, $h \neq 0$) Therefore, $(\nabla^2_{u^2} J^1)^{-1}$ and $(\nabla^2_{v^2} J^2)^{-1}$ exist, and by the implicit function theorem on Banach spaces (Kantorovich and Akilov, 1964, p 689) there exist $L_1$ $V \to U$ and $L_2$ $U \to V$, both continuously Fréchet differentiable (locally), such that

$$\nabla_u J^1(L_1(v), v) = 0, \quad \nabla_v J^2(u, L_2(u)) = 0,$$

is some open neighborhood of some given pair $(\tilde{u}, \tilde{v}) \in U \times V$ Since this is defined for every such pair, and $L_1$ and $L_2$ are unique in each case because of strong convexity, it follows that $L_1$ and $L_2$ can be extended to the entire domain as continuously

Fréchet differentiable mappings

Hence,

(i) a Nash equilibrium solution $(u, v)$ must satisfy

$$u^N = L_1(v^N) \quad \text{and} \quad v^N = L_2(u^N),$$

(ii) Algorithm (2 1) is equivalent to

$$u_{k+1} = L_1(v_k) \quad \text{and} \quad v_{k+1} = L_2(u_{k+1}) \qquad \text{for all } v_0 \in V, \tag{2 3}$$

and (2 1′) is equivalent to

$$v_{k+1} = L_2(u_k) \quad \text{and} \quad u_{k+1} = L_1(v_{k+1}) \qquad \text{for all } u_0 \in U, \tag{2 3′}$$

where $L_1\ V \to U, L_2\ U \to V$ are continuously Fréchet differentiable mappings

The question of existence and stability of $\{u^N, v^N\}$ can now be addressed

## 3 EXISTENCE OF A STABLE NASH EQUILIBRIUM

The following theorem concerns the existence, uniqueness and stability of a Nash equilibrium solution

*Theorem* 1 Suppose that $J^1(u,v)$ and $J^2(u,v)$ are strongly convex in $u$ and $v$, respectively, satisfying (2 2a) and (2 2b), and are second order Fréchet differentiable Furthermore, assume that $J^1(u,v)$ and $J^2(u,v)$ satisfy one of the following two conditions for some positive $\alpha < 1$ and $\beta < 1$

(i) $$\|T_{uu}\| = \|[(\nabla^2_{u^2}J^1)^{-1}\nabla^2_{uv} \times J^1]_{(u,v)}[(\nabla^2_{v^2}J^2)^{-1}\nabla^2_{vu}J^2]_{(\bar{u},v)}\| = \alpha < 1$$

for all $\bar{u} \in U$ and with $v = L_2(\bar{u}), u = L_1L_2(\bar{u})$,

(ii) $$\|T_{vv}\| = \|[(\nabla^2_{v^2}J^2)^{-1}\nabla^2_{vu} \times J^2]_{(u,v)}[(\nabla^2_{u^2}J^1)^{-1}\nabla^2_{uv}J^1]_{(u,\bar{v})}\| = \beta < 1,\text{*}$$

for all $\bar{v} \in V$ and with $u = L_1(\bar{v}), v = L_2L_1(\bar{v})$ Then, there exists a unique stable Nash equilibrium solution, i e for any $u_0 \in U$ or $v_0 \in V$, the sequence $\{u_k, v_k\}$ generated by (2 1) or (2 1′) converges, and the limit point is a Nash equilibrium solution □

*Remark* 3 1 Note that for the algorithm to be implemented it is not necessary that the decision makers know each other's cost functionals, as long as the tentative decisions are transmitted Hence, this scheme is implementable in the context of incomplete games □

*Remark* 3 2 The contraction condition (i) or (ii) in Theorem 1 is tight in the sense that for the particular case when $J^1 = J^2 = J$, and $J$ is duadratic, it reduces to the known condition of strong convexity of $J$ in $(u,v) \in U \times V$ □

*Proof of Theorem* 1 Here the theorem is proved under (i) The proof under (ii) follows a similar line of argument, and is therefore omitted

By (2 3′), for any given $\bar{u} \in U$, $u = L_1(v) = L_1(L_2(\bar{u}))$ Let $T\ U \to U, T = L_1L_2$ Then if $T$ is a contraction mapping, the result will follow from the Banach contraction mapping theorem (Luenberger, 1969)

Now for any $\bar{u} \in U$, by the chain rule of Fréchet derivatives,

$$T'(\bar{u}) = L_1'(v)L_2'(\bar{u}) \quad \text{where } v = L_2(\bar{u}) \tag{3 1}$$

But from the definition of $L_1, L_2$,

$$\nabla^2_{u^2}J^1(u,v)L_1'(v) + \nabla^2_{uv}J^1(u,v) = 0,$$
$$\Rightarrow L_1'(v) = -(\nabla^2_{u^2}J^1(u,v))^{-1}\nabla^2_{uv}J^1(u,v),$$

$$\nabla^2_{vu}J^2(\bar{u},v) + \nabla^2_{v^2}J^2(\bar{u},v)L_2'(\bar{u}) = 0,$$
$$\Rightarrow L_2'(\bar{u}) = -(\nabla^2_{v^2}J^2(\bar{u},v))^{-1}\nabla^2_{vu}J^2(\bar{u},v),$$

hence (3 1) becomes

$$T'(\bar{u}) = (\nabla^2_{u^2}J^1(u,v))^{-1}\nabla^2_{uv}J^1(u,v)(\nabla^2_{v^2}J^2(\bar{u},v))^{-1} \times \nabla^2_{vu}J^2(\bar{u},v) \equiv T_{uu}(\bar{u}),$$

where $v = L_2(\bar{u}), u = L_1(v)$

Then, by the mean value theorem, for any $u', u'' \in U$, and by condition (i) in the theorem,

$$\|T(u') - T(u'')\| \leqslant \|u' - u''\| \times \sup_{0<\theta<1} \|T'(u'' + \theta(u' - u''))\| \leqslant \alpha\|u' - u''\| \tag{3 2}$$

Clearly, (3 2) implies that $T$ is a contraction mapping from a complete space into itself This completes the proof of the theorem □

