Asymptotic Agreement and Convergence of Asynchronous Stochastic Algorithms

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Abstract—**In this paper, we present results on the convergence and asymptotic agreement of a class of asynchronous stochastic distributed algorithms which are in general time-varying, memory-dependent, and not necessarily associated with the optimization of a common cost functional. We show that convergence and agreement can be reached by distributed learning and computation under a number of conditions, in which case a separation of fast and slow parts of the algorithm is possible, leading to a separation of the estimation part from the main algorithm.**

I. Introduction

CONSIDER a system of processors which are connected by partial communication links and are engaged in distributed computation. Each processor has both private and shared information, and acquires new information via communications. A typical example is the distributed computation of solutions of fixed-point equations of the type $u = T(u)$, where $u \in U = \Pi_{i=1}^{K} U^i$ has K components, with the ith component being $u^i \in U^i$; such equations arise in the context of decision making, optimization, nonzero-sum games, etc. An iterative algorithm can be implemented by using several computers, each having the responsibility of computing only one component of u. There are two ways of implementing such a distributed algorithm. The standard way is to do it synchronously, according to a prefixed ordering of computation and information transmission. For example, to implement the algorithm $u_n = F(u_{n-1})$, at time n, each processor transmits the most recently computed value of the corresponding component to the rest of the processors by broadcasting it upon completing its nth computation. At the $n + 1$th iteration, each computation uses the most recently broadcast values of the $K - 1$ components.

There are two main drawbacks of synchronous computation. First, in some cases, especially in large scale networks, it is not real time implementable. For example, this may occur in noncooperative decision making, where it may be impossible to preagree on the ordering of computation and communication simply because of the noncooperative nature of decision making, bounded rationality of the decision makers, and the cost of maintaining fail-proof communication links. In a computer network, this real time implementability issue arises because of the nonrobustness of the algorithm, since convergence to a fixed point may fail if the communication channel is not perfect. The second drawback of synchronous computation is that it may constrain the overall speed of computation and rate of convergence when several very slow computers participate in the computation, along with fast ones. Therefore, there is a need to run the distributed system asynchronously.

Chazan and Miranker [1] were among the first to study asynchronous computation, but their investigation was restricted to the asynchronous distributed algorithm (ADA) for linear mappings of the form $u = Au + b$. Baudet [2] and Bertsekas [3] extended these results to nonlinear mappings on finite-dimensional spaces (say, $u \leftarrow T(u): R^K \rightarrow R^K$). They showed that if the product space U is endowed with the supremum norm under which T is a contraction, and if some kind of a so-called "continuous updating and transmission" condition is satisfied, then any ADA must converge to the solution of the nonlinear equation $u = T(u)$.

When the underlying system is stochastic (such as in stochastic teams, noncooperative stochastic games, dynamic routing algorithms in computer networks, etc.) the counterpart of the earlier equation is $u = T_t(u, F_t)$, where F_t is the information σ-field at time t. Here T_t is usually time-varying and memory-dependent. In this case the previous results cannot be applied for convergence analysis; furthermore, new issues arise, one of which is that of "asymptotic agreement" [4]. In a recent work [5], Borkar and Varaiya have shown that if agents of a team communicate to each other their most recent computations of the conditional expectation of a fixed random variable, and each agent updates his conditional mean based on the most recently received new information, then decisions converge (by the martingale convergence theorem), and convergence is to a common random variable provided that communications take place infinitely often. Subsequently, Tsitsiklis and Athans [6] extended this result to general convex team problems, and obtained convergence results (and asymptotic agreement) in the mean square sense, and in some cases with probability 1, using a descent function approach. Further results on asymptotic agreement in distributed detection (such as agreement on a single hypothesis) were obtained by Washburn and Teneketzis in [7].

In a general context, the common cost functional assumption may not hold. In the context of noncooperative decision making, for example, stochastic nonzero-sum games and even team problems where different agents develop different perceptions of the underlying probability space, require specification of more than one cost functional. Furthermore, even if the stochastic algorithm (of the type given above) is not related to any game problem, it may still not be possible to associate a common cost functional with it, because of the distributed nature of the computation and implementation. Available results on convergence analysis do not apply to such general stochastic distributed algorithms, because they are not contraction-type algorithms as in [3], and they do not enjoy the martingale and common descent properties of the algorithms studied in [5] and [6], respectively. This paper develops a new line of approach to study the stochastic ADA, and establishes convergence and asymptotic agreement (in a sense to be defined later) for various scenarios of distributed computation.

The organization of the paper is as follows. In Section II, we provide a mathematical description of the underlying model for a class of stochastic asynchronous distributed algorithms. In Section III, we study the convergence of a more general class of ADA's using random contraction mappings. In Section IV, we present results on asymptotic agreement, and exhibit a separation of time scales in computation and convergence. The paper ends with a Conclusions section, where we discuss some possible extensions and challenges for the future, and an Appendix.

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II. A Stochastic Model for Distributed Computation

In this section, we extend the models of [2], [5], and [6] to stochastic asynchronous distributed algorithms which are general time-varying, memory-dependent and not necessarily arising from the optimization of a common cost functional. Such algorithms arise, for example, in stochastic noncooperative games, when decision makers update on their actions (iterations in the action space) using the most recently available information. We will see that such algorithms do not fit into the models and the analyses available in the literature, and require different lines of approach than those used heretofore.

In what follows, let the underlying probability space be (Ω, F, P), and let the product decision (or computation) space be $U = \Pi_{i=1}^{K} U^i$, which is a complete metric space endowed with a norm $\|\cdot\|$ to be specified later. For each $u \in U$, we use the representation $u = (u^1, \cdots, u^K)$, $u^i \in U^i$, where each u^i will be called a *component* of u.