Now, a related question is the following If some other synchronous or asynchronous algorithm is used, are the conditions in Theorem 1 still sufficient? The answer is in the affirmative, mainly because there are only two players

* Here, by an abuse of notation, $\|\ \|$ is used to denote appropriate operator norms (Luenberger, 1969) If $U$ and $V$ are finite dimensional spaces, $\|\ \|$ can be taken to be the spectral norm, and then the two conditions (i) and (ii) become equivalent

*Theorem* 2 For any asynchronous algorithm of the type

$$u_{k+l_k} = L_1(v_k)^*$$
$$v_{k+\iota_k} = L_2(u_k), \quad k = 0, 1,$$

where the integer sequences $\{l_k\}$ and $\{\iota_k\}$ are positive and bounded, the sequence generated by the algorithm converges to the unique Nash equilibrium solution for all $v_0 \in V$ or $u_0 \in U$ provided that the conditions in Theorem 1 are satisfied

*Proof* Let $k' = k + \iota_k$, and $k'' = k' + l_k \equiv k + \jmath_k$, where $\{\jmath_k\}$ is a positive and bounded integer sequence because of the positivity and boundedness of $\{\iota_k\}$ and $\{l_k\}$ Then, the asynchronous algorithm can equivalently be written as

$$u_{k+\jmath_k} = L_1(L_2(u_k)) \equiv T(u_k)$$

Now, since $\jmath_k > 0$ and is uniformly bounded, there is a one-to-one correspondence between the sequence $\{0, \jmath_0, \jmath_{\jmath_0}, \quad\}$ and $\{0, 1, 2, \quad\}$, and therefore Theorem 1 directly applies here to yield the desired result □

*Remark* 3 3 Theorem 1 provides a uniqueness condition which leads to convergence and stability There is an earlier result in the literature (see Rosen, 1965) which pertains to another condition on the uniqueness and existence of Nash solution and convergence of a gradient-type algorithm for constrained convex Nash games Under the assumption that equilibrium is achieved inside the constraint set, one sufficient condition for uniqueness given in that paper is (for a two-player game and with $U = R^{p_1}, V = R^{p_2}$)

$$\begin{bmatrix} \nabla^2_{u^2} J^1(u,v) & \nabla^2_{uv} J^1(u,v) \\ \nabla^2_{vu} J^2(u,v) & \nabla^2_{v^2} J^2(u,v) \end{bmatrix} + \begin{bmatrix} \nabla^2_{u^2} J^1(u,v) & \nabla^2_{uv} J^2(u,v) \\ \nabla^2_{vu} J^1(u,v) & \nabla^2_{v^2} J^2(u,v) \end{bmatrix} > 0,$$
$$\text{for any } (u,v) \in U \times V \qquad (3\,3)$$

For this problem, condition (i) given in Theorem 1 becomes

$$\|(\nabla^2_{u^2} J^1(u,v))^{-1} \nabla^2_{uv} J^1(u,v) \times (\nabla^2_{v^2} J^2(\bar{u},v))^{-1} \nabla^2_{vu} J^2(\bar{u},v)\| < 1$$
$$\text{for any } \bar{u} \in U, v = L^2(\bar{u}), u = L^1(v) \qquad (3\,4)$$

In general, it is not possible to compare (3 3) and (3 4) But for $p_1 = p_2 = 1$, and with $J^1, J^2$ quadratic, (3 3) and (3 4) can be compared, and some insight into these two conditions can be gained Toward this end, let

$$J^\iota = \frac{1}{2}u^2 R^\iota_{11} + v^2 R^\iota_{22} + uR^\iota_{12}v + C^\iota_1 u + C^\iota_2 v$$

Then condition (3 3) becomes

$$4(R^1_{11}R^2_{22} - R^1_{12}R^2_{12}) > (R^1_{12} - R^2_{12})^2 \quad (3\,3')$$

where $R^\iota_{jk}$ is a scalar for $\iota, \jmath, k = 1, 2$ and $R^1_{11} > 0$, $R^2_{22} > 0$ Condition (3 4), on the other hand, is

$$|R^{1\;-1}_{11} R^1_{12} R^{2\;-1}_{22} R^2_{12}| < 1,$$
$$\text{i e} \quad |R^1_{11}R^2_{22}| - |R^1_{12}R^2_{12}| > 0 \qquad (3\,4')$$

Clearly, in this case, (3 3') is more restrictive than condition (3 4') since the right hand side of (3 3') is non-negative and could, in fact, be made positive But in general, it is difficult to compare the two conditions □

Note that in the general formulation, the authors did not impose any structural restrictions on the Hilbert spaces $U$ and $V$, hence, the preceding results also apply to stochastic problems with static information patterns An example on this will now be provided, as an illustration of the use of Theorem 1

*Example* 3 1 The formulation in Başar (1985) will be followed and the notation of that paper used Let $(\Omega, F)$ be the underlying measurable space and $P^1$ and $P^2$ be the subjective probability measures perceived by DM1 and DM2, respectively Each DM$\iota$ has a private measurement $y^\iota$ on the random variable $x$, where $x$ and $y^\iota$, $\iota = 1, 2$, are random vectors on the two probability spaces $(\Omega, F, P^\iota)$ $\iota = 1, 2$ Define

$$J^\iota(\gamma^1, \gamma^2) = E^\iota\{g^\iota(x, \gamma^1(y^1), \gamma^2(y^2))\}, \quad \text{for } \iota = 1, 2,$$

where $E^\iota$ denotes the expectation corresponding to $P^\iota$ and $\gamma^\iota$ is the strategy of DM$\iota$ belonging to the Hilbert space $\Gamma^\iota, \iota \in N$

With the above formulation, the following immediate consequence of Theorem 1 will now be given

*Corollary* 1 For the static two-person stochastic game introduced above, suppose that $J^1(\gamma^1, \gamma^2)$ and $J^2(\gamma^1, \gamma^2)$ are strictly convex in $\gamma^1$ and $\gamma^2$, respectively, are second-order continuously Fréchet differentiable, and satisfy conditions similar to (2 2a)

* The interpretation here is that DM2's decision at time instant $k$ (i e $v_k$) is used by DM1 in his computation only at the time instant $k + l_k > k$, a similar interpretation applies to the second recursion

and (2 2b), with $(\gamma^1, \gamma^2)$ replacing $(u, v)$ Assume that the following conditions are satisfied for at least one $i, i, j = 1, 2, i \neq j$.

$$\|T_{ii}\| = \|[(E^i\{\nabla^2_{\gamma^{i2}} g^i(x, \gamma^1(y^1), \gamma^2(y^2)) | y^i\})^{-1}$$

$$\times E^j\{\nabla^2_{\gamma^i\gamma^j} g^i(x, \gamma^1(y^1), \gamma^2(y^2)) | y^i\}]$$

$$\times [(E^j\{\nabla^2_{\gamma^{j2}} g^j(x, \bar{\gamma}^i(y^1), \gamma^2(y^2)) | y^j\})^{-1}$$

$$\times (E^j\{\nabla^2_{\gamma^j\gamma^i} g^j(x, \bar{\gamma}^1(y^1), \gamma^2(y^2)) | y^2\})]\| = \alpha < 1$$

Then, the stochastic game admits a unique stable Nash equilibrium solution □

*Remark* 3 4 Even though the condition of the corollary looks rather simple on the surface, the inverse operators in $T_{11}$ and $T_{22}$ may, in fact, be very difficult to evaluate, since they involve some integral operators Nevertheless, the corollary does indicate the possibility that a policy space iteration algorithm could converge to the unique Nash equilibrium solution even in general convex stochastic games, with the players having different probabilistic models of the decision process □

If conditions (a) and (b) in Theorem 1 cannot be satisfied globally, then the global stability of Nash equilibrium solution is not guaranteed But with a similar proof, conditions for local stability and local uniqueness can be established The following corollary does precisely that, and provides a localization of the results of Theorem 1

*Corollary* 2 Suppose that there exist closed subsets $U_L \subset U$ and $V_L \subset V$ such that

(i) for any $(u, v) \in U_L \times V_L, J^1(u, v), J^2(u, v)$ are strictly convex in $u$ and $v$, respectively, and are second-order continuously Fréchet differentiable,
(ii) for any $u \in U_L, (u, L_2(u)) \in U_L \times V_L$, and for any $v \in V_L, (L_1(v), v) \in U_L \times V_L$,
(iii) either one of (a) or (b) is satisfied for some positive $\alpha < 1$ or $\beta < 1$

(a) $\|T_{uu}\| = \alpha < 1$, for all $\bar{u} \in U_L$, and with $v = L_2(\bar{u})$, $u = L_1L_2(\bar{u})$,*
(b) $\|T_{vv}\| = \beta < 1$, for all $\bar{v} \in V_L$, and with $u = L_1(\bar{v})$, $v = L_2L_1(\bar{v})$

Then there exists a unique locally stable Nash equilibrium solution in $U_L \times V_L$ □

*Remark* 3 5 Conceptually, it is not difficult to generalize Theorem 1 and Corollary 1 to games with more than two players, see Section 6 for one such extension The main difficulty in such a generalization is that one has to deal with cumbersome notation, and the expressions become quite complicated □

## 4 INACCURATE SEARCH ALGORITHMS AND CONVERGENCE ISSUES

In the previous section, it was assumed that at each stage of the iterated algorithm, *exact* solutions of the underlying optimization problems could be obtained by each decision maker But this is not always possible, especially in cases when $J^1$ and $J^2$ are highly nonlinear, in which case the decision makers have to resort to inaccurate search using any one of the available numerical algorithms (such as Newton's algorithm, steepest descent algorithm, etc ) In this section, a desirable property of such an algorithm is identified, a specific inaccurate search algorithm is proposed, and finally the convergence of such an algorithm under the hypotheses of Theorem 1 is proved

First, the inaccurate search algorithm to be used by the decision makers is introduced

Consider (2 1), i e

$$v_{k+1} = \arg\min_{v \in V} J^2(u_k, v),$$

$$u_{k+1} = \arg\min_{u \in U} J^1(u, v_{k+1}), \quad \text{for any } u_0 \in U,$$

which is equivalent to (under the convexity and differentiability assumptions)

$$v_{k+1} = L_1(u_k), \quad u_{k+1} = L_1(v_{k+1})$$

or

$$u_{k+1} = L_1L_2(u_k), \quad \text{for any } u_0 \in U$$

Suppose that at stage $k$, DM1 and DM2 compute $u_{k+1}, v_{k+1}$ using any algorithm $A_k(v)$ and $B_k(u)$, respectively, which satisfy the following property

*Property* 4 1 Given any $(u, v) \in U \times V$, let $\hat{u} = L_1(v), \hat{v} = L_2(u)$, and $A_k(v), B_k(u), k = 1, 2,$ , be such that regardless of the starting points $\hat{u}^{(0)} \in U, \hat{v}^{(0)} \in V$,

$$\lim_{n \to \infty} A_k^{(n)}(v)(\hat{u}^{(0)}) = L_1(v) = \hat{u} \qquad (4\ 1a)$$

$$\lim_{m \to \infty} B_k^{(m)}(u)(\hat{v}^{(0)}) = L_2(u) = \hat{v} \qquad (4\ 1b)$$

* Here $T_{uu}$ and $T_{vv}$ are defined as in (i) and (ii) of Theorem 1

where $A_k(v), B_k(u)$ are some point-to-set mappings, and

$$A_k^{(n)}(v) = A_k(v)S_v A_k(v) \quad S_v A_k(v)$$
$$B_k^{(m)}(u) = B_k(u)S_u B_k(u) \quad S_u B_k(u),$$

with $S_v, S_u$ being some set-to-point mappings which choose arbitrarily a single element out of their domains