We consider a distributed system, comprising a set of $\bar{K} = \{1, 2, \cdots, K\}$ processors. We view each processor as a computing center or a decision maker, and denote the ith one by Pi. Processors communicate with each other through message transmission and reception. We assume that the communication channels are perfect with probability 1, and each processor knows the identity of the communicating party.

The memory content of each processor is modeled in terms of buffers. Each Pi has $K + 1$ buffers $B^i(j)$, $j = 0, 1, \cdots, K$. For $j \in \bar{K}$, let $B_t^i(j)$ denote the content of Pi's buffer at time t, corresponding to the information received up to and recalled at time t regarding computation (or actions) of P_j. For $j = 0$, $B_t^i(0)$ corresponds to "private measurements" received from the environment. The portion of the total information up to time t, which is still stored in Pi's memory at t, is denoted by $\eta_t^i = \cup_{j=0}^{K} B_t^i(j)$. We view η_t^i as a random quantity on the basic probability space, generating the (smallest) σ-algebra $F_t^i \subset F$.

Each processor has three modes of activities; namely, *reception, computation, and transmission*. Let $\{r_n^{ij}\}_{n\geq 1}$ be an increasing sequence of random times, where r_n^{ij} denotes the time instance when Pi receives the nth message w_n^{ij} from Pj, $j = 0, 1, \cdots, K$; $i \in \bar{K}$, where w_n^{i0} represents Pi's own observation of new data, and w_n^{ii} denotes his own computation. The message w_n^{ij}, for $j \neq i, j \in \bar{K}$, involves only computations of other processors, and not their private measurements, as will be made more clear in the sequel. For ease of presentation, we will assume that the transmission and reception of each message are done instantaneously. The reception is performed under a queueing discipline and each message is put into corresponding buffers. If a buffer has finite capacity, and is full at the time a message is received, then the oldest message in that buffer is discarded, to open space for the newcomer. Let $\{t_n^i\}_{n\geq 1}$ be an increasing sequence of random times, denoting the time instances when Pi computes. Pi computes only upon receiving a new message from Pj, for some $j \in \bar{K}, j \neq i$. For each $t > 0$, let $u_{it}^j \in U^j$ be the most current information P_i has in his buffer on the computations of P_j. (This information may have been received by P_i directly from P_j, or through P_k, $k \neq j, i$, prior to t.) In the computation mode, Pi generates a new u^i according to the asynchronous distributed algorithm (ADA):

$$u_n^i = \Psi_{t_n^i}^i(u_{it_n^i}^1, \cdots, u_{it_n^i}^K, \eta_{t_n^i}^i) \tag{2.1}$$

where $\Psi_{t_n^i}^i$, $i \in \bar{K}$ are some appropriate operators to be specified later.

Let $\{s_n^i\}_{n\geq 1}$ be an increasing sequence of random times when P_i transmits a message to some other processor(s). Let $\{s_{n_j}^i\}$ be a subsequence corresponding to times when P_i transmits to P_j. We assume that every such transmission is preceded by computation, and the message to P_j (from P_i) at time $s_{n_j}^i = r_m^{ji}$ is $w_m^{ji} \subset W_m^{ji} = \{u_{is_{n_j}^i}^k, k \in \bar{K}, k \neq j\}$, which is assumed to include $u_{t_p^i}^i$, where $p = \sup\{n: t_n < s_{n_j}^i\}$, (i.e., P_i's most recent computation). In this context, simultaneous transmission to more than one processor is possible; more precisely, to each s_n^i corresponds a subset $N_{s_n^i}$ of $\bar{K}$, chosen by a random law known to P_i, so that given $s_{n_j}^i$, there could exist $k \neq j$ where $k \in N_{s_{n_j}^i}$. Finally, we assume that the channels are perfect a.s., and introduce, for each $i \in \bar{K}$, a step function u_t^i, defined to be equal to u_n^i when restricted to the interval $[t_n^i, t_{n+1}^i)$. See Fig. 1 for a schematic description of this asynchronous communication and computation model.

To complete the mathematical description of the problem, we make the following standing assumptions.

Assumption A1: The times of computation for each processor are a.s. finite stopping times with respect to the information field of that processor.

Assumption A2: Each Pi communicates to every other Pj, $j \in \bar{K}, j \neq i$, infinitely often (i.o.), either directly or indirectly, i.e., for all $j \neq i, j \in \bar{K}$, either $Pi \to Pj$ i.o.,[1] or $Pi \to Pj$ i.o. through Pk.[2]

Remark 2.1: We have generalized the model given in [7] in the following aspects: 1) the operator $\Psi_{t_n^i}^i$ is time-varying; 2) $\Psi_{t_n^i}^i$ depends on memory through $\eta_{t_n^i}^i$, i.e., it depends on previous computation and private measurements which may possibly be dynamic. □

A Motivating Example 2.1: The above model could be used in a stochastic decision making context. Consider, for example, the following K-player static noncooperative stochastic game. Each player i (Pi) chooses his vector-valued action $u^i = \gamma^i(\eta^i)$ as the realization of a policy γ^i obtained by minimizing a cost functional

$$J^i(\gamma^1, \cdots, \gamma^K) = E\left(\frac{1}{2}\sum_{j=1}^{K}\sum_{k=1}^{K} u^{jT} R_{jk}^i u^k + \sum_{j=1}^{K} u^{jT} G_j^i x\right)^3 \tag{2.2}$$

over his policy set Γ^i, under the assumption that the remaining players' policies are fixed, and some static information is available on the random state of nature x, which is an m-dimensional random vector defined on the basic probability space (under possibly different probability measures for different P's). A set of strategies $\{\gamma^{i*} \in \Gamma^i, i \in N\}$ is then said to be in *Nash equilibrium* if

$$\gamma^{i*} = \arg\min_{\gamma^i \in \Gamma^i} J^i(\gamma^{i*}, \cdots, \gamma^{i-1*}, \gamma^i, \gamma^{i+1*}, \cdots, \gamma^{K*})$$

for all $i \in \bar{K}$ [9].