Note that (4 1) is satisfied if $A_k(v)$ and $B_k(u)$, and $J^1$ and $J^2$ satisfy the conditions of "global convergence theorem" used in nonlinear programming problems (Luenberger, 1973, p 120) Conditions (4 1) are reasonable assumptions due to the fact that most descent algorithms converge because $J^1$ and $J^2$ are strictly convex in $u$ and $v$, respectively Now, let $u_{k+1}^{(n)}(v, u^{(0)})$ and $v_{k+1}^{(m)}(u, v^{(0)})$ be defined as follows

$$u_{k+1}^{(n)}(v, u^{(0)}) \in A_k^{(n)}(v)(u^{(0)}),$$
$$\text{for any } u^{(0)} \in U, \quad \text{and } v \in V,$$

$$v_{k+1}^{(m)}(u, v^{(0)}) \in B_k^{(m)}(u)(v^{(0)}),$$
$$\text{for any } v^{(0)} \in V, \quad \text{and } u \in U$$

Then, the inaccurate search algorithm is given below

*Inaccurate Search Algorithm* Given two arbitrary positive sequences $\{\delta_k\}_{k=0}^{\infty}, \{\tilde{\delta}_k\}_{k=0}^{\infty}$,

(i) for $k = 0$, let $\bar{u}_k = u_0$, where $u_0 \in U$ is arbitrary,

(ii) for $k = 0, 1, 2,$ , let DM2 choose $\bar{v}_{k+1}$ as $\bar{v}_{k+1} = v_{k+1}^{(K)}(\bar{u}_k, v_{k+1}^{(0)})$, for any $v_{k+1}^{(0)} \in V$, where $K$ is chosen such that $\sup\{\|L_2(\bar{u}_k) - v\|\} < \delta_k$ and supremum is taken over all $v$ belonging to the point-to-set mapping $B_k^{(K)}(\bar{u}_k)(v_{k+1}^{(0)})$ Then, DM1 chooses $\bar{u}_{k+1}$ as

$$\bar{u}_{k+1} = u_{k+1}^{(\bar{K})}(\bar{v}_{k+1}, u_{k+1}^{(0)}),$$
$$\text{for any } u_{k+1}^{(0)} \in U,$$

where $\bar{K}$ is chosen such that $\sup\{\|L_1(\bar{v}_{k+1}) - u\|\} < \tilde{\delta}_k$,

(iii) update on $k$ and return to (ii) □

*Definition* 4 1 Let $H_k = A_k^{(\bar{K})}(\bar{v}_{k+1})S_u B_k^{(K)}(\bar{u}_k)$, $k = 0, 1,$ , Given a positive sequence $\{e_k\}_{k=0}^{\infty}$, let $\{\hat{u}_k\}$ be obtained as $\hat{u}_{k+1} \in H_k(\hat{u}_k)$, for all $k$, where the set $H_k$ (i e $\bar{K}$, $K$) is chosen such that for all possible $S_u$,

$$H_k(\hat{u}_k) = \{u \in U \ \|u - L_1 L_2(\hat{u}_k)\| < e_k\}, k = 0, 1, \tag{4 2}$$

Then, it can be said that $\{H_k\}$ yields an inaccurate search with error level $\{e_k\}$ □

*Remark* 4 1 Note that $e_k$ is the error made by a finite step approximation, i e the composite error made by using $B_k^{(K)}(u)$ and $A_k^{(\bar{K})}(v)$ rather than using $B_k^{(\infty)}$ and $A_k^{(\infty)}$ at stage $k$ Also note that $A_k, B_k$ and $H_k$ have been used, to allow for the situation where the decision makers use different algorithms at different stages □

*Remark* 4 2 The above inaccurate search algorithm is more realistic for real world implementation than some of the algorithms given previously, e g Rosen (1965), Cohen and Culıolu (1985), in two aspects

(i) In Rosen (1965), a gradient-type algorithm is given, in which each player uses the algorithm for only one iteration step at each stage $k$ rather than for $K$ or $\bar{K}$ steps as in the algorithm presented here The inaccurate search algorithm introduced above is more realistic, since at each stage $k$ it yields a decision closer to the optimum, while also accounting for the fact that absolute minimum may never be reached

(ii) In Cohen and Culıolu (1985), where the objective is off-line computation, a descent function which depends on both players' cost functionals is used However, when the goal is on-line computation of equilibria, it is more realistic to assume that the players do not know each other's cost functionals exactly (e g incomplete games), and hence cannot implement the type of algorithm given by Cohen and Culıolu (1985) Thus if one switches from pure computational algorithm to real world implementable algorithms in repeated games, it is more reasonable to use the above inaccurate search algorithm The algorithm also possesses some good convergence properties, as will be illustrated in the example of the next section □

The following result on the convergence of such an algorithm is now presented

*Theorem* 3 Suppose that the hypotheses of Theorem 1 are satisfied Then, for any positive sequence $\{e_k\}$ satisfying

$$\lim_{k \to \infty} e_k = 0 \tag{4 3}$$

and for any inaccurate search $\{H_k\}$ with error level $\{e_k\}$, the sequence $\{\bar{u}_k, \bar{v}_k\}$ generated by the inaccurate search algorithm converges to the unique Nash equilibrium solution for any initial choice $u_0 \in U$ □

*Remark* 4 3 Condition (4 3) clearly implies that as $k$ becomes larger, $H_k$ should approach the accurate search, which in turn means that $K \to \infty$ and

$\bar{K} \to \infty$ as $k \to \infty$ But if one is satisfied with the $\varepsilon$-Nash equilibrium solution, then, by continuity of $J^1$ and $J^2$, the solution can be obtained by holding $K$ and $\bar{K}$ finite, for any given $\varepsilon > 0$ □

*Proof of Theorem* 3 First note that with $(u^N, v^N)$ a Nash equilibrium solution, for all $k = 1, 2,$

$$\|\bar{u}_{k+1} - u^N\| = \|\bar{u}_{k+1} - L_1L_2(\bar{u}_k) + L_1L_2(\bar{u}_k) - u^N\|$$
$$\leqslant \|\bar{u}_{k+1} - L_1L_2(\bar{u}_k)\| + \|L_1L_2(\bar{u}_k) - u^N\|$$

Since $\bar{u}_{k+1} \in H_k(\bar{u}_k)$, and $L_1L_2$ is a contraction operator with parameter $\alpha < 1$ as in Theorem 1(i), the following holds

$$\|u_{k+1} - L_1L_2(\bar{u}_k)\| \leqslant e_k,$$
$$\| L_1L_2(\bar{u}_k) - u^N\| \leqslant \alpha\|\bar{u}_k - u^N\|$$

and hence

$$\|\bar{u}_{k+1} - u^N\| \leqslant e_k + \alpha\|\bar{u}_k - u^N\|, \quad \text{for all } k$$

Iterating on $k$ gives

$$\begin{aligned}\|\bar{u}_{k+1} - u^N\| &\leqslant e_k + \alpha[e_{k-1} + \alpha\|\bar{u}_{k-1} - u^N\|] \\ &= e_k + \alpha e_{k-1} + \alpha^2\|\bar{u}_{k-1} - u^N\| \\ &\leqslant e_k + \alpha e_{k-1} + \alpha^2 e_{k-2} \\ &\quad + \quad + \alpha^k e_0 + \alpha^{k+1}\|u_0 - u^N\| \\ &\triangleq s_k + \alpha^{k+1}\|u_0 - u^N\|\end{aligned}$$

where $s_k$ satisfies the linear difference equation

$$s_{k+1} = \alpha s_k + e_{k+1}$$

Since $0 < \alpha < 1$, and $e_k \to 0$ by (4 3), clearly $s_k \to 0$, and hence

$$\lim_{k\to\infty} \|\bar{u}_{k+1} - u^N\| = \lim_{k\to\infty} s_k + \|u_0 - u^N\| \lim_{k\to\infty} \alpha^{k+1} = 0$$

Therefore, $\bar{u}_k \to u^N$ and, by continuity, $\bar{v}_k \to v^N$ As a side remark note that condition (4 3) is not only sufficient but also necessary for $s_k \to 0$ □

## 5 AN ILLUSTRATIVE EXAMPLE ON NONQUADRATIC PROBLEMS A FISH WAR GAME

In this section a static repeated nonquadratic game is considered, to gain some insight into the conditions and algorithms given above

Suppose that there are two countries involved in a fish war (Levhari and Mirman, 1980), with the cost functions being

$$J^1 = -\log u - \beta_1 \log(x - u - v^{\mu_1})^\tau$$
$$J^2 = -\log v - \beta_2 \log(x - u^{\mu_2} - v)^\tau$$

where $0 < \beta_i, \mu_i \geqslant 1, 0 < \tau < 1, 0 < x < \infty, i = 1, 2,$ and

$$(u, v) \in D = \{(u, v)\ u \geqslant 0,\ v \geqslant 0,\ u + v^{\mu_1} \leqslant x,\ u^{\mu_2} + v \leqslant x\}$$

Here, $u$ and $v$ are the current consumption levels of country 1 and country 2, respectively, and $\beta_i$, $i = 1, 2$ are discount factors Each country attempts to minimize his own cost $J^i$ which is coupled with the other country's decisions The variable $x$ denotes the population of the fish in the region, which we assume to be constant throughout the repeated consumption and reproduction process Suppose that country $i$ does not know country $j$'s cost functional $J^j, i \neq j$, but it knows country $j$'s consumption levels at the previous stages and it chooses its optimal present consumption level according to $j$'s previous consumption level They repeat the process in this fashion until they reach a noncooperative equilibrium consumption pair (if it exists) This example is somewhat contrived but is designed to emphasize the essentials of the notion of stable Nash equilibrium and to illustrate the applicability of the previous results

First observe that the above formulation falls well within the framework of the learning and decision process (2 1) or (2 1′) Hence, Theorem 1 can be used to obtain a condition under which this game evolves finally to a Nash equilibrium solution

Assume that the game admits a Nash solution $(u^N, v^N) \in D$ (by appropriately choosing $\tau, \beta_i, \mu_i$ and $x$) Then,

$$\nabla_u J^1 = -\frac{1}{u} + \frac{\tau\beta_1}{x - u - v^{\mu_1}} = 0$$
$$\Leftrightarrow u = L_1(v) = \frac{x - v^{\mu_1}}{1 + \tau\beta_1}, \tag{5 1}$$

$$\nabla_v J^2 = -\frac{1}{v} + \frac{\tau\beta_1}{x - u^{\mu_2} - v} = 0$$
$$\Leftrightarrow v = L_2(u) = \frac{x - u^{\mu_2}}{1 + \tau\beta_2} \tag{5 2}$$

$$\nabla^2_{u^2} J^1 = \frac{1}{u^2} + \frac{\tau\beta_1}{(x - u - v^{\mu_1})^2} > 0,$$

$$\nabla^2_{uv} J^1 = \frac{\tau\beta_1\mu_1 v^{\mu_1 - 1}}{(x - u - v^{\mu_1})^2},$$

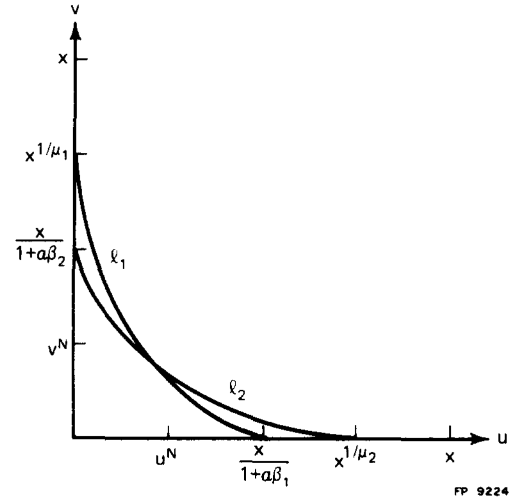


FIG 1 The reactions curves of the two players

$$\nabla^2_{v^2} J^2 = \frac{1}{v^2} + \frac{\tau\beta_2}{(x - u_1^{\mu} - v)^2} > 0,$$

$$\nabla^2_{vu} J^2 = \frac{\tau\beta_2\mu_2 u^{\mu_2 - 1}}{(x - u^{\mu_2} - v)^2}$$

From the above, we see that (a) $J^1$ and $J^2$ are strictly convex in $u$ and $v$, respectively, and $L_1$ and $L_2$ are differentiable, and (b) condition (ii) in Corollary 2 is automatically satisfied (see Fig 1)