Başar has discussed in [10] the convergence of a distributed algorithm for such a system, when the players iterate on the *policy space*. Such a scheme does not involve exchange of probabilistic information, and hence does not incorporate learning. One important issue is whether an iteration in *action space*, which necessarily involves an exchange of probabilistic information (both static and dynamic), converges to a unique set of action variables, and whether there exists a limiting common information field with respect to which all these action variables are measurable.

A natural $\Psi_{t_n^i}^i$ which corresponds to such an iteration (in action space) is

$$u_n^i = -R_{ii}^{i-1}\sum_{j\neq i} R_{ij}^i u_{it_n^i}^j + G_i^i E(x|F_{t_n^i}^i), \qquad \text{for all } i \in N \tag{2.3}$$

[1] A sufficient condition is $r_n^{ji} \to \infty$ as $n \to \infty$ a.s.

[2] A sufficient condition is $r_n^{ki} \to \infty$, $r_n^{jk} \to \infty$ as $n \to \infty$ a.s. and w_n^{jk} includes $u_{ks_{n_j}^k}^i$ for infinitely many n_j, with probability 1.

[3] Here superscript T denotes the transpose of an appropriate dimensional vector.

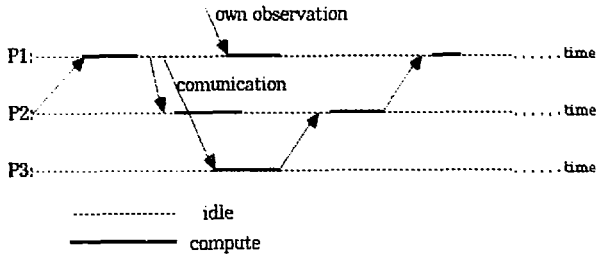

Fig. 1. The ADA algorithm.

and one possible synchronous algorithm is (obtained by taking $t_n^i = t_n$, independent of i):

$$u_{n+1} = Au_n + G\hat{x}_n, \qquad n = 0, 1, \cdots \tag{2.4}$$

where

$$A := \begin{bmatrix} 0 & D_{12}^1 & \cdots & D_{1K}^1 \\ D_{21}^2 & 0 & \cdots & D_{2K}^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ D_{K1}^K & D_{K2}^K & \cdots & 0 \end{bmatrix}, \quad D_{ij}^i := -R_{ii}^{i-1}R_{ij}^i, \quad \text{for all } i, j \in \bar{K},$$

$$G := \operatorname{diag}(G_1^1, \cdots, G_K^K),$$

$$\text{and } \hat{x}_{n+1}^T := (E[x^T|F_{t_n}^1], \cdots, E[x^T|F_{t_n}^K]). \quad \square$$

Returning to the ADA (2.1), let us now give the following definition to distinguish between the concepts of convergence and agreement.

Definition 2.1: The ADA *converges to an agreement* if 1) it converges, and 2) the limit u_∞^i is measurable with respect to F_∞^j for all $i, j \in \bar{K}$. □

Remark 2.3: In the context of decision making, and, in particular, Example 2.1, agreement means that $\{u_\infty^i, i \in \bar{K}\}$ is the solution of a static nonzero-sum game, in which each decision maker shares the same limiting common information $F_\infty := \cap_{i \in \bar{K}} F_\infty^i$. Note that $F_\infty^i - F_\infty$ is the information which is "private" to Pi, but irrelevant to the solution of the game. □

Now let us explain some of the difficulties associated with the general ADA when $\Psi_{t_n^i}^i$ is a general nonlinear operator. From (2.1), we see (also in view of (2.3) with the interpretation that $\eta_{t_n^i}^i$ is the random quantity which generates the σ-field $F_{t_n^i}^i$) that because of the time-varying and memory-dependent property, the contraction mapping argument of [3] cannot be applied here. Furthermore, since in (2.2), the cost functionals are inconsistent, a common descent function as in [6] does not exist, and clearly the algorithm is not a martingale as in [5]. This is an intrinsic feature of algorithms arising from noncooperative games.

In the following sections, we will show that a method which uses an appropriate combination of contraction arguments and martingale analysis exists for the class of problems with mappings of the type given by (2.1).

III. Convergence of General ADA's

In this section, we will discuss only the convergence issue of our general ADA. The results to be derived will be useful also for the agreement analysis of the next section.

Let us first introduce some definitions and notation.

Definition 3.1: An operator $\Phi: U X \Omega \to U$ is a *random operator* on U if $\Phi(u, \omega)$ is F-measurable. Furthermore, Φ is a *random contraction operator* on U if there exists a random variable α on (Ω, F), $0 \leqslant \alpha(\omega) < 1$ a.s., such that

$$P\{\omega : \|\Phi(u, \omega) - \Phi(v, \omega)\| \leqslant \alpha(\omega)\|u - v\|\} = 1$$

for all $u, v \in U$. □

Definition 3.2: With Φ defined as above, a random quantity $u^*: \Omega \to U$ is a *fixed point* of Φ if $u^* = \Phi(u^*, \omega)$ a.s. □

Results concerning random contractions can be found in [11] and [12], some of which are natural counterparts of known results on deterministic contractions. One such result, which we will be using in the sequel, is the following extension of the classical Banach contraction mapping theorem.