The contraction condition [condition (iii a) in Corollary (2)] is

$$|\bar{u}^{\mu_2 - 1}(x - \bar{u}^{\mu_2})^{\mu_1 - 1}| < \frac{(1 + \tau\beta_1)(1 + \tau\beta_2)^{\mu_1}}{\mu_1\mu_2},$$
$$\text{for all } \bar{u} \in [0, x^{1/\mu_2}], \qquad (5\ 3)$$

a sufficient condition for which is

$$x^{\frac{\mu_1\mu_2 - 1}{\mu_2}} < \frac{(1 + \tau\beta_1)(1 + \tau\beta_2)^{\mu_1}}{\mu_1\mu_2} \qquad (5\ 4)$$

Clearly, (5 4) defines a nonvoid set of $x$, $\mu_1$, $\mu_2$, $\beta_1$, $\beta_2$ and $\tau$

A special case of (5 4) is obtained when $\mu_1 = \mu_2 = 1$, in which case (5 4) reduces to $(1 + \tau\beta_1)(1 + \tau\beta_2) > 1$ This is a known result, since in this case $L_1, L_2$ are linear functions and the convergence condition is the same as in the case when $J^1$ and $J^2$ are some appropriate quadratic functions

Hence, if (5 4) is satisfied, then there exists a stable Nash equilibrium solution on $D$ Thus starting from any initial consumptions in $D$, the decision process generated by the two countries converges to the Nash equilibrium point

In what follows, some numerical results for the above Fish War problem will be presented using the following different schemes

(i) Accurate search algorithm (2 1′) (ASA)
(ii) Inaccurate search algorithm (see Section 3) with $A_k, B_k$ being
  (a) Newton's algorithm (ISNA)
  (b) gradient algorithm (ISGA)
(iii) Newton's algorithm (NA)

$$v_{k+1} = v_k - (\nabla^2_{vv} J^2(u_k, v_k))^{-1} \nabla_v J^2(u_k, v_k),$$
$$u_{k+1} = u_k - (\nabla^2_{uu} J^1(u_k, v_k))^{-1} \nabla_u J^1(u_k, v_k),$$

where $a_k$ is some appropriate step size

The convergence aspects of the above schemes will be analyzed and compared

In view of the authors' earlier discussion here, the permissible set of $\tau, \beta_1, \beta_2, x, \mu_1, \mu_2$ will be restricted to the region

$$D_a = \left\{(\tau, \beta_1, \beta_2, \mu_1, \mu_2, x)\ x > 1, x^{\frac{\mu_1\mu_2 - 1}{\mu_2}} < \frac{(1 + \tau\beta_1)(1 + \tau\beta_2)^{\mu_1}}{\mu_1\mu_2}\right\},$$

and a particular $(\tau, \beta_1, \beta_2, \mu_1, \mu_2, x)$ will be chosen from this set to test the above schemes, i e

$$(\tau, \beta_1, \beta_2, \mu_1, \mu_2, x) = (0\ 2852, 0\ 8, 0\ 48, 1\ 1, 1\ 2, 1\ 259) \in D_a$$

For this set of parameters, the unique Nash equilibrium solution is $(u^N, v^N) = (0\ 3, 0\ 9)$, and the corresponding pair of Nash costs is $(J^{1^N}, J^{1^N}) = (1\ 8159, 0\ 3920)$

The numerical results for the five different schemes described above are given in Tables 1 and 2 Comparing these tabulated numerical results, the following observations are made

(i) ASA is superior to all other algorithms (converges in 26 steps, faster than others) *
(ii) ISNA is superior to NA (ISNA converges in 28 steps, while NA converges in 40 steps)
(iii) ISGA is superior to GA (ISGA converges in about 80 steps while GA converges very slowly) Note that if a large step size is chosen, then both of these algorithms may converge faster, but ISGA is still superior to GA
(iv) Newton's algorithm is better than the gradient algorithm, since here $J^1$ and $J^2$ are strictly convex in $u$ and $v$, respectively

*Here and below, by "convergence", it is meant that $\{u_k, v_k\}$ converges to $(u^N, v^N)$ within a degree of accuracy of four significant places

TABLE 1. THE FISH WAR GAME OF SECTION 5: NUMERICAL VALUES FOR THE DECISION VARIABLES $(u, v)$ UNDER THE FIVE DIFFERENT SCHEMES

| | ASA | | ISNA* | | ISGA† | | NA | | GA‡ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $u$ | $v$ | $u$ | $v$ | $u$ | $v$ | $u$ | $v$ | $u$ | $v$ |
| 1 | 0.2000 | 0.2000 | 0.2000 | 0.2000 | 0.2000 | 0.2000 | 0.2000 | 0.2000 | 0.2000 | 0.2000 |
| 5 | 0.2731 | 0.9303 | 0.4251 | 0.7737 | 0.3159 | 0.8236 | 0.6688 | 0.5867 | 0.2018 | 0.2019 |
| 10 | 0.2943 | 0.9064 | 0.3405 | 0.8566 | 0.3512 | 0.8397 | 0.5191 | 0.7108 | 0.2039 | 0.2040 |
| 15 | 0.2988 | 0.9014 | 0.3097 | 0.8892 | 0.3480 | 0.8471 | 0.3956 | 0.8202 | 0.2058 | 0.2059 |
| 20 | 0.2997 | 0.9003 | 0.3021 | 0.8976 | 0.3426 | 0.8532 | 0.3278 | 0.8771 | 0.2075 | 0.2077 |
| 25 | 0.2999 | 0.9001 | 0.3004 | 0.8995 | 0.3376 | 0.8586 | 0.3064 | 0.8947 | 0.2091 | 0.2093 |
| 30 | 0.3000 | 0.9000 | 0.3001 | 0.8999 | 0.3332 | 0.8634 | 0.3014 | 0.8989 | 0.2105 | 0.2107 |
| 35 | 0.3000 | 0.9000 | 0.3000 | 0.9000 | 0.3271 | 0.8709 | 0.3003 | 0.8998 | 0.2118 | 0.2120 |
| 40 | 0.3000 | 0.9000 | 0.3000 | 0.9000 | 0.3195 | 0.8794 | 0.3000 | 0.9000 | 0.2130 | 0.2133 |
| 45 | | | | | 0.3129 | 0.8865 | | | 0.2141 | 0.2144 |
| 50 | | | | | 0.3082 | 0.8915 | | | 0.2151 | 0.2155 |
| 55 | | | | | 0.3050 | 0.8948 | | | 0.2161 | 0.2165 |
| 60 | | | | | 0.3030 | 0.8969 | | | 0.2170 | 0.2174 |
| 65 | | | | | 0.3018 | 0.8981 | | | 0.2178 | 0.2182 |

* In the algorithm, $\delta_k, \bar{\delta}_k$ are chosen as $\delta_k = 0.1 \times 0.9^k, \bar{\delta}_k = 0.9^k$ (see (2.3), inaccurate search algorithm and Corollary 2.3, here $\alpha = 0.9924$).

† Here, $\delta_k$ and $\bar{\delta}_k$ are chosen the same as above, the step size of the gradient algorithm is chosen as $a_k = 1/(100 + k)$.

‡ Step size is chosen the same as above, i.e. $a_k = 1/(100 + k)$. (For some larger step sizes, the algorithm diverges.)

TABLE 2. THE FISH WAR GAME OF SECTION 5: NUMERICAL VALUES FOR THE COST FUNCTIONS $(J^1, J^2)$ UNDER THE FIVE DIFFERENT SCHEMES

| | ASA | | ISNA* | | ISGA† | | NA | | GA‡ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $J1$ | $J2$ | $J1$ | $J2$ | $J1$ | $J2$ | $J1$ | $J2$ | $J1$ | $J2$ |
| 1 | 1.6364 | 1.6217 | 1.6364 | 1.6217 | 1.6364 | 1.6217 | 1.5248 | 1.5014 | 1.8159 | 0.3920 |
| 5 | 1.9313 | 0.3647 | 1.4323 | 0.5391 | 1.6086 | 0.4255 | 1.1738 | 0.9298 | 1.6281 | 1.6129 |
| 10 | 1.8303 | 0.3861 | 1.6683 | 0.4363 | 1.6151 | 0.4495 | 1.3261 | 0.6666 | 1.6189 | 1.6030 |
| 15 | 1.8209 | 0.3907 | 1.7774 | 0.4023 | 1.6380 | 0.4451 | 1.5721 | 0.5002 | 1.6107 | 1.5942 |
| 20 | 1.8170 | 0.3917 | 1.8073 | 0.3942 | 1.6572 | 0.4388 | 1.7371 | 0.4218 | 1.6033 | 1.5863 |
| 25 | 1.8162 | 0.3919 | 1.8141 | 0.3925 | 1.6745 | 0.4331 | 1.7970 | 0.3987 | 1.5967 | 1.5791 |
| 30 | 1.8160 | 0.3920 | 1.8155 | 0.3921 | 1.6901 | 0.4281 | 1.8118 | 0.3934 | 1.5906 | 1.5726 |
| 35 | 1.8159 | 0.3920 | 1.8158 | 0.3920 | 1.7148 | 0.4213 | 1.8150 | 0.3923 | 1.5851 | 1.5666 |
| 40 | 1.8159 | 0.3920 | 1.8159 | 0.3920 | 1.7433 | 0.4129 | 1.8157 | 0.3921 | 1.5801 | 1.5612 |
| 45 | | | | | 1.7678 | 0.4057 | | | 1.5755 | 1.5562 |
| 50 | | | | | 1.7853 | 0.4008 | | | 1.5712 | 1.5516 |
| 55 | | | | | 1.7971 | 0.3973 | | | 1.5673 | 1.5473 |
| 60 | | | | | 1.8046 | 0.3952 | | | 1.5636 | 1.5434 |
| 65 | | | | | 1.8091 | 0.3939 | | | 1.5602 | 1.5397 |

* In the algorithm, $\delta_k, \bar{\delta}_k$ are chosen as $\delta_k = 0.1 \times 0.9^k, \bar{\delta}_k = 0.9^k$ (see (2.3), inaccurate search algorithm and Corollary 2.3, here $\alpha = 0.9924$).

† Here, $\delta_k$ and $\bar{\delta}_k$ are chosen the same as above, the step size of the gradient algorithm is chosen as $a_k = 1/(100 + k)$.

‡ Step size is chosen the same as above, i.e. $a_k = 1/(100 + k)$. (For some larger step sizes, the algorithm diverges.)

In the above analysis and comparison, the search steps between $k$ and $k + 1$, for any $k$, have not been taken into account in the accurate search algorithm, i.e. it has been assumed that the steps for each $k$ ($k$-scale) correspond to slow time scales, while the steps involved between the pair $(k, k + 1)$ ($n$-scale) correspond to fast time scales. □

## 6. GAMES WITH MORE THAN TWO PLAYERS

In this section, Theorem 1 is extended to many-player Nash games. The extension follows basically the lines of reasoning similar to that for Theorem 1, the main hurdle being the complexity of notation to be used.

Let $N = \{1, \ldots, n\}$ be the player set. Suppose that DM$i$, with strategy space $U^i$, has a rational response function $L_i: \prod_{\substack{j \neq i \\ j \in N}} U^j \to U^i$ for $i = 1, \ldots, n$, and consider the following sequential updating scheme.