Lemma 3.1: Let U be a complete normed space, and Φ be a random contraction on U. Then there exists an a.s. unique fixed point u^* of Φ. Furthermore, the sequence $\{u_n\}$ generated by the algorithm $u_{n+1} = \Phi(u_n, \omega)$ converges almost surely to u^*. □

As a preliminary to our discussion of the ADA, we first consider in this section the synchronous version

$$u_{n+1} = \Psi_n(u_n, \eta_n), \qquad n = 0, 1, \cdots \tag{3.1}$$

for a general time-varying operator $\Psi_n: U X \Omega \to U$. The following result now follows.

Theorem 1: Suppose that there exists a random contraction operator Ψ on U, a complete normed space, such that

$$\limsup_{n \to \infty} \|\Psi_n(u, \eta_n) - \Psi(u, \omega)\| = 0 \qquad \text{a.s.} \tag{3.2}$$

for all $u \in U$ (where $\|\cdot\|$ denotes the norm on U). Then the sequence $\{u_n\}$ generated by (3.1) converges to the fixed point u^* of Ψ. □

Proof: The line of reasoning used in [13, p. 395] for the deterministic counterpart of this result can directly be applied to (3.1) to prove the theorem. □

Let us now replace the synchronous algorithm (3.1) with the ADA (2.1):

$$u_n^i = \Psi_{t_n^i}^i(u_{it_n^i}^1, \cdots, u_{it_n^i}^K, \eta_{t_n^i}^i), \qquad i \in \bar{K}. \tag{3.3a}$$

Furthermore, let us introduce the *max-norm* on the product space U by

$$\|u\| = \max_{1 \leqslant i \leqslant K} \{\|u^i\|_i\} \tag{3.3b}$$

where $u \in U$, $u^i \in U^i$ and $\|\cdot\|_i$ is any norm defined on U^i. One of our main results in this paper is the following theorem on the convergence of the algorithm (3.3a).

Theorem 2: Assume that Assumptions A1 and A2 are satisfied, U^i, $i \in \bar{K}$, are complete spaces with arbitrary (but fixed) norms, and U is endowed with the max-norm (3.3b). Suppose that there exists a random contraction operator Ψ on U, such that

$$\limsup_{t \to \infty} \|\Psi_t^i(u^1, \cdots, u^K, \eta_t^i) - \Psi^i(u^1, \cdots, u^K, \omega)\|_i = 0, \qquad \text{a.s.} \tag{3.4}$$

for all $u^j \in U^j$, $j \in \bar{K}$ and all $i \in \bar{K}$, where Ψ^i is the ith component of Ψ, corresponding to the decomposition on U.

Then, the sequence $\{u_n^i\}$ generated by (3.3a) converges almost surely to v^{i*}, $i \in \bar{K}$, where v^{i*} is the ith component of the unique fixed point of Ψ. □

Proof: Let $\{v_n^i\}_{n \geqslant 1}$ be a sequence in U^i, $i \in \bar{K}$, generated by (3.3a) with $\Psi_{t_n^i}^i$ replaced by the time-invariant operator Ψ^i for all $i \in \bar{K}$. Let v^* be the unique fixed point of Ψ, which exists by Lemma 3.1. Furthermore, introduce (as earlier), for each $i \in \bar{K}$, a step function v_t^i, defined to be equal to v_n^i when restricted to the interval $[t_n^i, t_{n+1}^i)$. (Recall that the sequence of increasing stopping times $\{t_n^i\}_{n \geqslant 1}$ denotes the time instances when P_i computes.)

For $t \geqslant 0$, let M_t be a nonnegative random variable on the basic probability space, defined by

$$M_t := \max_i \{\|v_t^i - v^{i*}\|_i\} \equiv \|v_t - v^*\|. \tag{3.5}$$

Let $s_0 > 0$ be arbitrary, and $s_0 < s_1 < s_2 \cdots < s_n < \cdots$ be an increasing sequence of stopping times such that in the interval $[s_{n-1}, s_n)$ each processor makes at least one computation which incorporates new messages received in the same interval from all other processors (either directly or indirectly). Mathematically speaking, for each $i \in \bar{K}$, there exists an integer m_i and $t^i \in [s_{n-1}, s_n)$ such that $s_n \geqslant t^i_{m_i} > t^i$, and for all $j \in \bar{K}, j \neq i$, $u^j_{t^i} \in B^i_{t^i}(j)$ implies $u^j_{t^i} \notin B^i_t(j)$ for any $t < s_{n-1}$. Note that such a sequence can be constructed recursively, by starting at s_0, under Assumption A2; in fact there exists such a "minimum" sequence of *stopping times* under Assumption A1 (we call a sequence $\{s_n\}_{n\geqslant 1}$ "minimum" if given any other sequence $\{s'_n\}_{n\geqslant 1}$ satisfying the above conditions, $s_n \leqslant s'_n$, $n = 1, 2, \cdots$, with strict inequality for at least one n). We now state a result on an important property of this sequence, which will be utilized in the sequel. A proof is provided in the Appendix.

Lemma 3.2: Let $s_0 \geqslant 0$ be an arbitrary time point such that $M_{s_0} < \infty$ a.s.. Let $\{s_n\}_{n\leqslant 1}$ be a "minimum" sequence of stopping times as introduced above. Then, for any nonnegative integer p,

$$M_t \leqslant \alpha(\omega)^p M_{s_0} \qquad \text{a.s. for all } t \geqslant s_P \tag{3.6}$$

where $\alpha(\omega)$ is the random variable which defines the contraction property of Ψ (cf. Definition 3.1). □

In words, the lemma above says that in each interval $[s_n, s_{n+1})$, the algorithm $v \leftarrow \Psi(v, \omega)$ generates a sequence which gets closer to v^* by a factor of at least $\alpha(\omega) \in [0, 1)$ a.s.

Now, let $\{u^i_n\}_{n\geqslant 1}$ be the sequence generated by the original algorithm (3.3a). It follows under condition (3.4) that, given any $\epsilon_0 > 0$, there exists a stopping time $T(\omega) > 0$ such that for all $t > T(\omega)$, and for all $u^j \in U^j$, $j \in \bar{K}$,

$$\|\Psi^i_t(u^1, \cdots, u^K, \eta^i_t) - \Psi^i(u^1, \cdots, u^K, \omega)\|_i < \epsilon_0, \qquad \text{a.s., for all } i \in \bar{K}.$$

Let s_0 in Lemma 3.2 be chosen such that $s_0 > T(\omega)$, and let $\{\epsilon_n\}_{n\geqslant 1}$ be a nonincreasing sequence of positive numbers converging to zero, such that for all $u^j \in U^j$, $j \in \bar{K}$, and for each $n \geqslant 1$,

$$\|\Psi^i_t(u^1, \cdots, u^K, \eta^i_t) - \Psi^i(u^1, \cdots, u^K, \omega)\|_i < \epsilon_n, \qquad \text{a.s.} \tag{3.7}$$

for all $t > s_n$, $i \in \bar{K}$.

Under (3.4), there exists, necessarily, a subsequence of positive integers $\{n\}_{n\geqslant 1}$, say $\{n_k\}_{k\geqslant 1}$, such that $0 \leqslant \epsilon_{n_{k+1}} < \epsilon_{n_k}$, $k = 1, 2, \cdots$. Let $\{t_{n_k}\}_{k\geqslant 1}$ be any increasing sequence of stopping times, with $t_{n_k} \in (s_{n_k}, s_{n_{k+1}}]$. Then, along this sequence,

$$\|u_{t_{n_k}} - v^*\| \leqslant \|u_{t_{n_k}} - v_{t_{n_k}}\| + \|v_{t_{n_k}} - v^*\|. \tag{3.8}$$

The first term on the right-hand side is equal to

$$\|u_{t_{n_k}} - v_{t_{n_k}}\| = \max_i \{\|\Psi^i_{t_i}(u^1_{it_i}, \cdots, u^K_{it_i}, \eta^i_{t_i}) - \Psi(u^1_{it_i}, \cdots, u^K_{it_i}, \omega)\|_i\}$$

for some $t_i > s_{n_{k-1}}$, $i \in \bar{K}$, where t_i denotes the most recent computation time for P_i, prior to t_{n_k}. In view of (3.7), this quantity can be bounded from above by $\epsilon_{n_{k-1}}$. Using this in (3.8), we have, also in view of (3.6),

$$\|u_{t_{n_k}} - v^*\| \leqslant \epsilon_{n_{k-1}} + M_{t_{n_k}} \leqslant \epsilon_{n_{k-1}} + \alpha(\omega)^{n_k} M_{s_0}.$$

Since $\alpha(\omega) \in [0, 1)$ a.s., M_{s_0} is finite a.s. and $\epsilon_{n_k} \to 0$ as $k \to \infty$, it readily follows that $\|u_t - v^*\| \to 0$ along $\{t_{n_k}\}$. But since $\{t_{n_k}\}$ was chosen arbitrarily, and since (3.5) and (3.7) hold for all sufficiently large t, $\|u_t - v^*\| \to 0$ a.s. □

One application of Theorems 1 and 2 is to the class of quadratic stochastic Nash games formulated in Example 2.1 of Section II. This result is presented below as a corollary.

Corollary 1: For the quadratic stochastic Nash game of Example 2.1, suppose that:

i) $F^i_t \subset F^i_s$, for all $t < s$;

ii) $\|A\| < 1$, where $\|A\|$ is the operator norm corresponding to the norm defined on U.

Then:

a) the synchronous algorithm (2.4) converges a.s. under any norm defined on U;

b) the ADA converges a.s. provided that the norm on U is the max-norm (3.3b). □

Remark 3.1: In the general ADA, $\Psi^i_{t^i_n}$ is time-varying, because it may depend on time through the joint distribution of random quantities of interest and $\eta^i_{t^i_n}$. Another case in which $\Psi^i_{t^i_n}$ may depend on time is when an approximation algorithm is used (such as inaccurate search), even though the original recursion may be time-invariant. Such algorithms use several intermediate steps of some stochastic approximation algorithm (such as Newton's algorithm, gradient algorithm) between any two successive steps of the main algorithm. Hence, to deal with general time-varying iterations and to develop a theory of convergence for these is important and is highly relevant to real world implementation. This point has been discussed further in [14] in the context of deterministic problems. □

In the next section we address the issue of asymptotic agreement.

IV. Asymptotic Agreement of ADA

Asymptotic agreement problem is, as discussed earlier, a very difficult one for the general ADA. To make the problem tractable, we restrict our attention in this section to a particular class of algorithms with separable mappings, and show that for this class of problems the part of the algorithm relating to memory can be separated out, and treated as an algorithm running on a faster time scale. Hence, the general convergence and agreement can be analyzed sequentially on two different time scales.