*Scheme 6.1*

$$u^1_{k+1} = L_1(u^2_k, \ldots, u^n_k),$$

$$\vdots$$

$$u^m_{k+1} = L_m(u^1_{k+1}, \ldots, u^{m-1}_{k+1}, u^{m+1}_k, \ldots, u^n_k),$$

$$\vdots$$

$$u^n_{k+1} = L_n(u^1_{k+1}, \ldots, u^n_{k+1}), \quad m \leqslant n,$$

for all $(u^2_0, \ldots, u^n_0) \in \prod_{i=2}^{n} U^i$. □

Successively substituting gives

$$u_{k+1}^2 = L_2(L_1(u_k^2, \ldots, u_k^n), u_k^3, \ldots, u_k^n) = \tilde{L}_2(u_k^2, \ldots, u_k^n),$$

$$u_{k+1}^m = L_m(L_1(u_k^2, \ldots, u_k^n), \tilde{L}_2(u_k^2, \ldots, u_k^n), \ldots, \tilde{L}_{m-1} \times (u_k^2, \ldots, u_k^n), u_k^{m+1}, \ldots, u_k^n) = \tilde{L}_m(u_k^2, \ldots, u_k^n),$$

$$u_{k+1}^n = L_n(L_1(u_k^2, \ldots, u_k^n), \ldots, \tilde{L}_{n-1}(u_k^2, \ldots, u_k^n)) = \tilde{L}_n(u_k^2, \ldots, u_k^n)$$

Now define $T \prod_{i=2}^{n} U^i \to \prod_{i=2}^{n} U^i$, by

$$T(\bar{u}) = \begin{bmatrix} \tilde{L}_2 \\ \vdots \\ \tilde{L}_n \end{bmatrix} \bar{u}, \quad \text{for any } \bar{u} \in \bar{U} = \prod_{i=2}^{n} U^i$$

Then, the problem reduces to that of finding the fixed point of the equation $u = T(u)$

Note also that a particular case of the above is when players have circular dependent rational responses, i e

$$u_{k+1}^1 = L_1(u_k^n), \quad \ldots,$$
$$u_{k+1}^m = L_m(u_{k+1}^{m-1}), \quad \ldots, u_{k+1}^n = L_n(u_{k+1}^{n-1}),$$

which is equivalent to

$$u_{k+1}^n = L_n L_{n-1} \cdots L_1(u_k^n) = T(u_k^n)$$

The operators $L_i$ are derived from the implicit functions $\nabla_{u^i} J^i(u^1, \ldots, u^n) = 0, i \in N$, respectively, as discussed in Section 2

Now the result for 3-player games will be stated as a corollary of Theorem 1 The $n$-player case follows the same derivation, but is much messier (Li, 1985)

*Corollary* 3 Suppose that $J^i$ is strongly convex in its $i$th argument, and is continuously second-order Fréchet differentiable, $i \in N$ Furthermore, assume that the following condition is satisfied

$$\|T\| = \alpha < 1 \quad \text{for some positive } \alpha,$$

$$\text{where } T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (6\ 1)$$

and

$$A = (\nabla_{22}^{-1} J^2 \nabla_{21} J^2)_{(u^1, u^2, \bar{u}^3)} (\nabla_{11}^{-1} J^1 \nabla_{12} J^1)_{(u^1, \bar{u}^2, \bar{u}^3)},$$

$$B = (\nabla_{22}^{-1} J^2 \nabla_{21} J^2)_{(u^1, u^2, \bar{u}^3)} (\nabla_{1}^{-1} J^1 \nabla_{13} J^1)_{(u^1, \bar{u}^2, \bar{u}^3)} - (\nabla_{22}^{-1} J^2 \nabla_{23} J^2)_{(u^1, u^2, \bar{u}^3)},$$

$$C = (\nabla_{33}^{-1} J^3 \nabla_{31} J^3)_{(u^1, u^2, u^3)} (\nabla_{11}^{-1} J^1 \nabla_{12} J^1)_{(u^1, u^2, u^3)} - (\nabla_{33}^{-1} J^3 \nabla_{32} J^3)_{(u^1, u^2, u^3)} (\nabla_{22}^{-1} J^2 \nabla_{21} J^2)_{(u^1, u^2, u^3)} \times (\nabla_{11}^{-1} J^1 \nabla_{12} J^1)_{(u^1, \bar{u}^2, u^3)},$$

$$D = (\nabla_{33}^{-1} J^3 \nabla_{31} J^3)_{(u^1, u^2, u^3)} (\nabla_{11}^{-1} J^1 \nabla_{13} J^1)_{(u^1, u^2, u^3)} - (\nabla_{33}^{-1} J^3 \nabla_{32} J^3)_{(u^1, u^2, u^3)} (\nabla_{22}^{-1} J^2 \nabla_{21} J^2)_{(u^1, u^2, u^3)} \times (\nabla_{11}^{-1} J^1 \nabla_{13} J^1)_{(u^1, \bar{u}^2, u^3)} + (\nabla_{33}^{-1} J^3 \nabla_{32} J^3)_{(u^1, u^2, u^3)} \times (\nabla_{22}^{-1} J^2 \nabla_{23} J^2)_{(u^1, u^2, u^3)},$$

for any

$$(\bar{u}^2, \bar{u}^3) \in U^2 \times U^3,$$

and with

$$u^1 = L_1(\bar{u}^2, \bar{u}^3), \quad u^2 = L_2(u^1, \bar{u}^3), \quad u^3 = L_3(u^1, u^2)$$

and the operator norm is the one induced by the norm in the product Banach space $U^2 \times U^3$

Then, there exists a unique Nash equilibrium solution, and furthermore, the sequence $\{u_k^1, u_k^2, u_k^3\}$ defined by Scheme 6 1 converges to the Nash equilibrium solution for any initial point

*Proof* This follows a similar line of proof to that of Theorem 1 □

## 7 CONCLUSIONS

In this paper, conditions for the existence, uniqueness and stability of Nash equilibrium solutions in general nonquadratic convex games have been obtained Under these conditions, the Nash equilibrium solution can be computed in a distributed manner, without requiring that the players know each other's cost functionals An inaccurate search algorithm has also been obtained, and a proof for its convergence to the unique Nash equilibrium has been provided, under a number of reasonable conditions An immediate consequence of this result is that the distributed computation of the Nash solution is robust against small inaccuracies in the computation during each phase The numerical studies included in the paper do indeed corroborate these theoretical findings, and also demonstrate the superiority of the authors' algorithms under different inaccurate search schemes over other ones which directly use Newton's method or gradient-type iterations

Several extensions of the results presented in this paper seem to be possible One of these would be to constrained Nash games, where the reaction functions would clearly not be differentiable, in this case a different proof will have to be devised in order to verify a result of the type given in Theorem 1 A second extension would be to discrete-time dynamic games, by carefully constructing the strategy spaces and decomposing the general conditions into stagewise components Finally, the general contraction mapping result here may be used in the study of asychronous distributed algorithms (Bertsekas, 1983, Li and Başar, 1986)

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