Toward this end, we first replace the ADA (2.1) with

$$u^i_n = T^i_{t^i_n}(u^1_{it^i_n}, \cdots, u^K_{it^i_n}) + Q^i(E(x|F^i_n)) \tag{4.1}$$

where $F^i_n := F^i_{t^i_n}$ and x is a random quantity defined on the basic probability space, taking values in a complete normed space X. We assume that $E\{\|x\|\} < \infty$, and $Q^i: X \to U^i$, $T^i_t: U \to U^i$ are general operators. Furthermore, in addition to Assumptions A1 and A2, we make the following assumptions.

Assumption A3: Informational requirements:

i) Buffers of processors are of infinite length, and $F^i_{n-1} \subset F^i_n$ for all $i \in \bar{K}$;

ii) $w^{ij}_n = W^{ij}_n$, and is F^i_n-measurable for all $j \in \bar{K}, j \neq i, i \in \bar{K}$.

Assumption A4: The event $\{Pi$ communicates to Pj i.o.$\}$ belongs to $F^i_\infty \cap F^j_\infty$, where $F^i_\infty := \cup_{n=1}^\infty F^i_n$, $i \in \bar{K}$.

Finally, we introduce the notation

$$Q := \operatorname{diag}(Q^1, \cdots, Q^K) \tag{4.2a}$$

$$\xi^i := E[x|F^i_\infty], \qquad i \in \bar{K} \tag{4.2b}$$

and let ξ be the random quantity whose ith component is ξ^i.

The main result of this section is then the following.

Theorem 3: Suppose that Assumptions A1–A4 are satisfied, and that the complete space U is endowed with the max-norm (3.3b). Furthermore, suppose that the following two conditions are satisfied.

i) There exists a contraction operator T on U such that

$$\limsup_{n\to\infty} \| T^i_{t^i_n}(u^1, \cdots, u^K) - T^i(u^1, \cdots, u^K)\|_i = 0$$

a.s. for all $i \in \bar{K}$, for all $u \in U$. (4.3)

ii) Q^i, $i \in \bar{K}$, are left invertible and continuous.

Then, the sequence $\{u^i_n, i \in \bar{K}\}$ generated by (4.1) converges to the unique fixed point of $T(u^1, \cdots, u^K) + Q(\xi)$, which is, furthermore, an asymptotic agreement of the ADA. □

Remark 4.1: The left invertibility condition is satisfied for the quadratic nonzero-sum game problem introduced in Example 2.1, if G^i_i, $i \in \bar{K}$, are left invertible. This condition is not too stringent; for example, if $x \in R$ and $u^i \in U^i = R^{p_i}$, then $G^i_i \in R^{p_i \times 1}$, and the left invertibility requires that there exists a $K^i \in R^{1\times p_i}$ such that

$$\sum_{j=1}^{p_j} k^i_j g^i_j = 1,$$

$$\text{where } K^i := (k^i_1, \cdots, k^i_{p_i}),\ G^i_i := (g^i_1, \cdots, g^i_{p_i})^T. \quad (4.4)$$

□

In view of this remark, we now have the following result on the linear quadratic stochastic Nash game.

Corollary 2: For the quadratic stochastic Nash game introduced in Example 2.1, suppose that Assumptions A1–A4 are satisfied, and that

i) condition (4.3) holds;

ii) $\|A\| < 1$, where $\|\cdot\|$ is the matrix norm induced by the max-norm (3.3b) on U.

Then, the ADA converges a.s. to an agreement. □

We now proceed with the proof of Theorem 3. First we state a lemma which will be useful in the sequel.

Lemma 4.1: Suppose that in (4.1), $T^i_{t^i_n} \equiv 0$, for all sequences $\{t^i_n\}$ and $i \in \bar{K}$, and that Q^i is the identity operator on X for all i. Furthermore, suppose that Assumptions A1–A4 are satisfied. Then, the ADA converges to an agreement, i.e.,

$$u^i_\infty = E\{x|F^1_\infty \cap \cdots \cap F^K_\infty\}, \qquad \text{for all } i \in \bar{K}.$$

Proof: See [5], [7] or the Appendix of [15]. □

Proof of Theorem 3: First, we note that, by Assumption A3 i), and $E(\|x\|) < \infty$, $\xi^i_n = E(x|F^i_n)$ is a uniformly integrable martingale for all $i \in \bar{K}$, and converges a.s. to some random variable ξ^i, defined in (4.2b) [14]. Introduce a random operator $\hat{T}: UX\Omega \to U$ whose ith component is

$$\hat{T}^i(u, \omega) := T^i(u^1, \cdots, u^K) + Q^i(\xi^i) \qquad \text{a.s. for all } u \in U.$$

Then clearly, by condition i) of the theorem, $\hat{T}$ is a random contraction operator, and furthermore, along the sequence $\{t^i_n\}$,

$$\limsup_{n\to\infty} \| T^i_{t^i_n}(u^1_{it^i_n}, \cdots, u^K_{it^i_n}) + Q^i(E(x|F^i_n)) - \hat{T}^i(u^1_{it^i_n}, \cdots, u^K_{it^i_n}, \omega)\|_i = 0$$

a.s. for all $i \in \bar{K}$, by (4.3) and continuity of Q^i.

Hence, by Theorem 2, the ADA converges almost surely to the fixed point of $\hat{T}$.

Now we claim that $\xi^i = \xi$, a.s. for all $i \in \bar{K}$, and ξ is F^j_∞-measurable for all $j \in \bar{K}$. For fixed $i, j \in \bar{K}$, $i \neq j$, suppose that at time r^{ij}_n, Pi receives the nth message w^{ij}_n from Pj. This message necessarily includes the most recent computation of Pj, say $u^j_{t^j_p}$, where $t^j_p := \sup_m \{t^j_m : t^j_m < r^{ij}_n\}$. This is Pj's pth computation, which has been executed according to

$$u^j_{t^j_p} \equiv u^j_p = T^j_{t^j_p}(u^1_{jt^j_p}, \cdots, u^K_{jt^j_p}) + Q^j(\xi^j_p).$$

Under Assumptions A2 and A3 i), there exists an integer p_j such that $t^i_{p_j} = \inf_m \{t^i_m \geqslant t^j_p : u^k_{jt^j_p} \in B^i_{t^i_{p_j}}(k)$ for all $k \in \bar{K}, k \neq i\}$. Hence, $Q^j(\xi^j_p)$ is $F^i_{t^i_{p_j}}$-measurable, and since Q^j is left invertible, ξ^j_p is also $F^i_{t^i_{p_j}}$-measurable. Note that if Pi had received $u^j_{t^j_p}$ not directly from Pj, but indirectly through, say, Pk, at some time $r^{ik}_q > t^j_p$, the same argument as above would apply, showing that ξ^i_p is $F^i_{t^i_{p_j}}$-measurable for some t_{p_j}.

Now, under Assumption A2, there exists almost surely an infinite sequence of random (computation) times $\{t^j_n\}$ for Pj, say $\{t^j_{n'}\}$, such that $t^j_{p'} > t^j_{m'}$ implies that Pi receives $u^j_{m'}$ before $u^j_{p'}$, either directly from Pj, or through some other processor Pk, which further implies that $F^i_{t^i_{p_j}} \supset F^i_{t^i_{m_j}}$. Since $\xi^i_n \to \xi^i$ along this new sequence $\{n'\}$, and $F^i_{t^i_{n_j}}$ is nondecreasing, it follows that ξ^j is F^i_∞-measurable.

Hence, we have a sequence $\{\xi^j_n, F^i_m, i, j \in \bar{K}; n, m \geqslant 1\}$ which defines an asynchronous estimation problem of the type covered by Lemma 4.1. We can thus invoke Lemma 4.1 to conclude that

$$\xi^i = E(x|F^1_\infty \cap \cdots \cap F^K_\infty) \qquad \text{for all } i \in \bar{K}. \qquad \square$$

Remark 4.2: From the proof of Theorem 3 we first note that the same result would hold true if we replace operator Q^i in (4.1) by a time varying operator Q^i_t, left-invertible continuous for all $t \geqslant 0$, satisfying the additional condition that there exists a left-invertible continuous operator Q^i such that

$$\limsup_{n\to\infty} \|Q^i_{t^i_n}(x) - Q^i(x)\|_i = 0$$

a.s. for all $i \in \bar{K}$, for all $x \in X$.

Second, we point out that in the proof we have actually separated the fast and the slow parts of the algorithm in (4.1) in the following sense: Under some "sufficient richness of information" conditions (e.g., Assumption A3, and ii) of Theorem 3), the estimation part $Q^i(E\{x|F^i_n\})$ converges to an agreement independently of the convergence of the decision sequence, and hence can be viewed as the fast convergent part. On the other hand, under the contraction conditions, the decisions computed by the algorithm are bounded when the fast part is still away from its limit, but they tend to approach the limit only after the fast part converges to some small neighborhood of its limit. □

V. Concluding Remarks

In this paper, we have generalized the available results on the convergence of contraction mapping algorithms given, for example, in [2], [3], and [13] to time-varying, memory-dependent stochastic ADA's. Also, we have analyzed the asymptotic agreement of asynchronous distributed algorithms which involve memory and arise from optimization of different cost functionals for different decision makers. We have shown that for a class of such algorithms, with a specific additive structure, we can separate out the estimation part from the main algorithm, thus decoupling the analyses of convergence and agreement. In view of these, the results of this paper could also be considered as extensions of the results of [5] and [6] to a different (more general) class of problems.

Some possible extensions of the results presented here are the following. First, we had studied earlier in [10] and [17], synchronous distributed algorithms (operating on the policy space) which arise in the context of stochastic games with quadratic cost functionals and inconsistent subjective probability measures. It seems possible to analyze ADA's for similar problems, but on the action space, within the framework developed in these two papers. Second, we have not considered

here mean square convergence of the ADA. As discussed earlier, [6] contains results on asymptotic agreement in mean square (m.s.) and in probability, and it would be important to extend the results in this paper to m.s. convergence mode so as to solve a larger class of problems in which we could possibly separate the team part from the main algorithm. The agreement problem for the general ADA considered in Section III (without the additive decomposition) seems to be a very difficult one as discussed in the paper, and it remains open for future research.

Appendix

Lemma 3.2: Let $s_0 \geqslant 0$ be an arbitrary time point such that $M_{s_0} < \infty$ a.s. Let $\{s_n\}_{n \geqslant 1}$ be a "minimum" sequence of stopping times as introduced above. Then, for any nonnegative integer p,

$$M_t \leqslant \alpha(\omega)^p M_{s_0} \qquad \text{a.s. for all } t \geqslant s_p \tag{A.1}$$

where $\alpha(\omega)$ is the random variable which defines the contraction property of Ψ (cf. Definition 3.1).

Proof: The proof proceeds by induction on p, by following the line of reasoning used in [2, p. 231] in the proof of Theorem 1.

First let $p = 0$, and note that (A.1) holds for $t = s_0$ (as an equality). We now assume that given an $s \geqslant s_0$, it holds for all $t \in [s_0, s]$, and prove that this implies validity also for $t > 0$. Toward this end, assume without any loss of generality, that no two processors compute simultaneously; i.e., there is *no* pair (i, j), $i \neq j$, and *no* pair (n, m), $n, m = 1, 2, \cdots$ such that $t_n^j = t_m^i$, and s is not a computation time of any processor. Let $q = \min_{n,j} \{t_n^j \geqslant s\} > s$, and say that this corresponds to a computation time of Pi. Then,

$$\begin{aligned}\|u_q - v^*\| &= \max\{\max_{j \neq i} \|u_s^j - v^{j*}\|_j, \|\Psi^i(u_{iq}^1, \cdots, u_{iq}^K, \omega) - v^{i*}\|_i\} \\ &\leqslant \max\{M_s, \|\Psi(u_{iq}^1, \cdots, u_{iq}^K, \omega) - v^*\|\} \\ &\leqslant \max\{M_s, \alpha(\omega) \max_k \|u_{iq}^k - v^{k*}\|_k\} \\ &\leqslant \max\{M_s, \alpha(\omega) M_s\} = M_s.\end{aligned}$$

In the above sequence of inequalities, the first one follows from the definition of max-norm (3.3b), the second one follows since Ψ is a random contraction mapping with multiplying factor $\alpha(\omega)$, and the last one follows because for each $k \in \bar{K}$, $u_{iq}^k = u_t^k$ for some $t < s$ (i.e., Pi's information at time q is based on computations made prior to s), and $M_t \leqslant M_{s_0}$ for all $t < s$ by our induction hypothesis. Hence, the first time a processor computes after s, the value of M_s does not increase. On the other hand, if all processors remain idle in a nonzero interval $[s, \bar{t})$, clearly M attains the same value M_s throughout the interval. Since at any point t in time, either all processors are idle or one processor is active, and M does not increase in either case, it follows readily that $M_s \leqslant M_{s_0}$ implies $M_t \leqslant M_s$ for all $t > s$. The result clearly also holds if s were a computation time for one processor, because then there is an idle period $(s, s']$, $s' > s$, following s. This proves (A.1) for $p = 0$.

To prove the inequality for a general $p > 0$, we proceed by induction, assuming that it is true for p and verify it for $p + 1$. Because of the way the sequence $\{s_n\}_{n \geqslant 1}$ was constructed, for each processor there is a computation in the interval (s_p, s_{p+1}) which uses u_s^j, for some $s^j \geqslant s_p$, $j \in \bar{K}$. Let Pi execute a computation at time $t_{n_p}^i \in (s_p, s_{p+1})$ under such an information. (Note that this has to be Pi's at least second computation in $[s_p, s_{p+1})$.) Then,

$$\begin{aligned}&\|\Psi^i(u_{it_{n_p}^i}^1, \cdots, u_{it_{n_p}^i}^K, \omega) - v^{i*}\|_i \\ &\leqslant \|\Psi(u_{it_{n_p}^i}^1, \cdots, u_{it_{n_p}^i}^K, \omega) - v^*\| \\ &\leqslant \alpha(\omega) \max_j \{\|u_{it_{n_p}^i}^j - v^{j*}\|_j\}\end{aligned} \tag{A.2}$$

where the first inequality follows from the definition of the max-norm (3.3b), and the second one follows since Ψ is a random contraction operator. Recall that $u_{it_{n_p}^i}^j$ is the most recent information available to Pi prior to $t_{n_p}^i$ on the computation of Pj, and because of the way the sequence $\{s_n\}_{n \geqslant 1}$ was constructed and the way $t_{n_p}^i$ was chosen, this value corresponds to a computation by Pj in the interval $[s_p, t_{n_p}^i)$, i.e., there exists an integer m_j such that $u_{it_{n_p}^i}^j \equiv u_{t_{m_j}^j}^j$, with $t_{m_j}^j \in [s_j, t_{n_p}^i)$. Let $t(p) := \inf_j \{t_{m_j}^j\} \geqslant s_p$. Then, (A.2) is bounded from above by

$$\leqslant \alpha(\omega) \max_j \{\|u_{t_{m_j}^i} - v^*\|\} = \alpha(\omega) \max_j \{M_{t_{m_j}^j}\}$$

which is further bounded above by the following, since $t_{m_j}^j \geqslant s_p$ for all $j \in \bar{K}$, and by hypothesis, (A.1) is true for p

$$\leqslant \alpha(\omega) \max_j \{\alpha(\omega)^p M_{s_0}\} = \alpha(\omega)^{p+1} M_{s_0}. \tag{A.3}$$

Hence,

$$\|u_{n_p}^i - v^{i*}\|_i \leqslant \alpha(\omega)^{p+1} M_{s_0} \qquad \text{a.s.},$$

and the reasoning employed in the proof of this result clearly shows that for any computation time $t_m^i > t_{n_p}^i$,

$$\|u_{t_m^i}^i - v^{i*}\|_i \leqslant \alpha(\omega)^{p+1} M_{s_0} \qquad \text{a.s.}$$

Furthermore, since $u_t^i \equiv u_{t_m^i}^i$ for $t \in [t_m^i, t_{m+1}^i)$,

$$\|u_t^i - v^{i*}\|_i \leqslant \alpha(\omega)^{p+1} M_{s_0} \qquad \text{a.s. for all } t \geqslant t_{n_p}^i$$

and since $i \in \bar{K}$ was arbitrary and $t_{n_p}^i < s_{p+1}$, for all $i \in \bar{K}$,

$$M_t = \max_i \|u_t^i - v^{i*}\|_i \leqslant \alpha(\omega)^{p+1} M_{s_0} \qquad \text{a.s. for all } t \geqslant s_{p+1}.$$

This then completes the induction process, and thus the proof of the lemma. □

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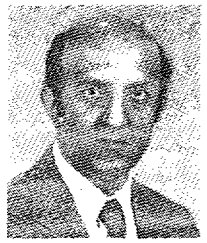

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